RADAR CODE DESIGN WITH A PEAK TO AVERAGE POWER RATIO CONSTRAINT: A RANDOMIZED APPROXIMATE APPROACH

A. De Maio*, Y. Huang†, M. Piezzo*, S. Zhang††, and A. Farina†††

*Università degli Studi di Napoli "Federico II", Dipartimento di Ingegneria Biomedica, Elettronica e delle Telecomunicazioni
Via Claudio 21, I-80125, Napoli, Italy
email: ademaio@unina.it, marco.piezzo@unina.it
†The Chinese University of Hong Kong, Department of Systems Engineering and Engineering Management
Shatin, Hong Kong, China
email: ywhuang@se.cuhk.edu.hk
††University of Minnesota, Program in Industrial and Systems Engineering
MN 55455, Minneapolis, USA
email: zhangs@umn.edu
†††SELEX Sistemi Integrati
via Tiburtina Km. 12.4, I-00131, Roma, Italy
email: afarina@selex-si.com

ABSTRACT

This paper considers the problem of radar waveform design in the presence of colored Gaussian disturbance under a Peak to Average power Ratio (PAR) and an energy constraint. Firstly, we focus on the selection of the radar signal optimizing the Signal to Noise Power Ratio (SNR) for a given target Doppler frequency (Algorithm 1). Then, we devise its phase quantized version (Algorithm 2), which forces the waveform phase to lie within a finite alphabet. Both the problems are formulated in terms of NP-hard non-convex quadratic optimization programs; in order to solve them, we resort to Semidefinite Programming (SDP) relaxation and randomization techniques, providing provable-quality sub-optimal solutions with a polynomial time computational complexity. Finally, we analyze the performance in terms of detection capability and robustness with respect to Doppler shifts.

1. INTRODUCTION

In the last decade, the growth in the digital technologies and computational speed gave a great contribution toward the radar waveform design [1, 2, 3, 4, 5], as well as the project of more involving code techniques [6, 7, 8].

In this paper, we move a further step toward this direction, and focus on design waveforms with a bounded transmitted Peak to Average power Ratio (PAR). This constraint, more general than unimodularity [9], is very reasonable for radar applications. Furthermore, it permits to constrain the excursions of the squared code elements around their mean value. This also allows to keep under control the dynamic range of the transmitted waveform which is an important practical issue (for the current technology) because high PAR values necessitate a linear amplifier having a large dynamic range and this may be difficult to accommodate. Finally, the PAR control is also a crucial task in OFDM (Orthogonal Frequency-Division Multiplexing) systems and the interested reader might refer to [10] and references therein where this issue is addressed.

First of all, we focus on the selection of the radar waveform optimizing the SNR in correspondence of a given expected target Doppler frequency, under a PAR and an energy constraint (Algorithm 1). Since Algorithm 1 does not impose any condition on the waveform phase (i.e. the waveform phase can range within the continuous interval $[0, 2\pi)$), we also devise its phase quantized version (Algorithm 2) which forces the waveform phase to belong to a finite alphabet.

Both the problems are formulated in terms of non-convex quadratic optimization problems with a finite number of quadratic constraints. We prove[1] that these problems are NP-hard and, hence, introduce design techniques, relying on Semidefinite Programming (SDP) relaxation and randomization, which approximate the optimal solution with a polynomial time computational complexity.

At the analysis stage, we assess the performance of the new techniques in terms of detection probability achievable by the Neyman-Pearson receiver and robust behavior of the detection performance with respect to the target Doppler frequency.

1.1 Notation

We adopt the notation of using boldface for vectors $\boldsymbol{a}$ and matrices $\boldsymbol{A}$. The i-th element of $\boldsymbol{a}$ and the (i, j)-th entry of $\boldsymbol{A}$ are respectively denoted by a_i and $\boldsymbol{A}_{ij}$. The transpose operator and the conjugate transpose operator are denoted by the symbols $(\cdot)^T$ and $(\cdot)^H$ respectively. $\mathrm{tr}(\cdot)$ is the trace of the square matrix argument, $\boldsymbol{I}$ and $\boldsymbol{0}$ denote respectively the identity matrix and the matrix with zero entries, while $\boldsymbol{e}_k$ is the vector with all zeros except 1 in the k-th position (their size is determined from the context). The letter j represents the imaginary unit (i.e. $j = \sqrt{-1}$), while the letter i often serves as index in this paper. $\mathbb{R}$ and $\mathbb{C}$ are respectively the set of real and complex numbers. For any complex number x, we use $\Re(x)$ and $\Im(x)$ to denote respectively the real and the imaginary parts of x; $|x|$ and $\arg(x)$ represent the modulus and the argument of x, and x^* stands for the conjugate of x. The Euclidean norm of the vector $\boldsymbol{x}$ is denoted by $\|\boldsymbol{x}\|$. The symbols $\odot$ represents the Hadamard element-wise product [11], $(\boldsymbol{A})^{(k)}$ denotes the Hadamard product of k copies of $\boldsymbol{A}$; $E[\cdot]$ stands for the expected value operator. The curled inequality symbol $\succeq$ (and its strict form $\succ$) is used to denote generalized inequality: $\boldsymbol{A} \succeq \boldsymbol{B}$ means that $\boldsymbol{A} - \boldsymbol{B}$ is an Hermitian positive semidefinite matrix ($\boldsymbol{A} \succ \boldsymbol{B}$ for positive definiteness). $\mathbf{diag}(\cdot)$ denotes the vector formed by the diagonal elements of matrix argument, whereas $\mathbf{Diag}(\cdot)$ indicates the diagonal matrix formed by the components of vector argument. Finally, $v(\cdot)$ denotes the optimal value of problem $(\cdot)$.

