A NOVEL FOURIER TRANSFORM ESTIMATION METHOD USING RANDOM SAMPLING

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ABSTRACT

This paper considers Fourier transform estimation of deterministic signals from a finite number of random samples. We refer to the recently reported methods by Masry facilitating significant acceleration of the convergence rates of the Fourier transform estimates with the growing number of samples. The acceleration does not start uniformly across all frequencies. It starts at DC and its close neighborhood. Then it spreads to higher frequencies once the average sampling rates significantly increase. In this paper we propose a modification of the signal sampling methods and appropriate to them data processing algorithms to allow moving away from zero the frequency about which the acceleration starts to practically any point in the frequency domain. We derive an expression of the mean-square error of the estimated spectrum as a measure of accuracy. Simulation results confirm the validity of the results presented in this paper.

1. INTRODUCTION

Nonuniform sampling combined with suitable algorithms can be a useful strategy to process sparse signals using a number of samples that is significantly reduced in comparison to classical DSP requirements. Considerable amount of work has been reported in the theory of nonuniform sampling. This work varies depending on several factors, such as previous knowledge/assumptions of the processed signal or its spectral support, the dedicated processing computations, the objective of signal processing, etc.

Estimating signal spectrum from randomly selected data is an efficient method for spectral analysis of signals with unknown spectral support. It is applicable to estimate the Fourier transform of windowed deterministic signals as well as the power spectrum density of random processes. With random sampling the distribution of the nonuniform sampling instants in the observation window is defined according to a probability density function(s) which is usually selected by the user. The advantage of such sampling schemes is that they often allow reducing the sampling rates well below the requirements set by the classical uniform sampling based DSP. Unbiased estimates of the complex-valued Fourier transform using a finite number of random samples of deterministic signals were introduced in [1]-[4].

In this paper we revisit the issue of estimating the Fourier transform of windowed deterministic signals from randomly sampled data. In particular, we refer to stratified estimates [3] and antithetical stratified estimates [4] which expedite significant acceleration of the convergence rates with the increasing number of samples. It has been proven that for sufficiently smooth signals when the number of samples N goes to infinity, the mean-square error decays at the rate of $1/N^3$ for stratified estimates and $1/N^5$ for antithetical stratified estimates. This compares favorably with the standard rate $1/N$ that is normally observed for the competing methods [1], [2] or for stratified and antithetical stratified, when N is small. However, the acceleration of the convergence rates of these estimates does not start uniformly across all frequencies. It originates at DC and its close neighborhood. Then when N grows, it spreads to higher frequencies.

It was shown in [3], [4] that the accuracy of stratified and antithetical stratified estimators depends on the value of the analysed frequency. However, there is no closed-form formula to determine the number of samples N at which the estimate starts to show the fast convergence. Simulation experiments showed consistent results that the average sampling density at which the fast convergence appears is always comparable to twice the value of the analysed frequency. Therefore, the benefits offered by the stratified sampling scheme come at a considerable price of having to significantly increase the sampling rates. This is against the original motivation of the use of nonuniform sampling.

In this paper we propose a modification to these methods; we change the scheme of selecting the sampling instants and the signal processing algorithms to allow moving the frequency about which the acceleration starts to almost any frequency of user's choice.

2. PROBLEM FORMALATION

Our objective in this paper is to use the samples of the signal to estimate its Fourier transform represented by

$$X_w(f)=\int_0^T x(t)\,w(t)\exp(-j2\pi ft)\,dt \qquad (1)$$

where $w(t)$ is a preselected tapering function and T is the observation interval. Three unbiased estimators of (1) were introduced and studied in [1]-[4]. All three use randomly selected sampling instants to probe the signal. The samples are then used to construct unbiased estimators of (1). The estimator defined in [1], [2] uses a sampling scheme, to which we refer as total random sampling. In this case N sampling instants t_n are independent, identically distributed (IID) random variables whose probability density function is $p(t)$. The estimator is defined as

$$\hat{X}_{wTR}(f)=\frac{1}{N}\sum_{n=1}^{N}\frac{x(t_n)w(t_n)\exp(-j2\pi ft_n)}{p(t_n)}. \qquad (2)$$

