

MULTITAPER ESTIMATION OF THE COHERENCE SPECTRUM IN LOW SNR

Johan Brynolfsson and Maria Hansson-Sandsten

Lund University,
Mathematical Statistics, Centre for Mathematical Sciences,
Box 118, SE-221 00 Lund, Sweden.

ABSTRACT

A pseudo coherence estimate using multitapers is presented. The estimate has better localization for sinusoids and is shown to have lower variance for disturbances compared to the usual coherence estimator. This makes it superior in terms of finding coherent frequencies between two sinusoidal signals; even when observed in low SNR. Different sets of multitapers are investigated and the weights of the final coherence estimate are adjusted for a low-biased estimate of a single sinusoid. The proposed method is more computationally efficient than data dependent methods, and does still give comparable results.

Index Terms— coherence, cross-spectrum, multitaper, sinusoids

1. INTRODUCTION

The coherence spectrum (CS) is a well established measure of the relation between two signals as a function of frequency. Assuming the cross-spectrum between the two stationary signals $x(t)$ and $y(t)$ is denoted $S_{XY}(f)$ and the auto-spectra are denoted $S_X(f)$ and $S_Y(f)$, the CS is defined as

$$C_{XY}(f) = \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)}. \quad (1)$$

There are multiple ways to estimate the CS where Welch's method was among the first and has been quite common since, [1–3], but the more advanced method of multitapering have also been used, [4–8]. Recently there has also been a surge of data dependent matched filter bank methods for estimation of sinusoids in heavy disturbance, e.g. [9, 10]. The Welch's method and multitaper methods suffer from high variance and spectral leakage whereas the data dependent filter bank methods have been shown to be more accurate when estimating sparse CS but are computationally heavy.

The concept of multitapers was invented by Thomson, [4], where the main idea is to reduce the variance of the spectral estimate by averaging several uncorrelated periodograms. The same data sequence is used for all periodograms but the

shape of the window changes in a way to give uncorrelated periodograms and thereby reduced variance. Other suggested multitaper methods are more suitable for non-smooth spectra and peaked spectra such as the sinusoidal tapers and the Peak Matched MultiTapers (PMMT) [5, 6].

We suggest an altered version of the computationally efficient multitaper method with lower variance and better concentration that is also comparable with the data dependent filter bank methods when estimating the CS for signal measurements with a low SNR. In this paper we show that the method is useful when investigating common frequencies between signals, rendering high accuracy and low variance.

2. MULTITAPER PSEUDO COHERENCE

Assume that we have two signals expressed in their tapered spectral representation

$$\begin{cases} X_k(f) = \sum_n x(n)h_k(n)e^{-i2\pi fn}, \\ Y_k(f) = \sum_n y(n)h_k(n)e^{-i2\pi fn}. \end{cases}$$

Estimating the spectra based on the entire observed signals using K different tapers, a variance reduction is achieved for a set containing K orthonormal tapers, i.e.,

$$h_k h_j^T = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

where $h_k = [h_k(0) \dots h_k(N-1)]$ and $(\cdot)^T$ is the transpose operator. The multitaper spectral estimate for the process $x(n)$, $\hat{S}_X(f)$, is formed as

$$\hat{S}_X(f) = \sum_{k=1}^K \alpha_k X_k(f) X_k(f)^H, \quad (2)$$

where $(\cdot)^H$ denotes the Hermitian conjugate and α_k are the weighting factors. For energy conservation, it must hold that

$$\sum_{k=1}^K \alpha_k = 1, \quad (3)$$

and by limiting the weights such that $\alpha_k \geq 0 \forall k$, one ensures the desired positivity of the spectral estimate. The classic way to estimate the multitaper coherence spectrum (MT-CS) using K number of tapers is written as

Thanks to the Swedish Research Council for funding.

$$C_{XY}^{MT-CS}(f) = \frac{|\sum_k \alpha_k X_k(f) Y_k^H(f)|^2}{(\sum_l \alpha_l X_l(f) X_l^H(f)) (\sum_j \alpha_j Y_j(f) Y_j^H(f))}. \quad (4)$$

We propose a slight change to this expression where the absolute value over the cross-spectrum is simply moved inside the sum.

$$C_{XY}^{MT-PCS}(f) = \frac{\sum_k \alpha_k^2 |X_k(f) Y_k^H(f)|^2}{(\sum_l \alpha_l X_l(f) X_l^H(f)) (\sum_j \alpha_j Y_j(f) Y_j^H(f))}. \quad (5)$$

We denote this method Multitaper Pseudo Coherence Spectrum (MT-PCS), which is a biased estimator that essentially treats the two signals as statistically independent. For strong dependencies between two signals, such as common sinusoids oscillating with the same frequency, the MT-PCS will retain large coherence, similar as for the usual coherence. The advantage of the MT-PCS estimate is that it will give a lower coherence level than the MT-CS, and additionally lower variance (shown below), for independent as well as dependent noise disturbances.