2. SYSTEM MODEL AND FORMULATION OF THE PROBLEMS

Let us focus on a monostatic radar transmitting a linearly encoded pulse train and consider the signal model of [6], where the N-dimensional column vector $\boldsymbol{v} = [v(t_0), v(t_1), \ldots, v(t_{N-1})]^T$ of the observations is expressed as $\boldsymbol{v} = \alpha \boldsymbol{c} \odot \boldsymbol{p} + \boldsymbol{w}$, with α a pa-

[1]The proofs are herein omitted for lack of space.

rameter accounting for channel propagation and target backscattering effects, $\boldsymbol{c}$ the N-dimensional column vector containing the code elements, $\boldsymbol{p} = [1, e^{j2\pi\nu_d}, \ldots, e^{j2\pi(N-1)\nu_d}]^T$ the temporal steering vector, ν_d the normalized Doppler frequency, and $\boldsymbol{w} = [w(t_0), w(t_1), \ldots, w(t_{N-1})]^T$ the zero-mean complex circular Gaussian vector of the disturbance samples, with known positive definite covariance matrix $E[\boldsymbol{w}\boldsymbol{w}^H] = \boldsymbol{M}$.

We are looking for codes optimizing the SNR under a constraint on the transmitted energy, namely $\|\boldsymbol{c}\|^2 = N$, and forcing an upper bound to the PAR, namely PAR $\triangleq \left[\max_{i=1,\ldots,N} |c_i|^2\right] / \left[\frac{1}{N}\|\boldsymbol{c}\|^2\right] = \max_{i=1,\ldots,N} |c_i|^2$, where $\boldsymbol{c} = [c_1, \ldots, c_N]^T \in \mathbb{C}^N$. Evidently, a bound on the PAR is tantamount to imposing a more general constraint than the phase-only condition, which can be obtained letting PAR=1.

Remind that [6] SNR $= |\alpha|^2 \boldsymbol{c}^H \boldsymbol{R}\boldsymbol{c}$, where $\boldsymbol{R} = \boldsymbol{M}^{-1} \odot (\boldsymbol{p}\boldsymbol{p}^H)^*$. Note that $\boldsymbol{R}$ is positive definite since $\boldsymbol{x}^H\boldsymbol{R}\boldsymbol{x} = (\boldsymbol{x} \odot \boldsymbol{p})^H \boldsymbol{M}^{-1} (\boldsymbol{x} \odot \boldsymbol{p}) > 0$ for any $\boldsymbol{x} \neq \boldsymbol{0}$ (which is equivalent to $\boldsymbol{x} \odot \boldsymbol{p} \neq \boldsymbol{0}$). Hence, for a given normalized target Doppler ν_d, we can formulate the Waveform Design Problem (WDP) in terms of the following complex quadratic optimization program

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R}\boldsymbol{c} \\ \text{s.t.} & \text{PAR} = \max_{i=1,\ldots,N} |c_i|^2 \leq \gamma \\ & \|\boldsymbol{c}\|^2 = N \end{array} \tag{1}$$

(PAR constrained WDP) where $1 \leq \gamma \leq N$ rules the maximum allowable PAR. The resulting waveform optimizes the radar performance in correspondence of the specific design Doppler.

Since in problem (1) the waveform phase can range within the continuous interval $[0, 2\pi)$, it is of interest to consider also its phase quantized version, forcing the waveform phase to belong to a finite set. This observation leads to PAR constrained and phase quantized WDP

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R}\boldsymbol{c} \\ \text{s.t.} & \text{PAR} = \max_{i=1,\ldots,N} |c_i|^2 \leq \gamma \\ & \arg c_i \in \{0, \frac{1}{M}2\pi, \ldots, \frac{M-1}{M}2\pi\}, i = 1, \ldots, N \\ & \|\boldsymbol{c}\|^2 = N \end{array} \tag{2}$$

(where the number of quantization levels M is an integer such that $M \geq 2$), with reference to case of known normalized target Doppler.