In [3] a stratified sampling scheme is used. To this end the interval $[0,T]$ is divided into N nonoverlappings subintervals whose sizes are τ_n. One sampling instant t_n is selected at random inside each subinterval. The spectrum is then estimated using

$$\hat{X}_{wS}(f)=\sum_{n=1}^{N}\tau_n x(t_n)w(t_n)\exp(-j2\pi ft_n). \qquad (3)$$

The antithetical stratified sampling introduced in [4] divides the interval $[0,T]$ into $N/2$ subintervals and then two sampling instants are selected in each of them. The first sampling instant t_n is chosen randomly in the same way as stratified sampling whereas the second one is the symmetrical reflection of the first one about the centre point of their subinterval, i.e. random sampling instant t_n is accompanied by another sampling instant at $2c_n-t_n$ where c_n is the centre point of the n-th subinterval.

When assessing the quality of a particular method of spectrum estimation a number of criteria could be taken into account. Complexity of calculations, required prior knowledge about the processed signal and hardware constraints are some examples. In our case we concentrate mostly on one aspect, i.e. the speed of convergence of those estimates to target (1). This is assessed by analysing the mean-square error of the estimators as a function of the number of processed samples N.

The three estimators are unbiased and their variances represent the estimation mean-square-error. When N goes to infinity the variances of all three estimators converge to zero. Hence, the estimators provide consistent estimation of the signal spectrum. However what differs them is the speed of convergence. The total random estimator's error decays at a constant rate proportional to $1/N$. The proportionality given does not depend explicitly on the frequency although it is slightly smaller at frequencies where $|X_w(f)|$ is large. The remaining two estimators' errors decay at rates that could be described by $1/N^{\alpha(N,f)}$ where α grows with increasing N from 1 to 3 in the case of stratified sampling and even 5 in the case of antithetical stratified sampling. These results are however adversely affected by the fact that $\alpha(N,f)$ does not increase uniformly across all frequencies for relatively small N. It reaches its larger values at $f=0$ and its close neighborhood. The higher the frequency is, the larger N must be to see $\alpha(N,f)$ significantly exceeding 1. This fact is illustrated in Fig. 1 where plots of $\alpha(N,f)$ against N are shown for various frequencies f. In the case of analysing bandpass signals, such as radio-frequency signals, this phenomenon leads to unreasonably high sampling rates or missing on high convergence rates in frequency ranges that are most meaningful for the user. A conclusive solution to resolve this problem is to downconvert the signal to baseband and only then sample it. In this paper we propose an alternative solution. We show how to modify the stratified scheme so that the fast convergence rates originate for small values of N not at DC but at practically any arbitrarily selected frequency f_c. By choosing this frequency in the vicinity of the signal spectrum, the advantages of fast convergence rates can be achieved using low sampling rates. In this paper, we restrict our attention to stratified sampling only. The proposed modifications can be also adapted to antithetical stratified sampling.

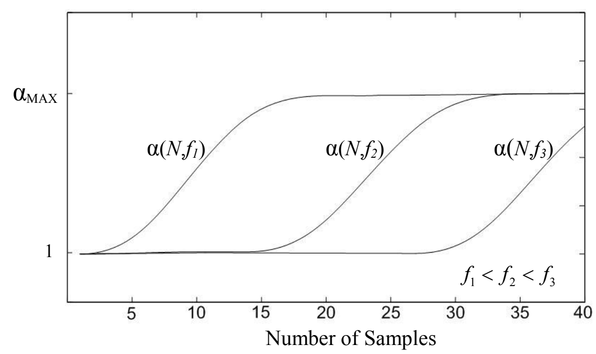

Fig. 1. Illustration of the dependence of $\alpha(N,f)$ for stratified and antithetical stratified estimators on the frequency and the number of samples

