THE BAYESIAN UNLUCKY BROKER

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ABSTRACT

We formulate, analyze, and solve a novel topic in detection theory, here referred to as the unlucky broker problem. Suppose you have a standard statistical test between two hypotheses, leading to the optimal Bayesian decision made by exploiting a certain dataset. Later, suppose that part of the data is lost, and we want to remake the test by using the surviving data and the previous decision. What is the best we can do?

Such problem, first considered in [1], is faced by standard tools from detection theory. We afford the general form of the optimal detectors, and discuss their operative modalities, emphasizing the intriguing insights hidden in the solution.

1. INTRODUCTION

Suppose that a wireless sensor network is engaged in a binary detection task. Each node of the system collects measurements about the state of the nature ($\mathscr{H}_0$ or $\mathscr{H}_1$) to be discovered. A common fusion center receives the observations from the sensors and implements an optimal Bayesian test, exploiting its knowledge of the a-priori probabilities of the hypotheses. Later, the priors used in the test are revealed to be inaccurate and a refined pair is made available. Unfortunately, at that time, only a subset of the original data is still available, along with the original decision. The *unlucky broker problem* is that of refining the original decision, taking advantage of the new pair of priors and of the surviving data.

A further example, that reveals the origin of the name *unlucky broker*, can be found in everyday life. Bernard is a broker and his job is to recommend a portfolio assessment to his customers. He relies on two data sets: one in the public domain and another made of certain confidential information he has received. To suggest the appropriate investments, Bernard must decide between a positive or a negative market trend, each characterized by an a-priori probability. He makes his decision by minimizing the risk of making a wrong prediction.

Just before visiting his customers, Bernard is informed that his a priori information on the market trend were unreliable and he receives a refined version of them. At this point, Bernard wants to revise his own decision to minimize the risk in light of the new information he has available. Unfortunately, the unlucky Bernard has lost the files containing the confidential information: his new decision must be based only on the public domain data set and on his original decision.

In statistical terms, the scenario just considered can be abstracted as in Fig. 1. A certain entity, we call it S_A, observes the data $(\mathbf{x}, \mathbf{y})$ and implements an optimal Bayesian test, exploiting the a-priori probabilities of the hypotheses, π_{0A} and π_{1A}, yielding a decision $\delta(\mathbf{x}, \mathbf{y}) = 0, 1$ at (the best)

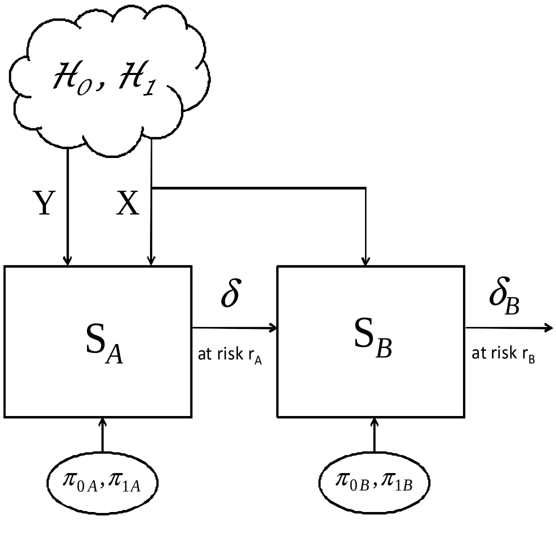

Figure 1: Notional scheme of the unlucky broker problem.

Bayes risk level r_A. Another entity, S_B, which has a refined version of priors, π_{0B} and π_{1B}, observes only a portion of the original data (the vector $\mathbf{x}$), receives from S_A the binary value $\delta(\mathbf{x}, \mathbf{y})$ and makes the best Bayes decision $\delta_B(\mathbf{x}, \delta(\mathbf{x}, \mathbf{y}))$ at a Bayes risk r_B.

In the described system architecture, we address the following basic questions. What is the best decision S_B can make, by exploiting the observation vector $\mathbf{x}$ and the binary decision $\delta(\mathbf{x}, \mathbf{y})$? What about the behavior of the optimal detector at site S_B? Should S_B simply retain the previous decision δ, or should it ignore that, and use only the currently available data set for a completely new decision? Or, what else?

The notional scheme in Fig. 1 allows us to figure out that a similar problem arises in decentralized detection with tandem (serial) architecture; this topic is widely investigated in the literature, see, e.g., [2–8]. In our case, however, the decision δ does depend upon $\mathbf{x}$, and this makes the problem essentially different from that considered in the literature.