3. PAR CONSTRAINED WDP

Problem (1) can be equivalently reformulated as

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R}\boldsymbol{c} \\ \text{s.t.} & |c_i|^2 \leq \gamma, i = 1, \ldots, N \\ & \|\boldsymbol{c}\|^2 = N. \end{array} \tag{3}$$

In this section, we consider (3) with $\gamma > 1$ (the case $\gamma = 1$, which proves to be NP-hard, has already been studied in [12, 13]), which means that the norm constraint does not vanish. Clearly, problem (3) is a non-convex QCQP with multiple constraints[2]. We claim that problem (3) with γ greater than one is NP-hard by a reduction from an even partition problem which is known to be NP-complete.

Proposition 3.1. *The radar code design problem* (3) *is NP-hard with parameters* $\boldsymbol{R} \succeq \boldsymbol{0}$ *and* $\gamma > 1$.

Due to Proposition 3.1, the radar code design problem (3) is unlikely to admit a polynomial time solution method (which means (3) is computational intractable in general). Thus, we will make efforts toward the design of an approximation algorithm for (3).

[2] For a QCQP, non-convexity does not imply that it is hard to solve; it turns out that, if the number of constraints is not too high, the QCQP can be solved efficiently; in other words, the SDP relaxation of it is tight. See [14, 15].

3.1 Approximation algorithm via semidefinite programming relaxation and randomization

To get an approximate solution (alternatively termed as a suboptimal solution) of (3), we consider its SDP relaxation:

$$\begin{array}{ll} \max_{\boldsymbol{C}} & \operatorname{tr}(\boldsymbol{R}\boldsymbol{C}) \\ \text{s.t.} & C_{ii} \leq \gamma, i = 1, \ldots, N \\ & \operatorname{tr}(\boldsymbol{C}) = N \\ & \boldsymbol{C} \succeq \boldsymbol{0}, \end{array} \tag{4}$$

which proves to be solvable[3] by the strong duality theorem [16, Theorem 1.7.1]. Evidently, problem (4) with the additional rank constraint Rank $(\boldsymbol{C}) = 1$ is equivalent to (3).

However, often, it is not the case that Rank $\boldsymbol{C}^\star$ is one, which means that the SDP relaxation (4) is not tight for (3). Therefore, we resort to a Gaussian randomization procedure [17, 18] to produce, in polynomial time, an approximate solution to the NP-hard optimization problem (3), based on the optimal solution $\boldsymbol{C}^\star$ of the SDP relaxation problem (4). Such a procedure requires the definition of a suitable *ad hoc* covariance matrix of the Gaussian distribution, so that the entire randomization procedure could lead to a feasible solution of the original problem with probability one, and could provide mathematical tractability in assessing the quality of the resulting solution. For this purpose, let us denote by

$$\boldsymbol{d} = \sqrt{\mathbf{diag}(\boldsymbol{C}^\star)}, \tag{5}$$

(where $\sqrt{(\cdot)}$ denotes the element-wise square root) and by $\boldsymbol{d}^-$

$$(\boldsymbol{d}^-)_i = \begin{cases} 1/d_i, & \text{if } d_i > 0 \\ 1, & \text{if } d_i = 0 \end{cases} \quad i = 1, \ldots, N. \tag{6}$$

Additionally, let

$$\boldsymbol{D} = \mathbf{Diag}(\boldsymbol{d}), \boldsymbol{D}^- = \mathbf{Diag}(\boldsymbol{d}^-), \tag{7}$$

and observe that, from (5)-(7),

$$(\boldsymbol{D}^-\boldsymbol{D})_{ii} = \begin{cases} 1, & \text{if } d_i > 0 \\ 0, & \text{if } d_i = 0 \end{cases} \quad i = 1, \ldots, N.$$

Hence, the entries of the matrix

$$\tilde{\boldsymbol{C}}^\star = \boldsymbol{C}^\star + (\boldsymbol{I} - \boldsymbol{D}^-\boldsymbol{D}) \tag{8}$$

comply with

$$(\tilde{\boldsymbol{C}}^\star)_{ik} = \begin{cases} (\boldsymbol{C}^\star)_{ik}, & \text{if } i \neq k \\ (\boldsymbol{C}^\star)_{ii}, & \text{if } (\boldsymbol{C}^\star)_{ii} > 0 \\ 1, & \text{if } (\boldsymbol{C}^\star)_{ii} = 0 \end{cases} .$$