There are different sets of tapers to choose from when estimating spectra where the set of Thomson multitapers, which are optimized for white noise, is the most common one [4]. However, as we here want to identify peaks in the spectrum we will turn to the sinusoidal tapers and the Peak Matched multitapers (PMMT) [5, 6] as these have better performance for varying spectra. The choice of weights and number of tapers used, K , must also be addressed. The standard method is to choose tapers such that a certain band-width is covered and then using equally distributed weights, i.e. $\alpha_k = 1/K$ $k = 1, \dots, K$. Another way is to do some optimization, typically minimizing mean squared error (MSE) or a weighted version of the MSE [6].

3. PROPERTY ANALYSIS

Below some properties of the MT-PCS estimator is derived. It is assumed the noise in the two signals are statistically independent.

3.1. Expected Value

As the auto-spectra are calculated in the same way in both estimates, i.e. the denominators in equations (4) and (5), only the properties of the cross-spectrum estimate will be investigated here. Below, the following notations will therefore be used:

$$\begin{aligned} \hat{S}_{XY}^{MT-CS}(f) &= \left| \sum_k \alpha_k X_k(f) Y_k^H(f) \right|^2 \\ \hat{S}_{XY}^{MT-PCS}(f) &= \sum_k \alpha_k^2 |X_k(f) Y_k^H(f)|^2. \end{aligned} \quad (6)$$

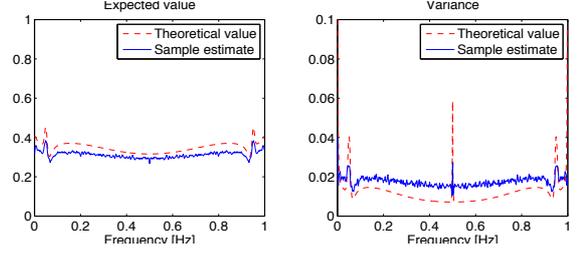


Fig. 1. Expected value and variance estimated from simulations compared with the approximated theoretical values.

As mentioned, this new estimator introduces some bias. However, how biased is determined by the choice of multitapers. This can be seen by expanding the expression for the cross-spectrum of the old method, MT-CS:

$$\begin{aligned} & \left| \sum_k \alpha_k X_k Y_k^H \right|^2 \\ &= \sum_j \sum_k \alpha_j \alpha_k X_j Y_j^H X_k^H Y_k \end{aligned} \quad (7)$$

$$= \sum_k \alpha_k^2 |X_k Y_k^H|^2 + \sum_{j \neq k} \alpha_j \alpha_k X_j Y_j^H X_k^H Y_k, \quad (8)$$

where we have released the notation including the frequency parameter. Note that the first sum of equation (8) can then be identified as $\hat{S}_{XY}^{MT-PCS}(f)$. Assuming that the two observed processes are statistically independent we then have that

$$\begin{aligned} \mathbb{E} \{ \hat{S}_{XY}^{MT-CS} \} &= \mathbb{E} \{ \hat{S}_{XY}^{MT-PCS} \} \\ &+ \sum_{j \neq k} \alpha_j \alpha_k \mathbb{E} \{ X_j X_k^H \} \mathbb{E} \{ Y_j^H Y_k \}, \end{aligned} \quad (9)$$

where $\mathbb{E} \{ \cdot \}$ denotes the expected value. It has been previously shown that for a Gaussian process, $x = [x(0) \dots x(N-1)]^T$, it holds that

$$\mathbb{E} \{ X_j(f) X_k(f)^H \} = h_j \Phi(f) R_x \Phi(f)^H h_k^T, \quad (10)$$

where $\Phi(f) = \text{diag}(1 e^{-i2\pi f} \dots e^{-i2\pi(N-1)f})$ and $R_x = \mathbb{E}[xx^T]$ is the covariance matrix [6, 11]. From this one can draw the conclusion that, if one chooses tapers that are as close to orthogonal with respect to the covariance matrices R_x and R_y as possible, the last term will be small.