The paper is organized as follows. In the next section the problem is formalized and the answers to the stated questions are given. Examples of applications are provided in Sect. 3, while Sect. 4 concludes the paper. An extended version of this work can be found in [1], where the focus on the Neyman-Pearson setting. Since many general results are similar, we refer the reader to [1] for many details and proofs that are here omitted. On the other hand, the in-depth investigation of the Bayesian unlucky broker, not given in [1], represents the original contribution of this work.

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2. STATEMENT OF THE PROBLEM & RESULTS

Let $\mathbf{X} = [X_1, X_2, \ldots, X_{N_x}]$ and $\mathbf{Y} = [Y_1, Y_2, \ldots, Y_{N_y}]$ be two independent continuous-valued random vectors. With reference to Fig. 1, S_A has to solve the following binary hypothesis test:

$$\begin{array}{lll} \mathscr{H}_0 & : & X_i \sim f_X(x;\mathscr{H}_0), \quad Y_j \sim f_Y(y;\mathscr{H}_0), \\ \mathscr{H}_1 & : & X_i \sim f_X(x;\mathscr{H}_1), \quad Y_j \sim f_Y(y;\mathscr{H}_1), \end{array} \tag{1}$$

where i and j span the entries of the observation vectors $\mathbf{X}$ and $\mathbf{Y}$, respectively. In the above, $f_X(x;\mathscr{H}_0)$ is the marginal probability density function (pdf, for short) of the variable X_i, under hypothesis $\mathscr{H}_0$. Similarly, $f_X(x;\mathscr{H}_1)$ is the pdf under $\mathscr{H}_1$, while $f_Y(y;\mathscr{H}_1)$ and $f_Y(y;\mathscr{H}_0)$ are the corresponding quantities for Y_j. We assume that both $\mathbf{X}$ and $\mathbf{Y}$ have mutually independent and identically distributed entries.

The optimal Bayes strategy (see, e.g., [9–11]) for (1) is

$$\delta(\mathbf{x},\mathbf{y}) = \begin{cases} 1, & L(\mathbf{x},\mathbf{y}) \geq \gamma, \\ 0, & L(\mathbf{x},\mathbf{y}) < \gamma, \end{cases}$$

where

$$\begin{aligned} L(\mathbf{x},\mathbf{y}) &= \sum_{i=1}^{N_x} \ln \frac{f_X(x_i;\mathscr{H}_1)}{f_X(x_i;\mathscr{H}_0)} + \sum_{j=1}^{N_y} \ln \frac{f_Y(y_j;\mathscr{H}_1)}{f_Y(y_j;\mathscr{H}_0)} \\ &= L_x(\mathbf{x}) + L_y(\mathbf{y}) \end{aligned} \tag{2}$$

is the log-likelihood ratio between $\mathscr{H}_0$ and $\mathscr{H}_1$, and the threshold γ is defined as

$$\gamma = \ln \frac{\pi_{0A}(C_{10} - C_{00})}{\pi_{1A}(C_{01} - C_{11})}, \tag{3}$$

where C_{ij} is the cost incurred by choosing the hypothesis $\mathscr{H}_i$ when $\mathscr{H}_j$ is true and $\pi_{1A} = 1 - \pi_{0A}$. Let us consider an uniform cost assignment, that is $C_{ij} = 0$ if $i = j$ and $C_{ij} = 1$ if $i \neq j$. Thus, the threshold in (3) becomes $\gamma = \ln \pi_{0A}/(1 - \pi_{0A})$, and the Bayes risk r_A is [10]

$$r_A = \pi_{0A} P_f + (1 - \pi_{0A})(1 - P_d), \tag{4}$$

where P_f and P_d are the false alarm and detection probabilities of the optimal Bayes test based upon $(\mathbf{X},\mathbf{Y})$, respectively.