By the construction of $\tilde{\boldsymbol{C}}^\star$, we see that the diagonal elements $\tilde{\boldsymbol{C}}^\star$ are positive and that $\tilde{C}^\star_{ii} = 1$ provided that $C^\star_{ii}$ vanishes. Exploiting the above definitions and observations, we have further important properties about $\tilde{\boldsymbol{C}}^\star$:

Proposition 3.2. *Let* $\boldsymbol{C}^\star$ *be a positive semidefinite matrix and* $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$, $\tilde{\boldsymbol{C}}^\star$ *be defined as* (5)-(7), (8)*, respectively. Then, the matrix* $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-$ *enjoys the following properties:*

(i) $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^- \succeq \boldsymbol{0}$;

(ii) the diagonal elements of $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-$ *are one.*

According this proposition, $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-$ can be a suitable choice for the covariance matrix of a Gaussian distribution to be adopted in our randomized approximation algorithm. Indeed, if we take a random vector $\boldsymbol{\xi} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-)$, with probability one $(\sqrt{C^\star_{11}}\frac{\xi_1}{|\xi_1|}, \ldots, \sqrt{C^\star_{NN}}\frac{\xi_N}{|\xi_N|})$ is feasible for the PAR constrained

Algorithm 1 Gaussian randomization procedure for radar code design problem (3)

Input: $\boldsymbol{R}$, γ;
Output: a randomized approximate solution $\boldsymbol{c}$ of (3);
1: solve the SDP (4) finding $\boldsymbol{C}^\star$;
2: define $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$ according to (5)-(7);
3: draw a random vector $\boldsymbol{\xi} \in \mathbb{C}^N$ from the complex normal distribution $\mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{D}^-(\boldsymbol{C}^\star + (\boldsymbol{I} - \boldsymbol{D}^-\boldsymbol{D}))\boldsymbol{D}^-)$;
4: let $c_i = \sqrt{C^\star_{ii}} e^{j \arg \xi_i}$, $i = 1, \ldots, N$.

WDP (1). Therefore, in order to produce an approximate solution (i.e., a suboptimal solution, or a feasible solution) of (3), we propose the following randomization procedure (in Algorithm 1).

We remark that in practice the randomization steps 3 and 4 can be repeated many times, in order to obtain a solution with better quality. As it can be directly seen, the computational cost of Algorithm 1 is dominated by solving SDP (4) which has a complexity of $O(N^{3.5}\log(1/\varepsilon))$ [15], given a solution accuracy $\varepsilon > 0$. Moreover, it is possible to prove that the algorithm presents the following approximation bound:

$$E[\boldsymbol{c}^H \boldsymbol{R}\boldsymbol{c}] = \operatorname{tr}(\boldsymbol{R}(\boldsymbol{D}F(\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-)\boldsymbol{D})) \geq \frac{\pi}{4}\operatorname{tr}(\boldsymbol{R}\boldsymbol{C}^\star) \geq \frac{\pi}{4}v((3)) \quad (9)$$

where $\tilde{\boldsymbol{C}}^\star$ is defined in (8), $\boldsymbol{c}$ is the randomized solution output by Algorithm 1, and the function $F(\cdot)$ is such that

$$E[\boldsymbol{z}\boldsymbol{z}^H] = F(\boldsymbol{Z}) = \frac{\pi}{4}\boldsymbol{Z} + \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)} (\boldsymbol{Z}^T \odot \boldsymbol{Z})^{(k)} \odot \boldsymbol{Z} \succeq \frac{\pi}{4}\boldsymbol{Z}$$

with $\boldsymbol{Z} \succeq \boldsymbol{0}$ and $\boldsymbol{z}$ a randomized vector generated setting $z_i = e^{j\arg\xi_i}$, $i = 1, \ldots, N$, where $\boldsymbol{\xi} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{Z})$.

4. PAR CONSTRAINED AND PHASE QUANTIZED WDP

In this section, we consider the synthesis of an approximation algorithm for (2), equivalently reformulated as:

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H\boldsymbol{R}\boldsymbol{c} \\ \text{s.t.} & |c_i|^2 \leq \gamma \\ & \arg c_i \in \{0, \frac{1}{M}2\pi, \ldots, \frac{M-1}{M}2\pi\}, \ i = 1, \ldots, N \\ & \|\boldsymbol{c}\|^2 = N. \end{array} \quad (10)$$

Clearly, when M goes to infinity, (10) becomes (3). We claim that problem (10) is also NP-hard:

Proposition 4.1. *The phase quantized code design problem* (10) *is NP-hard with parameters* $\boldsymbol{R} \succeq \boldsymbol{0}$ *and* $\gamma > 1$.

Due to the hardness of problem (10), similar to Algorithm 1, we propose a randomized approximation algorithm based on the SDP relaxation technique (as explained in Algorithm 2). Notice that the SDP relaxation problem for (10) is (4) as well.