3.2. Variance

The two main advantages with our proposed MT-PCS estimate compared to MT-CS is the lowered variance and the better localization. Again, by only considering the numerator in the CS estimates, and remembering equation (8), one can see that the MT-CS estimate equals the MT-PCS estimate plus an extra term. It will therefore hold that

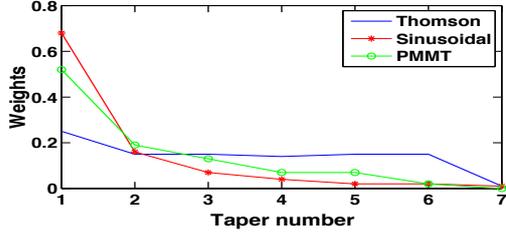


Fig. 2. Resulting weights for the three sets of tapers.

$$\mathbb{V} \left\{ \left| \sum_k \alpha_k X_k(f) Y_k^H(f) \right|^2 \right\} \geq \mathbb{V} \left\{ \sum_k \left| \alpha_k X_k(f) Y_k^H(f) \right|^2 \right\}, \quad (11)$$

as, even though the extra terms may have expected value close to zero, they will still add variance. Below $\mathbb{V} \{ \cdot \}$ denotes variance and $\mathbb{C} \{ \cdot, \cdot \}$ denotes covariance. For completeness we will derive an analytic expression of the variance of the cross-spectrum estimate for two independent Gaussian signals. The variance then is

$$\begin{aligned} & \mathbb{V} \left\{ \hat{S}_{XY}^{MT-PCS} \right\} \\ &= \mathbb{V} \left\{ \sum_k \alpha_k^2 \left| X_k(f) Y_k^H(f) \right|^2 \right\} \\ &= \sum_j \sum_k \alpha_j^2 \alpha_k^2 \mathbb{C} \left\{ X_j Y_j^H Y_j X_j^H, X_k Y_k^H Y_k X_k^H \right\}. \end{aligned} \quad (12)$$

The covariance for each (j, k) combination is

$$\begin{aligned} & \mathbb{C} \left\{ X_j Y_j^H Y_j X_j^H, X_k Y_k^H Y_k X_k^H \right\} \\ &= \mathbb{E} \left\{ X_j Y_j^H Y_j X_j^H X_k^H Y_k Y_k^H X_k \right\} - \\ & \quad - \mathbb{E} \left\{ X_j Y_j^H Y_j X_j^H \right\} \mathbb{E} \left\{ X_k^H Y_k Y_k^H X_k \right\}. \end{aligned} \quad (13)$$

As the processes $X(f)$ and $Y(f)$ are assumed to be independent, the expected value of all mixed combinations of X and Y equals zero. Below we make use of Isserlis' theorem, [12], which states that if X_1, X_2, X_3, X_4 are circularly symmetric zero-mean Gaussian random variables it holds that

$$\begin{aligned} & \mathbb{E} \{ X_1 X_2 X_3 X_4 \} \\ &= \mathbb{E} \{ X_1 X_2 \} \mathbb{E} \{ X_3 X_4 \} + \mathbb{E} \{ X_1 X_3 \} \mathbb{E} \{ X_2 X_4 \} \\ & \quad + \mathbb{E} \{ X_1 X_4 \} \mathbb{E} \{ X_2 X_3 \}. \end{aligned} \quad (14)$$

Using this, the two parts of the covariance can be calculated as

$$\begin{aligned} & \mathbb{E} \left\{ X_j Y_j^H Y_j X_j^H X_k^H Y_k Y_k^H \right\} \\ &= \mathbb{E} \left\{ X_j X_j^H X_k X_k^H \right\} \mathbb{E} \left\{ Y_j Y_j^H Y_k Y_k^H \right\} \\ &= \left(\mathbb{E} \left\{ X_j X_j^H \right\} \mathbb{E} \left\{ X_k X_k^H \right\} + \mathbb{E} \left\{ X_j X_k^H \right\} \mathbb{E} \left\{ X_k X_j^H \right\} \right) \\ & \quad + \mathbb{E} \left\{ X_j X_k \right\} \mathbb{E} \left\{ X_j^H X_k^H \right\} \left(\mathbb{E} \left\{ Y_j Y_j^H \right\} \mathbb{E} \left\{ Y_k Y_k^H \right\} + \right. \\ & \quad \left. + \mathbb{E} \left\{ Y_j Y_k^H \right\} \mathbb{E} \left\{ Y_k Y_j^H \right\} + \mathbb{E} \left\{ Y_j Y_k \right\} \mathbb{E} \left\{ Y_j^H Y_k^H \right\} \right), \end{aligned} \quad (15)$$

