A SYNTHESIS METHOD FOR ROBUST FREQUENCY-INVARIANT VERY LARGE BANDWIDTH BEAMFORMING

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ABSTRACT

In this paper a mixed stochastic and analytic method is proposed to synthesize a robust filter-and-sum beamforming system working with an array that is, at the same time, superdirective and undersampled. The method allows to obtain a broadband beam pattern with an optimal trade-off among directivity and frequency-invariance, over a very large bandwidth. The method jointly optimizes the sensors' positions (providing an aperiodic layout) and the coefficients of the filters used to process the signals, deploying a limited number of sensors. The simulation results show the performance and the robustness to array imperfections of the obtained solutions, highlighting the improvement over the literature methods.

1. INTRODUCTION

Systems using sensor arrays are often involved in processing broadband signals. In some cases, it is important that the performance of the array processor should be adequately constant over the entire frequency band of the signals. If the array processor is a beamformer [1], its performance is mainly measured by the beam pattern, so a frequency-invariant beam pattern (FIBP) is required. A FIBP allows one to receive the broadband signals without any distortion, even if they come from directions different from the steering direction. In the last decades, some papers have addressed the general structure of a broadband filter-and-sum beamformer, proposing methods to optimise the beamformer in such a way that a FIBP is obtained [1-6].

However, in some applications, like hearing aids [7], autonomous underwater vehicles [8], antenna arrays for miniaturized wireless radio systems [9] or phased arrays for wideband radar systems [10], strong constraints are present on the maximum aperture of the array. As a consequence, the condition in which the array aperture, D, is shorter than some of the involved wavelengths λ is frequently unavoidable. In this case, the generation of a superdirective beam pattern, achieved by synthesizing specific apodization functions [10], is essential, and the robustness to array imperfections and random errors becomes a very crucial point. Recently, a few approaches have been proposed [11-14] that can be used to synthesize the filters' coefficients necessary to yield a FIBP by using a superdirective array, assuring a sufficient robustness against errors in the array characteristics.

In general, the number of elements of an array strongly affects the array cost and the complexity of the conditioning and processing circuits. Therefore, it is very useful to succeed in decreasing the number of elements, while keeping the same spatial aperture of the array. To reduce the number of elements and to prevent grating lobes, one may increase the spacing, breaking, at the same time, the periodicity of the elements' positions. This operation leads to aperiodic arrays, where the average space between the elements, d, is larger than $\lambda/2$, i.e., larger than the Nyquist limit. In the literature, different approaches [15-18] have been proposed to optimize the elements' positions and the apodization functions of aperiodic arrays, working with narrowband [15-16] or wideband [17-18] signals.

When working with very large bandwidth signals (e.g., audio signals), the task of obtaining a FIBP with a satisfying directivity, while limiting at the same time the array aperture and the number of sensors, can be achieved only relying on an array structure which is, at the same time, aperiodic and superdirective. This means that at the lowest frequencies of the signal band, the array aperture is shorter than the wavelength (i.e., $D < \lambda$), whereas at the highest frequencies of the signal band, the same array is undersampled (i.e., $d > \lambda/2$). To the best of our knowledge, no work has been published, that is aimed at designing a robust frequency invariant beamformer applied to an aperiodic superdirective array. To bridge this literature gap, in this paper we propose a mixed stochastic and analytic synthesis method that, for a given array aperture and number of sensors, produces a robust FIBP optimizing both the filters' coefficients and the sensors' positions. The method is slightly computationally expensive, since the stochastic procedure, based on simulated annealing (SA), is employed only for the positions optimization, while, at each iteration, the filters' coefficients are analytically calculated. Unlike other methods for array synthesis, the critical and time-consuming operation of choosing a desired beam pattern (DBP) is avoided: the key optimization criterion lies in finding the broadband beam pattern which assures the best trade-off among directivity and frequency invariance. Finally, the robustness against errors in the sensor characteristics is achieved by optimizing the mean performance calculated over all the possible array characteristics taking into account the statistics of the sensor errors.

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Although the technique addressed in this paper is mainly focused on audio processing, it can also be effective in different application fields, like those addressed in [8-10].

The paper is organized as follows: after the introduction, in section 2 the proposed method is described; in section 3 the results obtained by the proposed method applied to a microphone array are shown and compared with the ones obtained through literature methods; finally, in section 4, some conclusions are drawn.