3. THE PROPOSED METHOD

This method is of particular interest for signals with spectrum concentrated in the neighborhood of some high frequency f_c. These signals can be modeled using their in-phase $x_I(t)$ and quadrature components $x_Q(t)$:

$$x(t)=x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t\ . \qquad (4)$$

These components can be sampled separately and directly from the passband signal. To do so, we define two grids of sampling points. The first grid is defined in such a way that the captured samples of the signal at its time instants are samples of the in-phase component of the signal. This grid has sampling points taking place according to

$$t_I = \frac{m}{2f_c} \quad m = 0,1,2,...,M_I - 1 \tag{5}$$

where $M_I = \lfloor 2Tf_c \rfloor$, consequently,

$$x(t_I) = (-1)^m x_I(t_I) \; . \tag{6}$$

The second grid is designed to capture the quadrature component of the signal. The grid is defined as follows

$$t_Q = \frac{2m+1}{4f_c} \quad m = 0,1,2,...,M_Q - 1 \tag{7}$$

where $M_Q = \left\lfloor \frac{4Tf_c - 1}{2} \right\rfloor$, hence

$$x(t_Q) = (-1)^{m+1} x_Q(t_Q) \tag{8}$$

M_I and M_Q are of the same value or the difference of one, therefore, for demonstration we assume that they are equal and denoted by M.

The in-phase and quadrature components of the signal represent its complex envelope

$$\gamma(t) = x_I(t) + jx_Q(t) \, . \tag{9}$$

The Fourier transform of the signal can be written in terms of the Fourier transform of its complex envelope $\Gamma(f)$:

$$\begin{aligned} X_{WD}(f) &= \frac{1}{2}\Gamma(f - f_c) \\ &= \frac{1}{2}\left[X_I(f - f_c) + jX_Q(f - f_c) \right] \end{aligned} \tag{10}$$

where

$$X_I(f - f_c) = \frac{T}{M}\sum_{l=1}^{M} x_I(t_l) w(t_l) \exp\left(-j2\pi(f - f_c)t_l\right) \tag{11}$$

$$X_Q(f - f_c) = \frac{T}{M}\sum_{l=1}^{M} x_Q(t_l) w(t_l) \exp\left(-j2\pi(f - f_c)t_l\right) . \tag{12}$$

$w(t)$ represents a window function which is used to reduce the effect of truncation.

Capturing signal samples for all time instants in both grids entails collecting an enormous amount of data. For signals with specific spectral supports, uniform skip of grid points can be deployed to obtain the spectrum with no aliasing and reduced uniform sampling density [5]. A much more challenging problem is to design a sampling scheme with reduced sampling density without a prior exact knowledge of the support of the spectrum. We propose a method of collecting samples at randomly selected time instants in both grids with average sampling density below Nyquist.

The randomly collected samples of $x_I(t)$ and $x_Q(t)$ are used to define estimators of the Fourier transform of the signal:

$$\hat{X}_{IQs}(f) = \frac{1}{2}\left[\hat{X}_I(f - f_c) + j\hat{X}_Q(f - f_c) \right] \tag{13}$$

where $\hat{X}_I(f - f_c)$ and $\hat{X}_Q(f - f_c)$ denote estimators of $X_I(f - f_c)$ and $X_Q(f - f_c)$, respectively. To select K random points from the in-phase grid, we divide the observation interval into K non-overlapping equal-sized subintervals with M_k grid points within the k-th subinterval. Then, a grid point is chosen randomly and independently within each subinterval. Those grid points are randomly selected by assigning a random variable $a_{k,l}$ $l = 1,2,...,M_k$ to each grid point within the k-th subinterval. These random variables have the probability to be one or zero according to

$$P(a_{k,l} = 1) = \frac{1}{M_k} \tag{14}$$

$$P(a_{k,l} = 0) = \frac{M_k - 1}{M_k} \, . \tag{15}$$

Using K samples to estimate the in-phase component which is usually chosen to be $N/2$, where N is the total number of samples we aim to collect from the signal, the estimator can now be written as follows

$$\hat{X}_I(f_v) = \sum_{k=1}^{K} v_k \hat{X}_{I,k}(f_v) \tag{16}$$

where $f_v = f - f_c$, $v_k = M_k / M$ and

$$\hat{X}_{I,k}(f_v) = T\sum_{l=1}^{M_k} a_{k,l} x_I(t_{k,l}) w(t_{k,l}) \exp\left(-j2\pi f_v t_{k,l}\right) . \tag{17}$$

Similar approach can be deployed to select the time instants in the quadrature grid. Also, the quadrature estimator can be written in a similar fashion to the in-phase estimator and it has similar statistical characteristics. Therefore, we analyse only the in-phase estimator here.