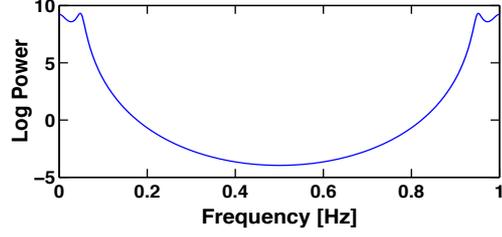


Fig. 3. Log power spectrum of the AR noise added to analyzed signal.

Frequency [Hz]	SNR
0.108	-19.24
0.15	-10.19
0.2	-3.04
0.205	-2.46

Table 1. Oscillating frequencies in the simulated example with their respective local SNR.

and the second term as

$$\begin{aligned} & \mathbb{E} \left\{ X_j(f) Y_j^H(f) Y_j(f) X_j^H(f) \right\} \\ &= \mathbb{E} \left\{ X_j(f) X_j^H(f) \right\} \mathbb{E} \left\{ Y_j(f) Y_j^H(f) \right\} \\ &= \mathbb{E} \left\{ |X_j(f)|^2 \right\} \mathbb{E} \left\{ |Y_j(f)|^2 \right\}. \end{aligned} \quad (16)$$

By plugging in the results from equations (13,15,16) and (10) into equation (12) we have an analytic expression for the variance of the cross-spectrum estimate. In [6] it is then also proven that the variance of the auto-spectrum for the process $x(n)$ is

$$\mathbb{V} \left\{ \hat{S}_x(f) \right\} = \sum_k \sum_j \alpha_j \alpha_k \left| h_j \Phi(f) R_x \Phi(f)^H h_k^T \right|^2. \quad (17)$$

An approximation of the variance of the MT-PCS estimate can then be made for the full MT-PCS estimate using Taylor expansion for moments of functions of random variables.

4. SIMULATIONS AND NUMERICAL RESULTS

4.1. Expected Value and Variance

Simulations were first made containing pure noise to show that the calculations in section 3 are correct. We made 1000 monte-carlo simulations of the AR noise with spectral density seen in figure 3 and estimated the mean and the variance of the MT-PCS estimate. The result is plotted together with their respective theoretical values in figure 1. Note that the theoretical values are Taylor expansion approximations and the two curves do therefore not match completely.