2. PROPOSED METHOD

2.1 Filter-and-Sum Beamforming

In filter-and-sum beamforming, tapped delay line architectures, where each array element feeds a transversal filter and the filter outputs are summed up to produce the beam signal, are typically exploited to design a broadband spatial filter [1]. Let us consider a linear array composed of N omnidirectional, point-like sensors, each connected to an FIR (Finite Impulse Response) filter composed of L taps. The far-field beamformer response, i.e. the actual beam pattern (ABP), is a function of the direction of arrival (DOA) and of the frequency, and can be expressed [1] as follows:

$$BP(\theta, f) = \sum_{n=0}^{N-1}\sum_{l=0}^{L-1} \mathrm{w}_{n,l} A_n \cdot \exp[-\, j2\pi f \cdot (\mathrm{d}_n \sin\theta / c + l\mathrm{T}_c)] \quad (1)$$

where f is the frequency, θ is the arrival angle belonging to the interval [-90°, 90°], c is the speed of the acoustic waves in the medium, T_c is the sampling interval of the FIR filters, d_n is n-th element position along the array, $\mathrm{w}_{n,l}$ represents the l-th tap coefficient of the n-th filter and $A_n = a_n \cdot \exp(-\gamma_n)$ represents the n-th sensor characteristic including the gain a_n and the phase γ_n, both of them supposed to be frequency-invariant. The L coefficients of the N FIR filters are independently adjustable, and can be arranged in the row vector $\mathbf{w}$ of length $M = NL$. Analogously the N sensors' positions can be arranged in a vector $\mathbf{d}$ of length N.

2.1.1 Beamforming performance analysis

The beamformer performance can be derived from the directivity and the white noise gain (WNG). The directivity indicates the improvement in the signal-to-noise ratio (SNR) provided by the array, as compared with a single omnidirectional sensor, for an isotropic noise field and plane waves [10,19]. The WNG indicates the improvement in the SNR provided by the array, as compared with a single omnidirectional sensor, for sensor self-noise, assumed to be spatially white [19]. The inverse of the WNG is called "sensitivity factor" [19] and corresponds to the sensitivity of the array beam pattern to array imperfections (e.g., element position errors and element response errors). Consequently, an excessive decrease in the WNG value cannot be accepted. The equations for the computation of the directivity and WNG of a broadside linear array can be found in [19].

2.2 Proposed cost function

Let P be the odd number of points used in discretizing the DOA axis, from -90° to 90°, Q the number of points used in discretizing the frequency axis over the desired bandwidth, $BP_{pq}(\mathbf{w},\mathbf{d})$ the value of the broadband ABP in θ_p and f_q, computed by (1) using the tap coefficients in $\mathbf{w}$ and the element positions in $\mathbf{d}$, and BPd_p the value of the DBP calculated in θ_p for an arbitrary frequency (as the DBP is supposed to be frequency-invariant, it doesn't depend on the index q). Let be θ_S the steering angle, and let us organize the values of BPd_p, for $p = 1, 2, \ldots, S\text{-}1, S\text{+}1, \ldots, P$ into the vector $\mathbf{BP}d$ of length P-1. The DBP at the steering angle is not inserted into the above vector and is kept fixed at the normalized value 1.

A cost function well tailored to our aim is the following:

$$J(\mathbf{w},\mathbf{d},\mathbf{BP}d) = \sum_{p=1}^{P}\sum_{q=1}^{Q}\left|BP_{pq}(\mathbf{w},\mathbf{d}) - BPd_p\right|^2 + K\left|BPd_p\right|^2 \quad (2)$$

Such a cost function is made up of two terms: the first accounts for the adherence between ABP and DBP, in a least squares sense, for all the frequencies and angles of interest, and the second expresses the DBP energy. The relative weight of the two terms can be tuned by the parameter K. This cost function has to be minimized in respect to the FIR filter's coefficient $\mathbf{w}$, the elements positions $\mathbf{d}$ and the values of the DBP, contained in the vector $\mathbf{BP}d$, calculated for every discretized angle, except the steering one. Considering the constraint on the DBP at the steering angle and the definition of directivity [19], the minimization of the DBP energy is equivalent to the maximization of the DBP directivity, approximately calculated using a discrete number of angles.

