DIRECTION-OF-ARRIVAL ESTIMATION USING MULTI-FREQUENCY CO-PRIME ARRAYS

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ABSTRACT

In this paper, we present a new method for increasing the number of resolvable sources in direction-of-arrival estimation using co-prime arrays. This is achieved by utilizing multiple frequencies to fill in the missing elements in the difference coarray of the co-prime array corresponding to the reference frequency. For high signal-to-noise ratio, the multi-frequency approach effectively utilizes all of the degrees-of-freedom offered by the coarray, provided that the sources have proportional spectra. The performance of the proposed method is evaluated through numerical simulations.

Index Terms— Co-prime arrays, DOA estimation, difference coarray, multiple frequencies.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation is a major application of antenna arrays. Traditional subspace-based DOA techniques, such as MUSIC [1], can resolve up to $(N-1)$ sources when applied to an N-element uniform linear array (ULA). Different sparse array configurations have been introduced to increase the degrees-of-freedom (DOFs) and, as such, resolve more sources than the number of physical sensors. An effective configuration reduces the number of redundant virtual elements in the difference coarray [2]. Minimum Redundancy Arrays (MRAs) are a class of sparse arrays, which aims at maximizing the number of contiguous elements in the difference coarray for a given number of sensors [3]. Nested arrays are sparse arrays that can also increase the achievable DOFs [4]. In their basic configuration, nested arrays consist of a combination of two ULAs, where the inter-element spacing of the first array is equal to the unit spacing d_0 while the elements of the second ULA are separated by an integer multiple of d_0. The integer multiple is related to the number of sensors in the first ULA.

Recently, co-prime arrays have been proposed, which constitute yet another class of sparse arrays [5, 6]. A co-prime array consists of two ULAs, where the first array has M elements with spacing Nd_0 and the second array has N elements with spacing Md_0, with M and N being co-prime integers. A co-prime array can achieve $O(MN)$ DOFs using $(M+N)$ elements. For DOA estimation with co-prime arrays, the autocorrelation matrix of the data measurements is vectorized to emulate observations at the corresponding difference coarray, which has an extended aperture [4-6]. In this model, the sources are replaced by their power, which casts them as mutually coherent. As such, subspace-based techniques can no longer be directly applied. Spatial smoothing [7] is used to decorrelate the signals and restore the full rank of the resulting autocorrelation matrix. However, this technique employs only a part of the difference coarray, which contains no missing elements or holes, thereby resulting in significantly decreased DOFs than those available for DOA estimation. A sparsity-based approach for increased DOFs was presented in [8].

In this paper, the problem of reduced DOFs is alleviated by operation at multiple frequencies. Multiple frequencies can be used to fill in the missing coarray points [9]. As a result, a longer filled difference coarray is obtained, which permits additional sources to be resolved using traditional high-resolution DOA estimation techniques. The extra DOFs offered by the use of multiple additional frequencies come with certain restrictions on the sources' spectra [10].

The remainder of the paper is organized as follows. In Section 2, we review the single-frequency based high-resolution DOA estimation using co-prime arrays. Section 3 discusses the use of multiple frequencies to fill the holes in the coarray and presents the *virtual* correlation matrix, which provides increased DOFs for DOA estimation using co-prime arrays. Supporting simulations results are provided in Section 4 and conclusions are drawn in Section 5.

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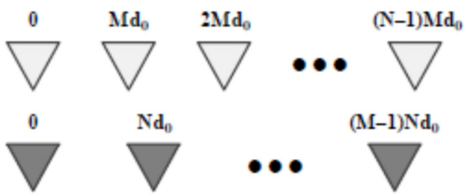

Fig. 1. Co-prime array configuration

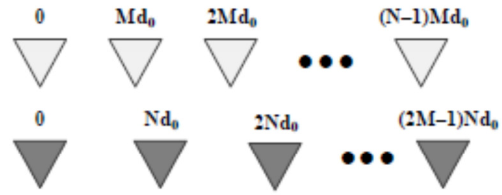

Fig. 2. Extended co-prime array configuration

2. HIGH-RESOLUTION DOA ESTIMATION USING CO-PRIME ARRAYS

A co-prime array consists of two uniformly spaced linear arrays, as shown in Fig. 1. The first array has N elements with Md_0 inter-element spacing, and the second has M elements with an inter-element spacing of Nd_0 [5]. M and N are co-prime numbers, and d_0 is the unit spacing, which is typically chosen as half-wavelength at the operating frequency. The positions of the array elements form the set

$$S = \{Mnd_0\} \cup \{Nmd_0\}, \tag{1}$$

where $0 \le n \le N-1$ and $0 \le m \le M-1$. The corresponding difference coarray has at least MN distinct elements between $-N(M-1)d_0$ and $M(N-1)d_0$. However, these elements are not contiguous.