We remark that, using the related idea in [18], the approximation algorithm is applicable to the following quadratic program:

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H\boldsymbol{R}\boldsymbol{c} \\ \text{s.t.} & \arg c_i \in \{0, \frac{1}{M}2\pi, \ldots, \frac{M-1}{M}2\pi\}, \ i = 1, \ldots, N \\ & [|c_1|^2, \ldots, |c_N|^2]^T \in \mathscr{F} \end{array} \quad (12)$$

[3] By saying "solvable", we mean the problem is feasible, bounded above (for maximization problem), and the optimal value is attained [16, page 13].

Algorithm 2 Gaussian randomization procedure for radar code design problem (10)

Input: $\boldsymbol{R}$, γ, M;
Output: a randomized approximate solution $\boldsymbol{c}$ of (10);
1: solve the SDP (4) finding $\boldsymbol{C}^\star$;
2: define $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$ according to (5)-(7);
3: draw a random vector $\boldsymbol{\xi} \in \mathbb{C}^N$ from the complex normal distribution $\mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{D}^-(\boldsymbol{C}^\star + (\boldsymbol{I} - \boldsymbol{D}^-\boldsymbol{D}))\boldsymbol{D}^-)$;
4: let $c_i = \sqrt{C^\star_{ii}}\mu(\xi_i)$, $i = 1, \ldots, N$. where $\mu(x)$ is defined as

$$\mu(x) = \begin{cases} 1, & \text{if } \arg x \in [0, 2\pi\frac{1}{M}) \\ e^{j2\pi\frac{1}{M}}, & \text{if } \arg x \in [2\pi\frac{1}{M}, 2\pi\frac{2}{M}) \\ \vdots & \\ e^{j2\pi\frac{M-1}{M}}, & \text{if } \arg x \in [2\pi\frac{M-1}{M}, 2\pi) \end{cases} . \quad (11)$$

where $\mathscr{F} \subseteq \mathbb{R}^N_+$ is a closed convex set. In this case, the convex relaxation of (12) is

$$\begin{array}{ll} \max_{\boldsymbol{C}} & \operatorname{tr}(\boldsymbol{R}\boldsymbol{C}) \\ \text{s.t.} & \operatorname{\mathbf{diag}}(\boldsymbol{C}) \in \mathscr{F} \\ & \boldsymbol{C} \succeq \boldsymbol{0} \end{array} \quad (13)$$

which can be solved efficiently due to the convexity of the problem. As to the approximation bound for Algorithm 2, it is possible to prove that

$$E[\boldsymbol{c}^H\boldsymbol{R}\boldsymbol{c}] \geq R(M) \times \operatorname{tr}(\boldsymbol{R}\boldsymbol{C}^\star) \geq R(M) \times v((10)) \quad (14)$$

where $\boldsymbol{c}$ is the randomized solution obtained through Algorithm 2, and

$$R(M) = \begin{cases} \frac{2}{\pi}, & \text{if } M = 2 \\ \frac{M^2\sin^2\frac{\pi}{M}}{4\pi}, & \text{if } M \geq 3 \end{cases} .$$

In words, Algorithm 2 is a randomized $R(M)$-approximation algorithm for (10), where some examples of $R(M)$ are $R(4) = 0.6366$, $R(8) = 0.7458$, $R(16) = 0.7754$, $R(32) = 0.7829$, $R(64) = 0.7848$, $R(128) = 0.7852$.

5. PERFORMANCE ANALYSIS

This section is devoted to the performance analysis of the proposed waveform design techniques in correspondence of different values for the design parameters (namely, the PAR constraint γ, the number of phase quantization levels M, etc.). To this end, we assume a disturbance covariance matrix $\boldsymbol{M} = \sum_{i=1}^{N_c}\beta_i\boldsymbol{p}(\nu_{d,i})\boldsymbol{p}(\nu_{d,i})^H + \beta_n\boldsymbol{I}$, which counts for both clutter and thermal noise, where the number of discrete clutter scatterers $N_c = 10$, their strength $\beta_i = \beta = 10^3$, $\nu_{d,i} = (i-1)/(2N_c)$, $i = 1, \ldots, 10$, and $\beta_n = 10^{-2}$.

The analysis is conducted in terms of P_d of the GLRT receiver [6] (or equivalently the standard matched filter with pre-whitening, followed by squared modulus operation and threshold comparison) for a prescribed target normalized Doppler frequency $\bar{\nu}_d$ (design parameter for Algorithms 1 and 2), namely $P_d(\alpha, \bar{\nu}_d) = Q\left(\sqrt{2|\alpha|^2\boldsymbol{c}^H\boldsymbol{R}(\bar{\nu}_d)\boldsymbol{c}}, \sqrt{-2\ln P_{fa}}\right)$, where $Q(\cdot,\cdot)$ is the Marcum Q function [19], assuming a false alarm probability $P_{fa} = 10^{-6}$. Additionally, due to the randomization procedures involved into Algorithms 1-2, the aforementioned performance metrics have been averaged over 500 independent trials.