The minimization process produces both the DBP which assures the best trade off between directivity and adherence to the ABP, and the filters' coefficients and sensors' positions which assure the best adherence to the optimized DBP.

Unlike other synthesis methods, the adherence between DBP and ABP is intended not only in modulus but also in phase: in order to avoid phase distortions on the acquired signals the phase of the obtained beam pattern should be a linear function of frequency, for each DOA. Consequently a proper linear phase term has to be imposed in the DBP [21].

2.2.1 Robust Cost function

The cost function presented above lies on the hypothesis that the sensors' characteristics are perfectly known. However, using small-size sensor arrays, the resulting beamformers are known to be highly sensitive to errors in the array characteristics, especially the sensor gain and phase. To overcome this drawback the strategy presented in [11] has been adopted. The idea is to optimise the mean performance, i.e., the weighted sum of the cost functions for all feasible sensors' characteristics using the probability density functions (PDFs) of the sensors' characteristics as weights. To this end a total cost function can be defined as:

$$J^{tot}(\mathbf{w},\mathbf{d},\mathbf{BP}d) = \int_{A_0}\ldots\int_{A_{N-1}} J(\mathbf{w},\mathbf{d},\mathbf{BP}d, A_0,\ldots,A_{N-1}) \cdot \\ \cdot f_A(A_0)\ldots f_A(A_{N-1}) dA_0 \cdots dA_{N-1} \quad (3)$$

where $J(\mathbf{w},\mathbf{d},\mathbf{BP}d, A_0,\ldots,A_{N-1})$ is the cost function defined in (2) for a specific set of sensor characteristics $\{A_0,\ldots, A_{N\text{-}1}\}$ and $f_A(A_n)$ represents the PDF of the random variable A_n for $n = 1,\ldots, N$. Regarding $f_A(A)$, we assume the following hypothesis: $f_A(A)$ is independent of frequency and DOA; all the sensor characteristics A_n are described by the same PDF $f_A(A)$; $f_A(A)$ is a joint PDF of the independent stochastic variables a (gain) and γ (phase), such that $f_A(A) = f_a(a)f_\gamma(\gamma)$ where $f_a(a)$ is the PDF of a and $f_\gamma(\gamma)$ is the PDF of γ; finally, $f_\gamma(\gamma)$ is an even function.

2.3 Cost function minimization

The robust cost function defined in (3) has to be minimized in respect of the three vectors **w**, **d** and **BP**d. If the vector **d** is kept fixed, the resulting function of **w** and **BP**d has only a global minimum which can be found analytically. On the contrary, keeping fixed **w** and **BP**d, no analytic expression is available for the minimum of the resulting function of **d**, due to the fact that the elements of **d** appear in the argument of the exponentials in the ABP equation (see (1)). Many local minima, leading to suboptimal solutions, are typically present, and for this reason SA has been adopted. SA is an iterative procedure aimed at minimizing an energy function $f(\mathbf{x})$, **x** being the vector of the state variables At each iteration, a small random perturbation is induced in the current state configuration $\mathbf{x}^i$, i being the iteration. If the new configuration, **x***, causes the value of the energy function to decrease, then it is accepted. Instead, if **x***causes the value of the energy function to increase, it is accepted with a probability dependent on the system temperature, in accordance with the Boltzmann distribution. The temperature is a parameter that is gradually lowered, following the reciprocal of the logarithm of the number of iterations. The higher the temperature, the higher the probability of accepting a perturbation causing a cost increase and of escaping, in this way, from unsatisfactory local minima. Further details can be found in [20].