A modification of the basic configuration of co-prime arrays was proposed in [6]. This modification doubles the number of elements in the second array, as shown in Fig. 2. The difference coarray of this configuration, shown in Fig. 3, can be expressed as

$$S_0 = \{Mnd_0 - Nmd_0\}, \tag{2}$$

where $0 \le n \le N-1$ and $0 \le m \le 2M-1$, and has contiguous elements between $-(MN+M-1)d_0$ and $(MN+M-1)d_0$.

Assuming that D narrowband sources with powers $[\sigma_1^2\ \sigma_2^2\ \ldots\ \sigma_D^2]$ impinge on the array from directions $[\theta_1\ \theta_2\ \ldots\ \theta_D]$, the received data vector at the co-prime array can be expressed as

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t), \tag{3}$$

where $\boldsymbol{s}(t) = [s_1(t)\ s_2(t)\ \ldots\ s_D(t)]^T$ is the source signal vector at snapshot t, and $\boldsymbol{n}(t)$ is the noise vector. The $(2M+N-1)\times D$ matrix $\boldsymbol{A} = [\boldsymbol{a}(\theta_1)\ \boldsymbol{a}(\theta_2)\ \ldots\ \boldsymbol{a}(\theta_D)]$ is the array manifold, whose columns are steering vectors of the co-prime array corresponding to the source directions. The steering vector corresponding to θ_d is given by

$$\boldsymbol{a}(\theta_d) = \left[e^{jk_0 x_1 \sin(\theta_d)}, \ldots,\ e^{jk_0 x_{2M+N-1} \sin(\theta_d)} \right]^T, \tag{4}$$

where $[x_1,\ x_2, \ldots,\ x_{2M+N-1}]$ are the positions of the co-prime array elements, and k_0 is the wavenumber at the operating frequency. With the assumption that the sources are uncorrelated and the elements of the noise vector are independent and identically distributed (i.i.d.) random variables following a complex Gaussian distribution, the autocorrelation matrix is obtained as

$$\boldsymbol{R}_{xx} = E[\boldsymbol{x}(t)\boldsymbol{x}(t)^H] = \boldsymbol{A}\boldsymbol{R}_{ss}\boldsymbol{A}^H + \sigma_n^2\boldsymbol{I}, \tag{5}$$

where $\boldsymbol{R}_{ss} = diag([\sigma_1^2\ \sigma_2^2\ \ldots\ \sigma_D^2])$ is the source correlation matrix and $\boldsymbol{I}$ is an identity matrix. In practice, the autocorrelation matrix is replaced by the following sample average,

$$\widehat{\boldsymbol{R}}_{xx} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}(t)\boldsymbol{x}(t)^H, \tag{6}$$

where T is the total number of snapshots.

The autocorrelation matrix is vectorized as [5]

$$\boldsymbol{z} = vec(\boldsymbol{R}_{xx}) = \widetilde{\boldsymbol{A}}\boldsymbol{p} + \sigma_n^2\tilde{\boldsymbol{\imath}}, \tag{7}$$

where $\tilde{\boldsymbol{\imath}}$ is the vector form of $\boldsymbol{I}$, $\widetilde{\boldsymbol{A}} = [\boldsymbol{a}(\theta_1)\otimes \boldsymbol{a}(\theta_1)\ \ldots\ \boldsymbol{a}(\theta_D)\otimes\boldsymbol{a}(\theta_D)]$, where the operator '$\otimes$' denotes the Kronecker product, and $\boldsymbol{p} = [\sigma_1^2\ \sigma_2^2\ \ldots\ \sigma_D^2]^T$. The vector $\boldsymbol{z}$ acts as the received signal vector of a longer array whose elements positions are given by the difference coarray. The sources are replaced by their powers and the noise is deterministic. Since the sources now act as coherent sources, subspace-based high-resolution methods, such as MUSIC, can no longer be applied directly to perform DOA estimation.