The in-phase estimator is unbiased according to the following calculations

$$\begin{aligned} E\left[\hat{X}_I(f_v)\right] &= \sum_{k=1}^{K} v_k E\left[\hat{X}_{I,k}(f_v)\right] \\ &= \sum_{k=1}^{K} \frac{M_k T}{M} \sum_{l=1}^{M_k} E(a_{k,l}) x_I(t_{k,l}) w(t_{k,l}) \exp\left(-j2\pi f_v t_{k,l}\right) \\ &= \frac{T}{M}\sum_{l=1}^{M} x_I(t_l) w(t_l) \exp\left(-j2\pi f_v t_l\right) = X_I(f_v) \, . \end{aligned} \tag{18}$$

The variance of the estimator can be found as follows

$$\text{var}\left[\hat{X}_I(f_v)\right] = \sum_{k=1}^{K} v_k^{\,2} \, \text{var}\left[\hat{X}_{I,k}(f_v)\right] \tag{19}$$

where

$$\begin{aligned} \text{var}\left[\hat{X}_{I,k}(f_v)\right] = T^2 \Bigg[& \sum_{l=1}^{M_k} x_I^{\,2}(t_{k,l}) w^2(t_{k,l}) \, \text{var}(a_{k,l}) \\ & + \sum_{l=1}^{M_k} \sum_{m=1, l \neq m}^{M_k} x_I(t_{k,l}) w(t_{k,l}) \exp\left(-j2\pi f_v t_{k,l}\right) \\ & x_I(t_{k,m}) w(t_{k,m}) \exp\left(j2\pi f_v t_{k,m}\right) \text{cov}(a_{k,l}, a_{k,m}) \Bigg] \end{aligned} \tag{20}$$

and

$$\mathrm{var}\left(a_{k,l}\right)=\left(1-\frac{1}{M_k}\right)\frac{1}{M_k}=\frac{\left(M_k-1\right)}{{M_k}^2}. \tag{21}$$

The probability of selecting two grid points in one subinterval is zero. Hence, the covariance is

$$\mathrm{cov}\left(a_{k,l},a_{k,m}\right)=E\left(a_{k,l}a_{k,m}\right)-E\left(a_{k,l}\right)E\left(a_{k,m}\right)=-\frac{1}{{M_k}^2} \tag{22}$$

Thus,

$$\begin{aligned}\mathrm{var}\left[\hat{X}_{I,k}\left(f_v\right)\right]=&\frac{T^2\left(M_k-1\right)}{{M_k}^2}\Bigg[\sum_{l=1}^{M_k}{x_I}^2\left(t_{k,l}\right)w^2\left(t_{k,l}\right)\\&-\frac{1}{M_k-1}\sum_{l=1}^{M_k}\sum_{m=1,m\neq l}^{M_k}x_I(t_{k,l})w\left(t_{k,l}\right)\exp\left(-j2\pi f_v t_{k,l}\right)\\&x_I(t_{k,m})w\left(t_{k,m}\right)\exp\left(j2\pi f_v t_{k,m}\right)\Bigg].\end{aligned} \tag{23}$$

By completing the square of the second term of (23), we obtain

$$\begin{aligned}\mathrm{var}\left[\hat{X}_{I,k}\left(f_v\right)\right]=&\frac{T^2}{M_k}\Bigg[\sum_{l=1}^{M_k}{x_I}^2\left(t_{k,l}\right)w^2\left(t_{k,l}\right)\\&-\frac{1}{M_k}\left|\sum_{l=1}^{M_k}x_I\left(t_{k,l}\right)w\left(t_{k,l}\right)\exp\left(-j2\pi f_v t_{k,l}\right)\right|^2\Bigg].\end{aligned} \tag{24}$$

The number of grid points inside each subinterval M_k is almost the same. Consequently, the variance is

$$\begin{aligned}\mathrm{var}\left[\hat{X}_I\left(f_v\right)\right]=&\frac{T^2}{KM}\sum_{l=1}^{M}{x_I}^2\left(t_l\right)w^2\left(t_l\right)\\&-\frac{T^2}{M^2}\sum_{k=1}^{K}\left|\sum_{l=1}^{M_k}x_I\left(t_{k,l}\right)w\left(t_{k,l}\right)\exp\left(-j2\pi f_v t_{k,l}\right)\right|^2.\end{aligned} \tag{25}$$