In our implementation the vector of state variables is the vector of sensors' positions **d**, while the energy function $J_{pos}^{tot}(\mathbf{d})$ is defined as the minimum of the robust cost function $J^{tot}(\mathbf{w}, \mathbf{d}, \mathbf{BP}d)$ in respect of **w** and **BP**d. Such a minimum is found in closed form, as it is described in the next subsection. At each iteration, the sensors of the array, except the first and last one, are visited once, following a random sequence. The position of each sensor is perturbed by letting it assume a random value, according to a uniform distribution, in the range defined by two adjacent sensors' positions. To take into account the physical dimensions of the sensors, a minimum distance among adjacent sensors is imposed. The energy function is evaluated for the new position vector **d*** and the perturbation is accepted or refused according to the SA procedure. If the initial temperature T_{start} and the number of iterations NUM_ITER are sufficiently high, the final state of **d** will be, in a statistical sense, close to the argument of the global minimum of the energy function. Since the analytical minimization is embedded into the SA procedure the global minimum of the energy function in respect of **d** will be identical to the global minimum of the robust cost function in respect **w**, **d** and **BP**d. On the contrary, it should be noted that a two step optimization in which the filter coefficient are optimized after the positions, or *vice-versa*, would lead (in general) to a sub-optimal solution. This combined strategy allows to greatly reduce the computational load associated with SA by diminishing the degrees of freedom of the iterative search.

```
FOR i = 1 TO NUM_ITER
    Ti = TEMP (Tstart, i)
    d^i = d^(i-1)
    FOR k = 1 TO N-2
        dk* = RND (d(k-1)^i + limit , d(k+1)^i - limit)
        d* = [ d0^i , … dk*, …]
        J_pos^tot (d*) = min_{w,BPd} J^tot (w, d*, BPd)
        ΔE = J_pos^tot (d*) - J_pos^tot (d^i)
        r = RND (0 ,1)
        IF = ΔE < 0 OR r > exp (-ΔE/κTι )
        THEN d^i = d*
        ENDIF
    ENDFOR
ENDFOR
```

- TEMP (T_{start}, i) updates the current temperature depending on the initial temperature T_{start} and the current i-th iteration.
- RND(arg1,arg2) selects randomly a real number in the range [arg1,arg2].
- limit is the minimum distance allowed among two adjacent sensors.
- k is the Boltzmann constant.

Figure 1 – Code-like implementation of the combined analytical-stochastic method

2.3.1 Closed form minimization

Let us consider the non-robust cost function defined in (2): for a fixed **d,** by substituting the ABP expression (1) in (2) and developing the squared moduli, it is possible to express the cost function as a quadratic form in respect of the compound vector $\mathbf{v} = [\mathbf{w}, \mathbf{BP}d]$. In particular:

$$J(\mathbf{w},\mathbf{d},\mathbf{BP}d) = \mathbf{v}\cdot\mathbf{M}(\mathbf{d})\cdot\mathbf{v}^{\mathrm{T}} - 2\mathbf{v}\cdot\mathbf{r}^{\mathrm{T}}(\mathbf{d}) + s \qquad (4)$$

where **M** is an $(NL+P\text{-}1)$ x $(NL+P\text{-}1)$ matrix, **r** is a $(NL+P\text{-}1)$ vector, and s is a scalar constant that can be ignored. The matrix **M** can be written as an element-wise product among two matrices $\mathbf{M}_1$ and $\mathbf{M}_2$: the first one depending only on the complex exponentials in (1), and the second one depending only on the array characteristics A_n. A similar statement can be made for the vector **r,** opportunely defining the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Now, it is straightforward to express the robust cost function in (3) as a quadratic form, by performing on (4) the multiple integrals over the array characteristics. In fact, the integrals act only on the matrix $\mathbf{M}_2$ and the vector $\mathbf{r}_2$, producing respectively the matrix $\tilde{\mathbf{M}}_2$ and the vector $\tilde{\mathbf{r}}_2$. These, under the hypotheses stated in subsection *2.2.2*, have the following structure:

$$\tilde{\mathbf{M}}_2 = \begin{bmatrix} P_a\mathbf{1}_{LxL} & \mu_a^2\mu_\gamma^2\mathbf{1}_{LxL} & \cdots & \mu_a^2\mu_\gamma^2\mathbf{1}_{LxL} & \\ \mu_a^2\mu_\gamma^2\mathbf{1}_{LxL} & P_a\mathbf{1}_{LxL} & \cdots & \mu_a^2\mu_\gamma^2\mathbf{1}_{LxL} & \mu_a\mu_\gamma\mathbf{1}_{NLxP-1} \\ \vdots & \vdots & & \vdots & \\ \mu_a^2\mu_\gamma^2\mathbf{1}_{LxL} & \mu_a^2\mu_\gamma^2\mathbf{1}_{LxL} & \cdots & P_a\mathbf{1}_{LxL} & \\ & \mu_a\mu_\gamma\mathbf{1}_{P-1xNL} & & & \mathbf{1}_{P-1xP-1} \end{bmatrix}$$