Spatial smoothing can be used to decorrelate the sources and restore the rank of the autocorrelation matrix of $\boldsymbol{z}$ [6, 7]. The elements of $\boldsymbol{z}$ which correspond to the coarray elements between $-(MN+M-1)d_0$ and $(MN+M-1)d_0$ are used to form a new vector $\boldsymbol{z}_f$ which can be expressed as

$$\boldsymbol{z}_f = \widetilde{\boldsymbol{A}}_f\boldsymbol{p} + \sigma_n^2\tilde{\boldsymbol{\imath}}_f, \tag{8}$$

where $\widetilde{\boldsymbol{A}}_f$ is the array manifold of the filled part of the coarray. $\tilde{\boldsymbol{\imath}}_f$ is a $(2MN+2M-1)\times 1$ vector whose $(MN+M)$-th element is equal to one and all its remaining elements are zeros. The filled virtual array is then divided into $(MN+M)$ overlapping subarrays, each having $(MN+M)$ elements. The element positions of the ith subarray are given by

$$\{(n+1-i)d_0, n = 0, 1, \ldots, MN+M-1\}. \tag{9}$$

The received signal vector at the ith subarray is denoted by $\boldsymbol{z}_{fi}$ and its elements consist of the $(MN+M-i+1)$th to $(2MN+2M-i)$th elements of $\boldsymbol{z}_f$. The autocorrelation

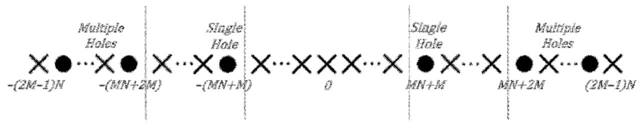

Fig. 3. Difference coarray of the co-prime array of Fig. 2.

matrix of each received signal vector is then formed following

$$\boldsymbol{R}_{fi} = \boldsymbol{z}_{fi}\boldsymbol{z}_{fi}^{H}. \tag{10}$$

The overall spatially smoothed correlation matrix is finally computed as the average of the autocorrelation matrices of the subarrays

$$\boldsymbol{R}_{zz} = \frac{1}{MN+M}\sum_{i=1}^{MN+M}\boldsymbol{R}_{fi}. \tag{11}$$

It can be shown that the rank of $\boldsymbol{R}_{zz}$ is equal to $(MN + M)$ [5], [6]. This means that up to $(MN + M - 1)$ sources can be estimated by applying high-resolution subspace techniques, such as MUSIC, on $\boldsymbol{R}_{zz}$.

3. MULTI-FREQUENCY DOA ESTIMATION USING CO-PRIME ARRAYS

The received signal at reference frequency ω_0 has the same form as (3),

$$\boldsymbol{x}(\omega_0) = \boldsymbol{A}(\omega_0)\boldsymbol{s}(\omega_0) + \boldsymbol{n}(\omega_0), \tag{12}$$

where $\boldsymbol{A}(\omega_0)$ is the array manifold at frequency ω_0. The (i,j)th element of $\boldsymbol{A}(\omega_0)$ can be expressed as

$$[\boldsymbol{A}(\omega_0)]_{i,j} = e^{jk_0 x_i sin(\theta_j)}. \tag{13}$$

If the physical co-prime array is operated at a second frequency $\omega_q = \alpha_q\omega_0$, the corresponding received signal vector can be expressed as

$$\boldsymbol{x}(\omega_q) = \boldsymbol{A}(\omega_q)\boldsymbol{s}(\omega_q) + \boldsymbol{n}(\omega_q), \tag{14}$$

where $\boldsymbol{A}(\omega_q)$ is the array manifold at frequency ω_q, with its (i,j)th element given by

$$[\boldsymbol{A}(\omega_q)]_{i,j} = e^{jk_q x_i sin(\theta_j)}, \tag{15}$$

In (15), k_q is the wavenumber at ω_q. Since $\omega_q = \alpha_q\omega_0$, k_q can be replaced by $\alpha_q k_0$ in (15) resulting in

$$[\boldsymbol{A}(\omega_q)]_{i,j} = e^{j\alpha_q k_0 x_i sin(\theta_j)}. \tag{16}$$

That is, the array manifold corresponding to ω_q is equivalent to the array manifold of a scaled version of the original co-prime array at the reference frequency ω_0. The position of the ith element in the equivalent array is given by $\alpha_q x_i$. This results in the difference coarray at ω_q to be a scaled version of the coarray at the reference frequency [9]. The corresponding coarray expands if ω_q is higher than ω_0, and contracts if ω_q is lower. As such, the multi-frequency coarray is the union of the coarrays corresponding to the various operating frequencies [9]. The sources are assumed to have a sufficient bandwidth to cover all the employed frequencies.