In Figure 1, we plot P_d, achieved using the code devised according to Algorithm 1, versus $|\alpha|^2$, for $N = 10$, some values of γ (precisely, $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2, 2.5\}$), and $\bar{\nu}_d = 0.1$. The curves highlight that greater and greater PAR parameters lead to better and better P_d values. Such behaviour was indeed expected, because increasing γ (namely, imposing a less restrictive PAR constraint on

Table 1: Average CPU time in seconds required to solve problems (4).

γ	1	1.3	1.6	1.9	2.5
SDP (4)	0.083	0.104	0.097	0.085	0.086

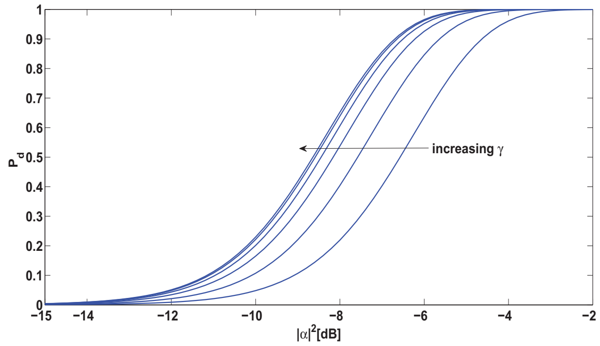

Figure 1: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $N = 10$ and $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2, 2.5\}$. Algorithm 1 - PAR constrained code.

the devised code) is tantamount to increasing the size of the feasible set of the problem. However, after a threshold value for γ, depending on the maximum eigenvalue of the covariance matrix $\boldsymbol{M}$, the PAR constraint becomes inactive and no additional performance improvements can be observed. Indeed, an optimal solution to (1) coincides with an optimal solution to

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{s.t.} & \|\boldsymbol{c}\|^2 = N, \end{array}$$

so that the optimal waveform becomes proportional to the eigenvector of $\boldsymbol{R}$ corresponding to the maximum eigenvalue.

In Figure 2, we plot P_d of the code designed according to Algorithm 2 versus $|\alpha|^2$ for $N = 10$, $\bar{\nu}_d = 0.1$, some values of the PAR parameter $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2\}$, and $M = 4$ levels for the phase quantization. As in Figure 1, increasing γ leads to better and better detection levels.

In Figures 3, we focus on corresponding approximation bounds of both the algorithms. We assume $N = 10$, $\bar{\nu}_d = 0.1$, $M = 4$ and compare the performance of Algorithms 1 and 2 with the P_d curves obtained exploiting their approximation bounds defined by (9) and (14) respectively (i.e. using (9) or (14) in the first argument of the Marcum Q function in place of the respective quadratic form). Each subplot refers to a specific value of the PAR parameter γ. The plots highlight that Algorithm 1 performs better than Algorithm 2, which quantizes the phase of the transmitted waveform on four different levels. The performance loss of the latter with respect to the former is kept within 1 dB, for $P_d = 0.9$, and is quite acceptable considering also the less demanding hardware implementation of a phase quantized waveform. It is also interesting to observe that the P_d curves obtained using the approximation bound provide a quite good approximation of the actual detection performance, for all the considered values of the parameter γ and for both the considered algorithms. As a matter of fact, the lower bound approximation is at most 2 dB far from the true P_d curve.

In the last part of this section, we investigate the effects of the number of quantization levels. Specifically, in Figure 4, we plot P_d versus $|\alpha|^2$ for $\bar{\nu}_d = 0.1$, $\gamma = 1.3$, and several values of M ($M \in \{2, 4, 8, 16\}$). As expected, increasing the number of quantization levels, leads to better and better performances until $M \leq 8$. Then, a saturation effect is experienced and the performance obtained by the phase quantized Algorithm 2 ends up coincident with that provided by Algorithm 1, which, as already pointed out, assumes code elements with phases ranging in a continuous interval.

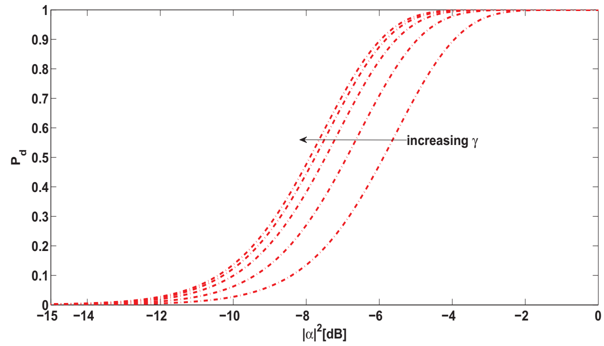

Figure 2: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $M = 4$, $N = 10$, and $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2\}$. Algorithm 2 - PAR constrained Phase quantized code.