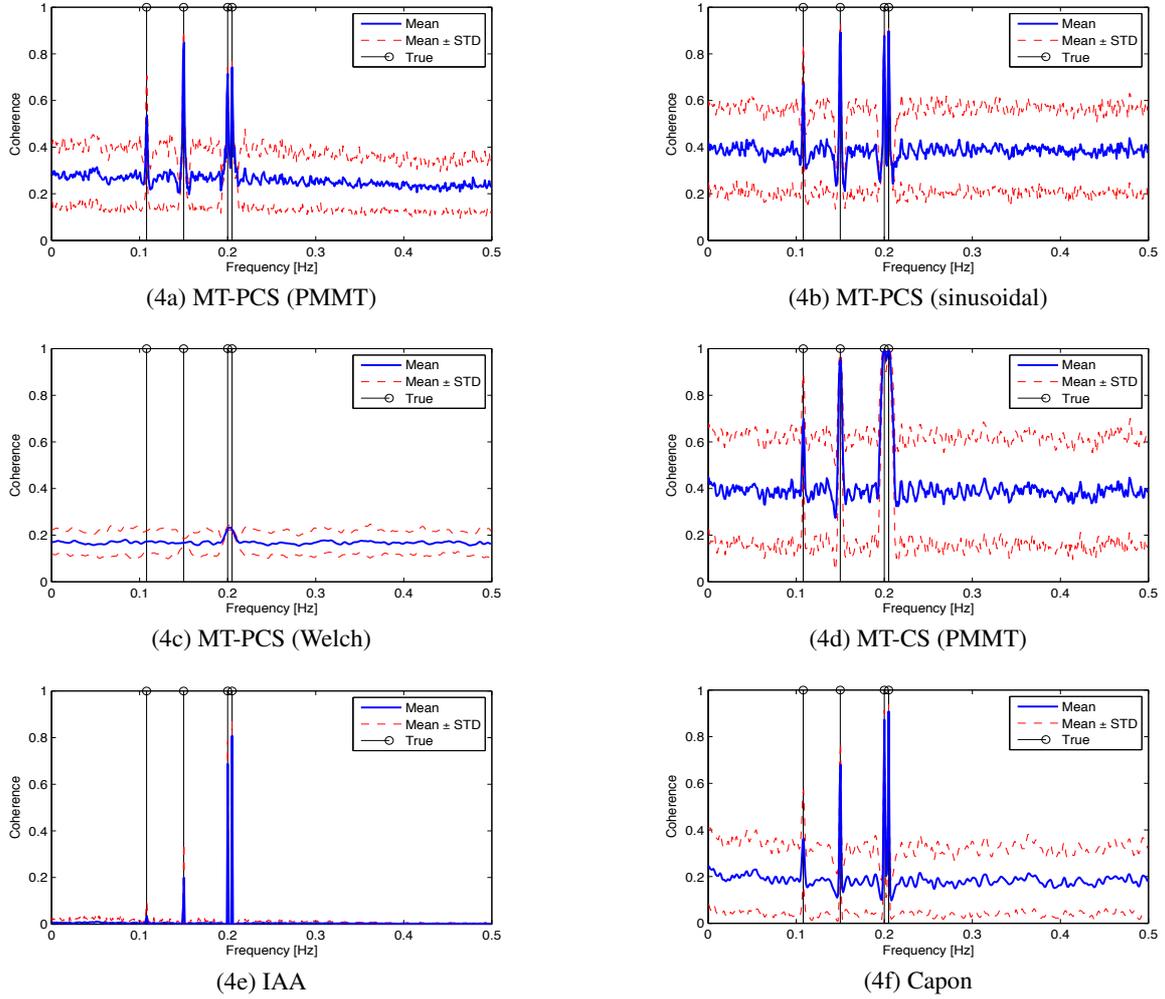


Fig. 4. The mean estimate and the mean \pm one standard deviation (dashed line) and the true frequencies (thin solid stems).

4.2. Tapers and Weights

As we are mainly interested in finding peaks in the CS, we adapt the weights to find a single frequency peak as well as possible. The assumption made then is that if the weights are suitable for estimation of a single peak, they are also suitable for multiple peaks. The main problem with parameter tuning in CS is that the weights occur both in the numerator and the denominator making the problem non-convex. Hence we find suitable parameters by a greedy grid search. A grid is set up for all possible α_k such that equation (3) and $\alpha_k \geq 0$ hold. The bias is then measured when estimating the CS for two signals, each 500 samples long, containing the same single frequency without additive noise. The weights with the lowest bias is chosen for further analysis and are shown in figure 2.

Note that the plots show the weights for seven tapers and the last weight is close to zero in all three cases, meaning more tapers than six will not significantly change the estimate.

4.3. Coherence estimation

Simulations are made with with two signals containing sinusoids in uncorrelated colored noise to show the advantageous properties of the MT-PCS. As we are considering colored noise we define the local SNR as

$$SNR(f) = 10 \log_{10} \left(\frac{P_{signal}(f)}{P_{noise}(f)} \right). \quad (18)$$

We will first compare the results from different sets of tapers, i.e. the sinusoidal, the PMMT, using the weights in figure 2 and 6 tapers. Also the Thomson multitapers were

evaluated, but they rendered no satisfactory results and are not presented here. The results are also compared to the Welch's method (equal weights) using 6 tapers. As test signals two real-valued signals oscillating at four frequencies, with independent additive colored noise with spectral distribution seen in figure 3, was used. The oscillating frequencies and their respective local SNR can then be seen in table 1. We simulated 100 realizations of the test signal with 500 samples in each realization. The CS was then estimated using the different methods and we present the mean estimate \pm one standard deviation. The results can be seen in figures 4a,4b and 4c. The PMMT and the sinusoidal tapers perform the best of the three, PMMT renders lower variance but also higher bias than the sinusoidal tapers. The Welch's method does not render any satisfactory results; this due to the high level of noise.