In order to fully exploit the DOFs offered by the co-prime array, some elements of the coarrays at the additional operating frequencies can be borrowed to fill in holes in the reference coarray. We show below how a virtual correlation matrix can be constructed using the co-prime array, which would be equivalent to that of a ULA with $(2M - 1)N + 1$ elements. This would permit DOA estimation of $(2M - 1)N$ sources instead of $(MN + M - 1)$ sources using the co-prime array with $(2M + N - 1)$ physical sensors.

For multi-frequency DOA estimation, we use the normalized correlation matrices [10]. The (i,j)th element of the $(2M + N - 1) \times (2M + N - 1)$ normalized correlation matrix $\bar{\boldsymbol{R}}_{xx}(\omega_q)$ at frequency ω_q is expressed as,

$$[\bar{\boldsymbol{R}}_{xx}(\omega_q)]_{i,j} = \frac{E\left\{[\boldsymbol{x}(\omega_q)]_i[\boldsymbol{x}^*(\omega_q)]_j\right\}}{\frac{1}{(2M+N-1)}E\{\boldsymbol{x}^H(\omega_q)\boldsymbol{x}(\omega_q)\}}, \tag{17}$$

where $[\boldsymbol{x}(\omega_q)]_i$ is the ith element of the data vector at frequency ω_q. This implies that the source and noise powers in the autocorrelation matrix representation of (5) are now replaced by the normalized powers, which are given by

$$\bar{\sigma}_k^2(\omega_q) = \frac{\sigma_k^2(\omega_q)}{\sum_{d=1}^{D}\sigma_d^2(\omega_q) + \sigma_n^2(\omega_q)} \tag{18}$$

$$\bar{\sigma}_n^2(\omega_q) = \frac{\sigma_n^2(\omega_q)}{\sum_{d=1}^{D}\sigma_d^2(\omega_q) + \sigma_n^2(\omega_q)} \tag{19}$$

We define a $(2M + N - 1) \times (2M + N - 1)$ matrix $\boldsymbol{C}(\omega_q)$ such that its (i,j)th element is given by

$$[\boldsymbol{C}(\omega_q)]_{i,j} = \alpha_q x_i - \alpha_q x_j, \tag{20}$$

i.e., the (i,j)th element of $\boldsymbol{C}(\omega_q)$ is the spatial lag or the supporting coarray element of the (i,j)th element of the correlation matrix $\bar{\boldsymbol{R}}_{xx}(\omega_q)$ at frequency ω_q. For illustration, consider $M = 2$ and $N = 3$. With $[0, 2d_0, 4d_0]$ and $[3d_0, 6d_0, 9d_0]$ as the two ULAs, the element positions of the co-prime array are $[0, 2d_0, 3d_0, 4d_0 6d_0, 9d_0]$. Then, the support matrix $\boldsymbol{C}(\omega_q)$ takes the form

$$\boldsymbol{C}(\omega_q) = \begin{bmatrix} 0 & -2 & -3 & -4 & -6 & -9 \\ 2 & 0 & -1 & -2 & -4 & -7 \\ 3 & 1 & 0 & -1 & -3 & -6 \\ 4 & 2 & 1 & 0 & -2 & -5 \\ 6 & 4 & 3 & 2 & 0 & -3 \\ 9 & 7 & 6 & 5 & 3 & 0 \end{bmatrix}\alpha_q d_0 \tag{21}$$

Note that $\boldsymbol{C}(\omega_q) = \alpha_q\boldsymbol{C}(\omega_0)$, where $\boldsymbol{C}(\omega_0)$ is the support matrix at the reference frequency ω_0. Let $\boldsymbol{C}_v(\omega_0)$ and

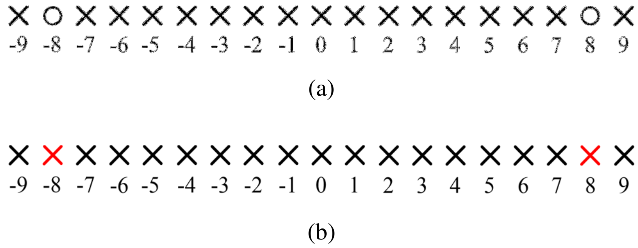

Fig. 4. (a) Difference coarray at the reference frequency $\boldsymbol{\omega_0}$ for $\boldsymbol{M = 2, N = 3}$, (b) Filled co-array using $\boldsymbol{\omega_1 = 8/9\, \omega_0}$.