$$\tilde{\mathbf{r}}_2 = \mu_a\mu_\gamma\mathbf{1}_{P-1+NLx1} \qquad (5)$$

where μ_a and P_a are respectively the mean value and the power of a, μ_γ is the mean value of $\cos(\gamma)$, while $\mathbf{1}_{XxY}$ denotes an X x Y identity matrix. Denoting with $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{r}}$

the element-wise multiplication of $\mathbf{M}_1$ with $\tilde{\mathbf{M}}_2$ and $\mathbf{r}_1$ with $\tilde{\mathbf{r}}_2$, respectively, the robust cost function can be expressed as:

$$J^{tot}(\mathbf{w},\mathbf{d},\mathbf{BP}d) = \mathbf{v}\cdot\tilde{\mathbf{M}}(\mathbf{d})\cdot\mathbf{v}^{\mathrm{T}} - 2\mathbf{v}\cdot\tilde{\mathbf{r}}^{\mathrm{T}}(\mathbf{d}) + s \qquad (6)$$

The argument of the global minimum of J^{tot} in respect of $\mathbf{v}$ can be found as:

$$\mathbf{v}_{opt} = \arg\min_{\mathbf{w},\mathbf{BP}d} J^{tot}(\mathbf{w},\mathbf{d},\mathbf{BP}d) = \tilde{\mathbf{M}}^{-1}\tilde{\mathbf{r}}^{\mathrm{T}} \qquad (7)$$

It is worth noting that the multiple integrals on the array characteristics need to be calculated just once, at the starting of the SA procedure. In fact, at each iteration, the updated vector $\mathbf{d}$ affects only the matrix $\mathbf{M}_1$ and the vector $\mathbf{r}_1$. This closed form minimization follows a similar procedure to that used in [11, 21] for different cost functions.

3. RESULTS AND DISCUSSION

As an example of application of the proposed method, let us consider a linear array, made up of 8 point-like omnidirectional microphones, with a spatial aperture of 20 cm. It has been designed to work in air where the sound speed is c = 340 m/s. Each microphone feeds a 70$^{\mathrm{th}}$-order FIR filter (i.e., having L = 71 taps) with a sampling frequency equal to 24 kHz. The frequency interval considered for the design of the FIBP ranges from 350 to 12000 Hz (i.e., more than 5 octaves) and is discretized by using Q = 100 equally spaced points. The DOA angle, θ, ranges between –90° and 90°; it is discretized by using P = 51 points that are equally spaced in the domain of $\sin\theta$. The steering angle has been fixed at broadside, i.e. $\theta_S = 0°$.

It is very important to note that the array aperture is shorter than the wavelengths up to 1700 Hz. Moreover, if the array is equispaced, the Nyquist limit (i.e., $d \leq \lambda/2$) is not respected for all the frequencies exceeding 5900 Hz.

In order to obtain a linear phase behavior for every DOA, the DBP phase has been set as a pure delay equal to half the filter's total delay.

The parameter K, setting the trade-off between directivity and frequency-invariance in the cost function, has been tuned to 0.01 in order to privilege the frequency-invariant behaviour. The PDFs of the microphone gain and phase are assumed to be Gaussian functions with a mean value respectively equal to 1 and 0 rad, and a standard deviation respectively equal to 0.0225 and 0.0225 rad.

Regarding the SA procedure an initial temperature of 20 and a number of iterations equal to 500 have been set. Such values allow a satisfying oscillation in the initial values of the energy function, so as to escape from local minima, and a stabilization in the final values. The initial position configuration has been chosen randomly, with a minimum distance allowed among adjacent microphones of 0.015 m. The overall procedure has been run several times, yielding similar results in terms of the minimum reached.

The resulting broadband beam pattern, normalized to the mean value in the steering direction, is displayed in Fig. 2: it has a main lobe of nearly constant width with nulls at about ±55°. The visible portion of the first side lobes rises at -16 dB at ±90°. The shape appears to be very uniform over frequency: only below 1500 Hz a slight broadening of the main lobe and an increase in the side lobe level is visible.