Finally, before concluding this section, we provide in Table 1 the average CPU time required to solve the SDP problem (4) which is the most computational expensive step of Algorithms 1 and 2 . All the experiments were conducted on a desktop computer equipped with a Intel Core 2 Quad Q9400 CPU (2.66 GHz). The results highlight that the computational time is quite modest and acceptable for all the considered values of γ. Nevertheless, it is also worth pointing out that the waveform design must not necessary be performed on-line. It can be also implemented off-line producing a waveform library [5] and then during the operation a waveform from the library is selected for that particular scenario.

6. CONCLUSIONS

In this paper, we have considered radar waveform design in the presence of colored Gaussian disturbance under a PAR and an energy constraint. First of all, we have focused on the selection of the radar signal optimizing the SNR in correspondence of a given expected target Doppler frequency (Algorithm 1).Then, since Algorithm 1 does not impose any condition on the waveform phase, we have also introduced its phase quantized version (Algorithm 2), forcing the waveform phase to belong to a finite alphabet. Both the problems have been formulated in terms of non-convex quadratic optimization programs with a finite number of quadratic constraints. Due to the NP-hard nature of the problems, we have introduced design techniques, relying on SDP relaxation and randomization techniques, which provide high quality sub-optimal solutions with a polynomial time computational complexity.

At the analysis stage, we have evaluated the performance of the devised algorithms, considering both the detection probability achieved by the Neyman-Pearson detector, as well as the effects of the possible phase quantization, showing the trade off existing between the number of quantization levels and some simplicity in circuitry implementation.

Possible future research tracks might concern the generalization of the waveform design problem so as to account for an additional similarity constraint with a known code sequence. This new approach will pave the way to a joint control of both the PAR and the waveform ambiguity function. Unfortunately, the additional constraint cannot be easily handled and the design of a solution method to the resulting optimization problems is still an open issue.

REFERENCES

[1] A. Farina, “Waveform Diversity: Past, Present, and Future”, *Third International Waveform Diversity & Design Conference*, Plenary Talk, Pisa, June 2007.

[2] M. Wicks, “A Brief History of Waveform Diversity”, *Proceedings of the IEEE 2009 Radar Conference*, Pasadena, May 2009.

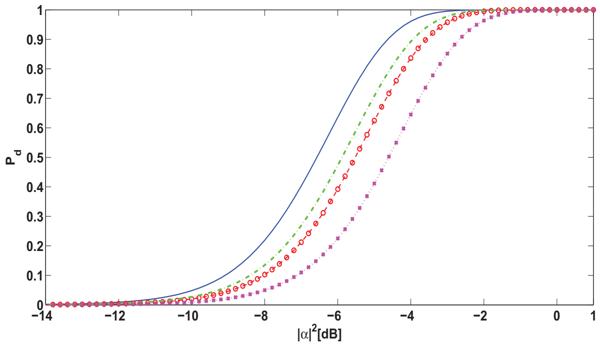

Figure 3a: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $M = 4$, $N = 10$ and $\gamma = 1$. Algorithm 1 - PAR constrained code (solid line). Approximation Bound of Algorithm 1 (dashed o-marked curve). Algorithm 2- PAR constrained Phase quantized code (dashed-dotted line). Approximation Bound of Algorithm 2 (dotted x-marked curve).

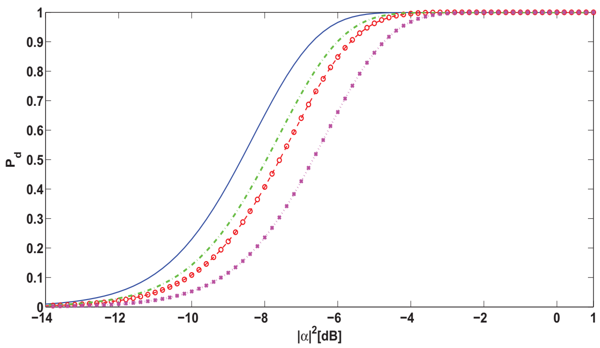

Figure 3b: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $M = 4$, $N = 10$ and $\gamma = 2.5$. Algorithm 1 - PAR constrained code (solid line). Approximation Bound of Algorithm 1 (dashed o-marked curve). Algorithm 2- PAR constrained Phase quantized code (dash-dotted line). Approximation Bound of Algorithm 2 (dotted x-marked curve).