We then compare the proposed MT-PCS estimator, using PMMT, to some of the previously proposed methods: The MT-CS, also using PMMT, the Capon estimator and the IAA with the recommended filter length $N/4$, [9, 10]. The mean estimated CS \pm one standard deviation for the different methods can be seen in figures 4d,4e and 4f. Note that the variance of the MT-PCS, seen in figure 4b, is lowered compared to the MT-CS, figure 4d, and the localization is also better as the peaks at 0.2 and 0.205 Hz are identified separately but also the thickness of the peak at 0.15 Hz is reduced. The data dependent filter bank methods, figures 4e and 4f, identifies the three higher frequencies exemplary but the frequency peak at 0.108 Hz, where the SNR is very low, is at average identified at a level close to the standard deviation meaning it will be hard to detect from one single realization; something MT-PCS handles better where the lowest peak is roughly 1.5 times the standard deviation.

It is also worthy to mention that the 100 simulations took 0.10 seconds to process for both MT-CS and MT-PCS whereas Capon took 2.98 seconds and IAA 5.30 seconds. The multitaper methods are then implemented using the fast Fourier transform and the data dependent filter bank methods are implemented using the computationally efficient methods presented in [9, 10].

5. CONCLUSIONS AND FUTURE WORK

We propose a multitaper Pseudo Coherence method (MT-PCS) for estimation of common frequencies between signals. The method proves to have lower variance and better localization than other Fourier transform based estimators. Further it is robust to high levels of noise and very computationally efficient in comparison to filter bank methods.

To improve the MT-PCS method even more, a better optimization method is needed for tuning the weighting factors as well as investigation of signal and noise optimal multitapers.

6. REFERENCES

- [1] P. D. Welch, "The use of fast Fourier transform for the estimation of power," *IEEE Trans on Audio and Electroacoustics*, vol. AU-15, no. 2, June 1967.
- [2] G.C. Carter and A.H. Nuttall, "Statistics of the Estimate of Coherence," *Proc. of the IEEE*, vol. 60, pp. 465–466, 1972.
- [3] R. Bortel and P. Sovka, "Approximation of statistical distribution of magnitude squared coherence estimated with segment overlapping," *Signal Processing*, vol. 87, pp. 1100–1117, 2007.
- [4] D.J. Thomson, "Spectrum estimations and harmonic analysis," *Proc. IEEE*, vol. 70, pp. 1055–1096, Sept. 1982.
- [5] K.S. Riedel, "Minimum bias multiple taper spectral estimation," *IEEE Trans. on Sig. Proc.*, vol. 43, no. 1, pp. 188–195, Jan. 1995.
- [6] M. Hansson, "Optimized weighted average of peak matched multiple window spectrum estimates," *IEEE Trans. on Sig. Proc.*, vol. 47, no. 4, pp. 1141–1146, April 1999.
- [7] A.T. Walden, "A unified view of multitaper multivariate spectral estimation," *Biometrika Trust*, vol. 87, no. 4, pp. 767–788, 2000.
- [8] M. Hansson-Sandsten, "Cross-spectrum and coherence function estimation using time-delayed Thomson multitapers," in *36th IEEE Intern. Conf. on Acoustics, Speech, and Signal Processing*, Prague, Czech Republic, 2011.
- [9] G. O. Glentis K. Angelopoulos and A. Jakobsson, "Computationally efficient capon- and APES-based coherence spectrum estimation," *IEEE Trans. on Sig. Proc.*, vol. 60, no. 12, pp. 6674–6681, Dec. 2010.
- [10] G. O. Glentis and A. Jakobsson, "Fast Algorithms for Iterative Adaptive Approach Spectral Estimation Techniques," in *17th International Conference on Digital Signal Processing*, Prague, May 22-27 2011.
- [11] E. McCoy A.T. Walden and D. B. Percival, "The Variance of Multitaper Spectrum Estimates for Real Gaussian Processes," *IEEE Trans. on Sig. Proc.*, vol. 42, no. 2, Feb. 1994.
- [12] L. Isserlis, "On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables," *Biometrika*, vol. 12, no. 1, pp. 134–139, 1918.