$\bar{\boldsymbol{R}}_v(\omega_0)$ be the support and the correlation matrices corresponding to the desired ULA with $(2M-1)N+1$ elements operating at a single frequency ω_0. Given that a sufficient number of frequencies are employed to fill all the holes in the coarray of the co-prime array, then

$$[\boldsymbol{C}_v(\omega_0)]_{i,j} = \left[\boldsymbol{C}(\omega_q)\right]_{p,r} \text{ for some } q,p,r \text{ and all } i,j \quad (22)$$

Let h be the map that arranges selected elements of the multi-frequency support matrices into the desired support matrix $\boldsymbol{C}_v(\omega_0)$. The virtual correlation matrix $\bar{\boldsymbol{R}}_v(\omega_0)$ can then be constructed from the multiple narrowband correlation matrices $\{\bar{\boldsymbol{R}}(\omega_q), \forall q\}$ using the same map. For a high signal-to-noise ratio (SNR), a sufficient condition for the virtual correlation matrix to be positive semi-definite is that the sources must have proportional spectra at the employed frequencies [10]. This condition ensures that the normalized power spectra at all frequencies are the same. Conventional high-resolution techniques can then be applied to the virtual correlation matrix to estimate up to $(2M-1)N$ sources.

A. A Note on Frequency Selection

The coarray of Fig. 3 corresponding to the co-prime array in Fig. 2 is symmetric. Considering only the portion corresponding to non-negative lags, we observe the following. The portion of the coarray, extending from $-(MN+M-1)$ to $(MN+M-1)$, is uniform and has no holes. There is a hole at $(MN+M)$, followed by another filled part from $(MN+M+1)$ to $(MN+2M-1)$. The final part of the coarray from $MN+2M$ to $(2M-1)N$ is non-uniform and contains multiple holes.

The two holes at $-(MN+M)$ and $(MN+M)$ can be filled using only one additional frequency. The additional frequency that minimizes the separation between the reference and the required frequency is given by

$$\omega_1 = \frac{MN+M}{MN+M+1}\omega_0. \quad (23)$$

Filling these two holes produces a filled coarray between $-(MN+2M-1)$ and $(MN+2M-1)$. As a result, M additional sources can be resolved by utilizing one additional frequency compared to the spatial smoothing based technique of [6]. The remaining holes in the difference coarray can likewise be filled through the use of additional frequencies. The choice and number of frequencies is tied to the non-uniformity pattern in the coarray beyond $\pm(MN+2M)d_0$, which varies from one co-prime configuration to the other. It is noted that the maximum frequency separation from the reference is determined by the distance of the farthest hole from its nearest filled neighbor and the location of the neighbor.

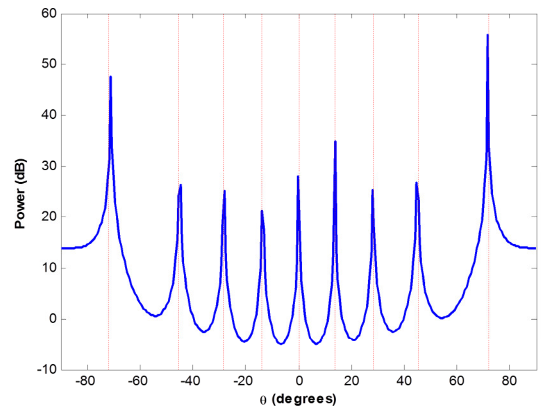

Fig. 5. MUSIC spectrum using two frequencies, D = 9 sources with proportional spectra.

4. SIMULATION RESULTS

We first consider a co-prime array configuration with 6 physical sensors, with M and N equal to 2 and 3, respectively. The first ULA consists of 3 elements positioned at $[0, 2d_0, 4d_0]$, where d_0 equals half-wavelength at the reference frequency ω_0. The second ULA also has 3 elements with positions $[3d_0, 6d_0, 9d_0]$. The difference coarray of this configuration is shown in Fig. 4. The coarray has two holes at $\pm 8d_0$ that can be filled using an additional frequency $\omega_1 = (8/9)\omega_0$. We consider 9 BPSK sources, with $\sin(\theta_d)$ uniformly distributed between -0.95 and 0.95. This is the maximum number of sources that can be resolved with the full coarray in Fig. 4. The total number of snapshots is 2000 and the average SNR is set to 0 dB at both frequencies. The sources have proportional spectra with $\sigma_d^2(\omega_1) = 3\sigma_d^2(\omega_0)$. Fig. 5 shows the estimated spatial spectrum using the proposed multiple frequency approach. The vertical lines indicate the actual locations of the sources. We can clearly see that the DOAs of all sources have been correctly estimated. The mean squared error (MSE) of the DOA estimates is determined to be 0.206. Note that the maximum number of sources that can be resolved with the spatial smoothing based approach is 7.