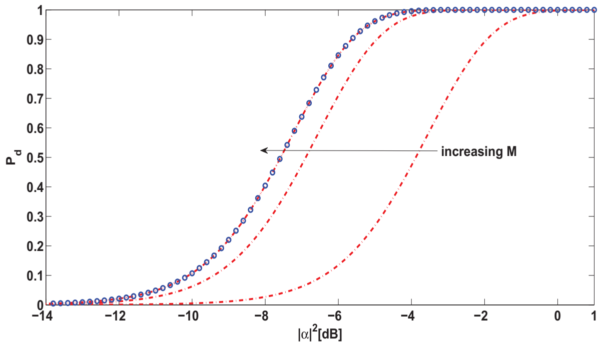

Figure 4: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $\gamma = 1.3$, and $M \in \{2, 4, 8, 16\}$. Algorithm 2 - PAR constrained Phase quantized code (dashed-dotted lines). Algorithm 1 - PAR constrained code (o-marked curve). Notice that the curve of Algorithm 1 overlaps with that referring to Algorithm 2 for $M = 8$ and $M = 16$.

[3] Special issue on "Adaptive Waveform Design for Agile Sensing and Communications", Edited by A. Nehorai, F. Gini, M. S. Greco, A. Papandreou-Suppappola, and M. Rangaswamy, *IEEE Journal of Selected Topics in Signal Processing*, Vol. 1, no. 1, pp. 2-213, June 2007.

[4] R. Calderbank, S. Howard and B. Moran, "Waveform Diversity in Radar Signal Processing", *IEEE Signal Processing Magazine*, Vol. 26, no. 1, pp. 32-41, February 2009.

[5] D. Cochran, S. Suvorova, S. Howard and B. Moran, "Waveform Libraries", *IEEE Signal Processing Magazine*, Vol. 26, no. 1, pp. 12-21, February 2009.

[6] A. De Maio, S. De Nicola, Y. Huang, S. Zhang, and A. Farina, "Code Design to Optimize Radar Detection Performance under Accuracy and Similarity Constraints", *IEEE Transactions on Signal Processing*, Vol. 56, no. 11, pp. 5618-5629, November 2008.

[7] A. De Maio, S. De Nicola, Y. Huang, D. Palomar, S. Zhang, and A. Farina, "Code Design for Radar STAP via Optimization Theory", *IEEE Transactions on Signal Processing*, Vol. 58, no. 2, pp. 679-694, February 2010.

[8] A. De Maio, Y. Huang, and M. Piezzo, "A Doppler Robust Max-Min Approach to Radar Code Design", *IEEE Transactions on Signal Processing*, Vol. 58, no. 9, pp. 4943-4947, September 2010.

[9] A. De Maio, S. De Nicola, Y. Huang, Z. Q. Luo, and S. Zhang, "Design of Phase Codes for Radar Performance Optimization with a Similarity Constraint", *IEEE Transactions on Signal Processing*, Vol. 57, no. 2, pp. 610-621, February 2009.

[10] T. Jiang and Y. Wu, "An Overview: Peak-to-Average Power Ratio Reduction Techniques for OFDM Signals," *IEEE Trans. on Broadcasting*, Vol. 54, no. 2, pp. 257-268, February 2008.

[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[12] S. Zhang and Y. Huang, "Complex Quadratic Optimization and Semidefinite Programming," *SIAM Journal on Optimization*, Vol. 16, no. 3, pp. 871 - 890, 2006.

[13] A. So, J. Zhang, and Y. Ye, "On Approximating Complex Quadratic Optimization Problems Via Semidefinite Programming Relaxations," *Mathematical Programming*, Series B, Vol. 110, no. 1, pp. 93 - 110, June 2007.

[14] Y. Huang and S. Zhang, "Complex Matrix Decomposition and Quadratic Programming", *Mathematics of Operations Research*, Vol. 32, no. 3, pp. 758-768, Aug. 2007.

[15] Z.-Q. Luo, W.-K. Ma, A.M.-C. So, Y. Ye, and S. Zhang, "Semidefinite Relaxation of Quadratic Optimization Problems," *IEEE Signal Processing Magazine*, Vol. 27, no. 3, pp. 20 - 34, May 2010.

[16] A. Nemirovski, "Lectures on Modern Convex Optimization", *Class Notes, Georgia Institute of Technology*, Fall 2005.

[17] M. X. Goemans and D. P. Williamson, "Improved Approximation Algorithms for Maximum Cut and Satisfiability Problem using Semi-Definite Programming," *Journal of the ACM*, Vol. 42, pp. 1115 - 1145, 1995.

[18] S. Zhang, "Quadratic Maximization and Semidefinite Relaxation," *Mathematical Programming*, Ser. A, Vol. 87, no. 3, pp. 453 - 465, 2000.

[19] S. M. Kay, *Fundamentals of Statistical Signal Processing: Detection Theory*, Englewood Cliffs, NJ: Prentice-Hall, 1998, vol. II.