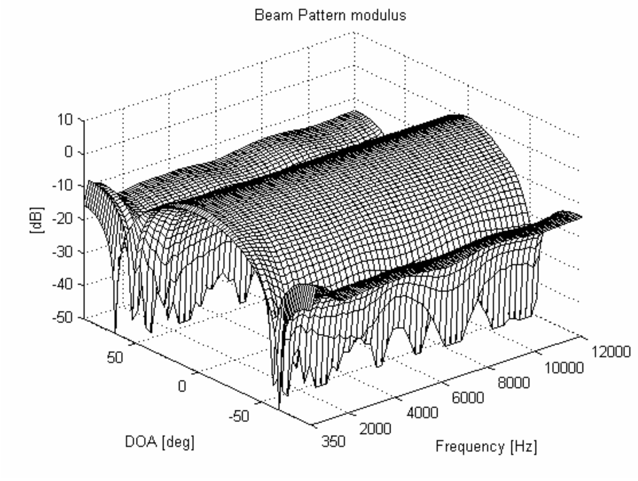

Figure 2 – Broadband beam pattern modulus vs. direction of arrival and frequency, obtained by the proposed method.

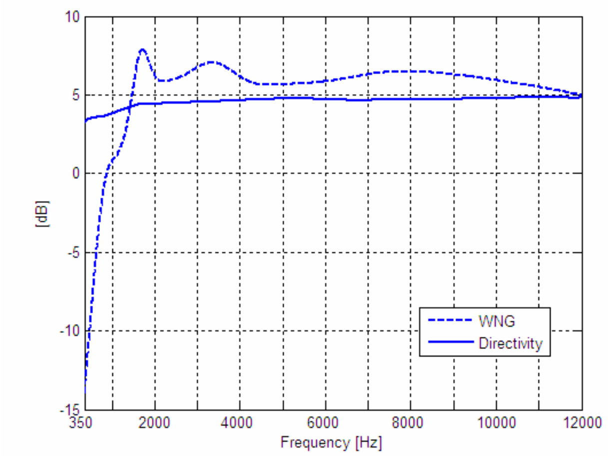

Figure 3 – Directivity and white noise gain vs. frequency for the broadband beam pattern shown in Fig. 2.

In Fig. 3 the directivity and the WNG of the obtained beam pattern are displayed versus frequency. The directivity has a constant enough profile with an average value of 4.60 dB and a minimum value of 3.2 dB at 350 Hz. The WNG is greater than 0 dB for frequencies higher than 880 Hz For lower frequencies, the white noise self-produced by the system is amplified by a limited amount, e.g., about 6 dB at 600 Hz and 14 dB at 350 Hz. These values are usually considered an index of satisfying robustness to the array imperfections, as can be seen observing that other robust synthesis methods, like [12,13], yield similar WNG profiles and values.

In Table I a comparison among the results of the FIBP design methods reported in literature is summarized. The bandwidth BW, measured in octaves, is referred to the range of frequencies over which a nearly frequency invariant behaviour has been obtained: our method allows to extend BW of about 2 octaves in respect to the best literature result [2]. Such an achievement is even more significant considering the reduced number of sensors N employed and the reduced array aperture D. The latter, in our case, is only 0.2 times the maximum involved wavelength λ_{max}. It is to note that a small D/λ_{max} characterizes the methods [12,13] dealing with superdirective arrays; on the contrary an high value of the mean sensor spacing normalized by the minimum wavelength d_{mean}/λ_{min} is typical of methods dealing with aperiodic undersampled arrays, like [2]. The proposed method is the only one which bears both this features.

Reference	BW [oct]	N	D/λ_{max}	d_{mean}/λ_{min}
[2]	3.3	17	2.5	1.47
[6]	1.16	14	3.17	0.55
[5]	1	15	3.5	0.5
[4]	2.6	21	1.66	0.5
[13]	3.0	8	0.4	0.66
[12]	2.8	8	0.18	0.18
Proposed method	**5.09**	**8**	**0.20**	**1.01**

Table I – Main features of the frequency invariant beamformers obtained by the literature and the proposed methods.

4. CONCLUSIONS

In this paper a method aimed at designing a robust frequency-invariant beamformer applied to a superdirective and, at the same time, undersampled array has been described. Such a method is based on the joint optimization of the sensors' positions and the filters' coefficients by means of a computationally advantageous mixed stochastic and analytic procedure. The results account for the ability of the method in achieving a frequency invariant beam pattern, with a satisfying directivity and robustness over a very large bandwidth of more than five octaves, while limiting the array aperture and the number of sensors employed. Future advancements will include the minimization of the array aperture and the number of sensors, by adding *ad hoc* terms in the cost function, and the extension of the robustness to other types of errors, such as sensor position errors.

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