Thus the variance of the IQ stratified estimator is

$$\begin{aligned}\mathrm{var}\left[\hat{X}_{IQs}\left(f\right)\right]=&\frac{1}{4}\left[\mathrm{var}\left[\hat{X}_I\left(f_v\right)\right]+\mathrm{var}\left[\hat{X}_Q\left(f_v\right)\right]\right]\\=&\frac{T^2}{4KM}\sum_{l=1}^{2M}x^2\left(t_l\right)w^2\left(t_l\right)\\&-\frac{T^2}{4M^2}\sum_{k=1}^{K}\left|\sum_{l=1}^{M_k}x_I\left(t_{k,l}\right)w\left(t_{k,l}\right)\exp\left(-j2\pi f_v t_{k,l}\right)\right|^2\\&-\frac{T^2}{4M^2}\sum_{k=1}^{K}\left|\sum_{l=1}^{M_k}x_Q\left(t_{k,l}\right)w\left(t_{k,l}\right)\exp\left(-j2\pi f_v t_{k,l}\right)\right|^2.\end{aligned} \tag{26}$$

The interesting difference in the IQ stratified estimator compared to the regular stratified estimator is that the calculated frequency is f_v instead of f. Hence, the fast convergence rates of the proposed estimates appear around $f_v=f-f_c=0$, i.e. around f_c. By choosing f_c in the vicinity of the signal's spectrum, fast convergence of the estimates can be achieved in frequency ranges that are most meaningful for the user using low sampling rates.

4. NUMERICAL EXAMPLE

We now provide a numerical example to demonstrate the efficiency of the proposed IQ stratified method and compare it to the existing regular stratified method [3]. We choose the following test signal:

$$x\left(t\right)=2B\mathrm{sinc}\left(2Bt\right)\cos\left(2\pi f_0 t\right). \tag{27}$$

A similar signal was used in [3]. We set the centre frequency $f_0=10^9\,\mathrm{Hz}$ and the one-sided bandwidth of the signal $B=10^6\,\mathrm{Hz}$. The signal is observed over a rectangular window of length $T=1/B=10^{-6}$ sec to capture most of the energy of the signal. Choosing the frequency f_c which defines the grids is important since f_c is the frequency around which the accelerated rates appear. Ideally, we would like to choose f_c to be in the centre of the spectrum of the processed signal. This leads to substantial reductions in the estimation errors at frequencies where the signal is present. However, with no prior knowledge of the centre frequency or for multiband signals choosing f_c in the neighborhood of the signal spectrum can still be beneficial. In Fig.2, the mean-square error in the estimated spectrum is shown using regular stratified method and the IQ stratified method with different frequency f_c. The results are obtained by averaging a large number of simulation experiments. The error is shown for frequency range that is shifted with f_0 and spreads over twice the bandwidth of the signal. All the estimators use the same number of samples, i.e. $N=20$. We notice that the proposed method delivers improved results over the regular stratified method. We also notice that moving the frequency f_c towards the centre of the spectrum causes further reduction in the estimation error at frequency ranges where the spectrum of the signal is present.

We note that in our method we sample the in- phase and quadrature components of the signal from grids of sampling instants. In the regular stratified estimate any time instant in the observation interval can be a sampling instant. As a consequence, our method estimates the discrete-time Fourier transform of the signal, whereas the regular stratified method estimates the continuous-time Fourier transform. However, since f_c is often high, the target of our estimator almost represents the Fourier transform estimated by the other method. Therefore, a comparison between the estimation errors of the two targets is valid. Nonetheless, in hardware implementations of the continuous-time sampling practical considerations impose confining the sampling instants to take place on a multiple of some time-interval [6].