In the second example, we study the effect of having sources with non-proportional spectra. The array configuration is the same as in the previous example. A total of 9 sources are again used in this example, but they no longer have proportional spectra. The power of each source at each frequency is chosen randomly between 0 and 1. Fig. 6 depicts the corresponding estimated spatial spectrum using 2

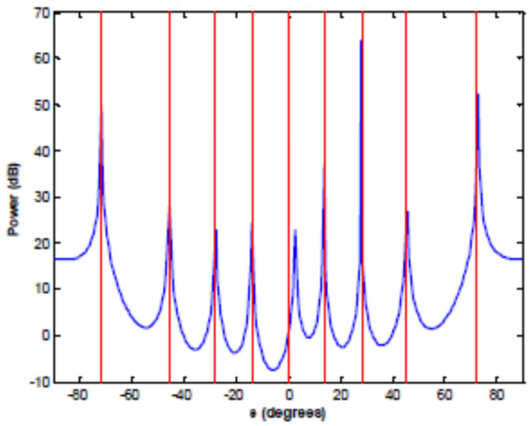

Fig. 6. MUSIC spectrum with two frequencies,
D = 9 sources with non-proportional spectra.

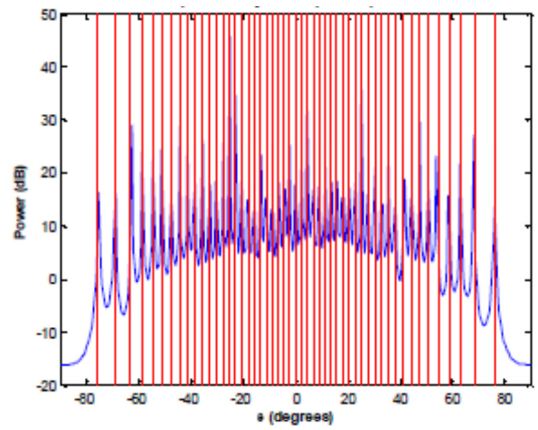

Fig. 7. MUSIC spectrum with multiple frequencies,
D = 51 sources with proportional spectra.

frequencies with the proposed scheme. Eight of the 9 sources are correctly estimated while the estimated DOA of the remaining source has a bias. The MSE in this case equals 1.2.

In the final example, we consider a different co-prime configuration with M and N equal to 5 and 7, respectively. The first ULA consists of 7 elements positioned at $[0, 5, 10, 15, 20, 25, 30]d_0$, and the second ULA has 9 elements with positions $[7, 14, 21, 28, 35, 42, 49, 56, 63]d_0$. Four additional frequencies are used to fill 8 holes in the difference coarray. Specifically, $\omega_1 = (40/41)\omega_0, \omega_2 = (45/46)\omega_0, \omega_3 = (47/48)\omega_0$, and $\omega_4 = (50/51)\omega_0$ are employed to fill the holes at $\{\pm 40, \pm 45, \pm 47, \pm 50\}d_0$. The resulting coarray has contiguous lags between $-51d_0$ and $51d_0$. 51 sources are considered with $\sin(\theta_d)$ uniformly distributed between -0.97 and 0.97. The sources have equal power spectra at all frequencies. The total number of snapshots is 2000 and the average SNR is set to 0 dB at all frequencies. Fig. 7 shows the estimated spatial spectrum, wherein the DOAs of all 51 sources have been accurately estimated. The MSE is determined to be 0.084 in this case.

5. CONCLUSION

A multi-frequency technique has been proposed for direction finding with enhanced degrees of freedom using co-prime arrays. Elements of the narrowband correlation matrices corresponding to different employed frequencies are utilized to create a virtual correlation matrix at the reference frequency. This virtual correlation matrix corresponds to a filled uniform array with the same coarray as that of the co-prime array. As such, the number of resolvable sources with high-resolution DOA techniques using co-prime arrays is increased. The performance of the proposed method was evaluated through simulation. For sources with proportional spectra, the DOAs are estimated with high accuracy.

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