In Fig. 3 and 4 we show the mean-square error of the two methods at particular frequency points with increasing the number of samples to illustrate the rate of convergence of the estimates. The mean-square error is shown at frequencies $f=10^9\,\mathrm{Hz}$ and $f=10^9+B/2=1.0005\times10^9\,\mathrm{Hz}$

using grids defined by $f_c = 10^9 + 2B$. In the shown figures, the rate at which the regular stratified estimates converges in the mean-square sense is $1/N$, whereas the endorsed estimates have a mean-square convergence rate of almost $1/N^3$ after collecting a small number of samples.

5. CONCLUSION

We proposed an efficient method of estimating the Fourier transform of radio-frequency signals with unknown support from a finite set of its randomly collected samples. This method is clearly advantageous over other random sampling approaches in terms of the rate of convergence and estimation error. The proposed method is particularly suitable to processing signals with energy concentrated in the neighborhood of a given high frequency and have tails that spread over long ranges of frequency. Unlike the classical DSP methods the proposed method requires no downconverting or filtering to truncate the unwanted tails prior to sampling. The authors would like to point out that the results of this paper can be adapted in a fairly straightforward way to antithetical stratified sampling and also be generalised to multiple dimensions. Another benefit of our method is that, in contrast to other random sampling approaches, it allows to accurately estimate weak components of the spectrum. However, these extensions and extra advantages are left to be studied and discussed in future work.

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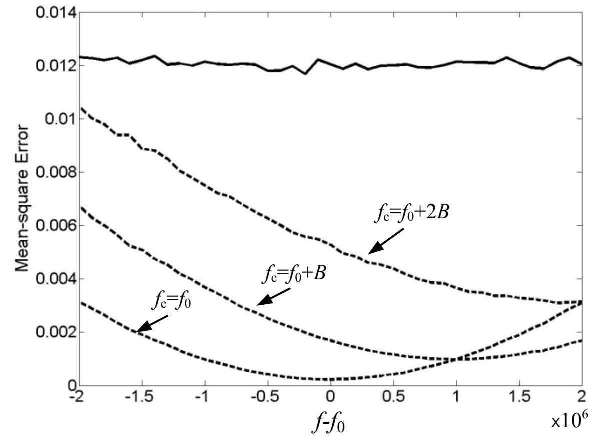

Fig. 2. Mean-square error in the estimated spectrum using regular stratified estimator (solid line) and IQ stratified estimator (dashed lines) with different f_c with $N = 20$

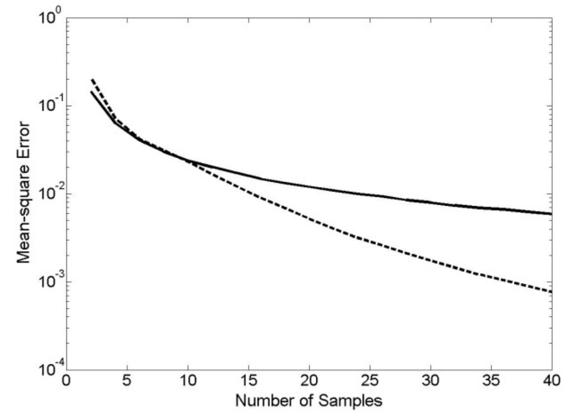

Fig. 3. Mean-square error in the estimated spectrum using regular stratified estimator (solid line) and IQ stratified estimator (dashed line) at frequency point $f = 10^9$ Hz

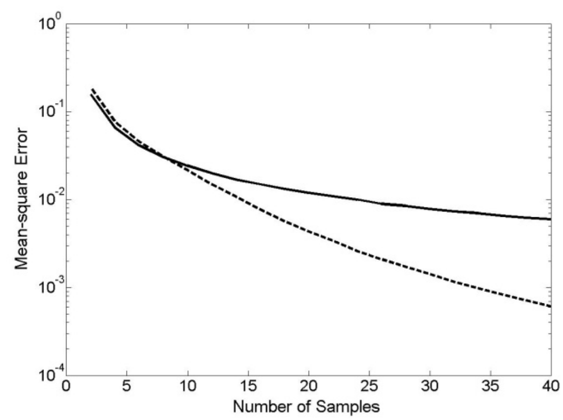

Fig. 4. Mean-square error in the estimated spectrum using regular stratified estimator (solid line) and IQ stratified estimator (dashed line) at frequency point $f = 10^9 + B/2 = 1.0005 \times 10^9$ Hz