Now, S_B — that models our unlucky broker — has available only the vector $\mathbf{x}$ and the decision $\delta(\mathbf{x},\mathbf{y})$, a binary variable, coming from S_A. The corresponding detection statistic can thus be written as the ratio [10]

$$T(\mathbf{x},\delta) = \ln \frac{P(\mathbf{x},\delta;\mathscr{H}_1)}{P(\mathbf{x},\delta;\mathscr{H}_0)}, \tag{5}$$

where $P(\mathbf{x},\delta;\mathscr{H}_0)$ and $P(\mathbf{x},\delta;\mathscr{H}_1)$ are the joint densities of the pair $(\mathbf{X},\delta(\mathbf{X},\mathbf{Y}))$ under $\mathscr{H}_0$ and $\mathscr{H}_1$, respectively. Let $P_{fy}(z)$ and $P_{dy}(z)$ be, respectively, the false alarm and detection probabilities of an optimal Bayesian test based only upon $\mathbf{y}$. By defining the random variable $L_x = L_x(\mathbf{X})$, whose realization is l_x, and introducing the functions

$$\begin{aligned} t_1(l_x) &= l_x + \ln \frac{P_{dy}(\gamma - l_x)}{P_{fy}(\gamma - l_x)}, \\ t_0(l_x) &= l_x + \ln \frac{1 - P_{dy}(\gamma - l_x)}{1 - P_{fy}(\gamma - l_x)}, \end{aligned} \tag{6}$$

the decision statistic in (5) can be rewritten (with slight abuse of notation) as

$$T(l_x,\delta) = \delta\, t_1(l_x) + (1-\delta)\, t_0(l_x), \tag{7}$$

as it can be easily checked by examining the two cases $\delta = 0, 1$.

The optimal decision rule for S_B is, clearly,

$$\delta_B(\mathbf{x},\delta(\mathbf{x},\mathbf{y})) = \begin{cases} 1, & T(l_x,\delta) \geq \gamma_B, \\ 0, & T(l_x,\delta) < \gamma_B, \end{cases} \tag{8}$$

where $\gamma_B = \ln \pi_{0B}/(1 - \pi_{0B})$ is the new threshold set by using the refined version of the priors. The corresponding Bayes risk is

$$r_B = \pi_{0B} P_{f,B} + (1 - \pi_{0B})(1 - P_{d,B}), \tag{9}$$

where $P_{f,B}$ and $P_{d,B}$ are the associated detection and false alarm probabilities.

All the above has been obtained by direct application of the standard tools from detection theory. More interesting are the following results and their implications. First, let $u(\cdot)$ denote the unit step function: $u(x) = 1$ for $x \geq 0$, and $u(x) = 0$ otherwise. In the following we will omit the arguments of $\delta_B(\mathbf{x},\delta(\mathbf{x},\mathbf{y}))$ and $\delta(\mathbf{x},\mathbf{y})$ for notational simplicity. The behavior of the detector S_B is summarized in the following results, whose proofs can be found in [1].

THEOREM. *The unlucky broker's optimal decision rule is*

$$\delta_B = \begin{cases} \delta\, u(t_1(l_x) - \gamma_B), & \text{for } \pi_{0B} > \pi_{0A}, \\ (1-\delta)\, u(t_0(l_x) - \gamma_B) + \delta, & \text{for } \pi_{0B} \leq \pi_{0A}. \end{cases} \tag{10}$$

•

COROLLARY. *Suppose that the functions $t_1(l_x)$ and $t_0(l_x)$ defined in (6) are invertible and strictly increasing, and let*

$$s_1 = t_1^{-1}(\gamma_B), \quad s_0 = t_0^{-1}(\gamma_B). \tag{11}$$

Then the unlucky broker's optimal decision rule becomes

$$\delta_B = \begin{cases} \delta\, u(l_x - s_1), & \text{for } \pi_{0B} > \pi_{0A}, \\ (1-\delta)\, u(l_x - s_0) + \delta, & \text{for } \pi_{0B} \leq \pi_{0A}. \end{cases} \tag{12}$$

•

The above theorem and its corollary allow us to identify the *modus operandi* of the optimal detector solving the unlucky broker problem. The behavior of the unlucky broker (entity S_B) is summarized as follows.

- When there is no refinement of the a-priori probabilities ($\pi_{0B} = \pi_{0A}$), the best solution for the unlucky broker is, clearly, to retain the original decision. It can be shown that this result is indeed embodied in the statement of the previous Theorem.
- When $\pi_{0B} > \pi_{0A}$, an original decision $\delta = 0$ (in favor of $\mathscr{H}_0$) is always retained by S_B, while a decision $\delta = 1$ (i.e., for $\mathscr{H}_1$) needs a double check: it is kept only if the function $t_1(l_x)$ is larger than, or equal to, the new threshold γ_B.
- Conversely, in the case $\pi_{0B} < \pi_{0A}$, a decision in favor of $\mathscr{H}_1$ is always retained, while a decision in favor of $\mathscr{H}_0$ is retained only if $t_0(l_x)$ is less than γ_B.

- In summary, when the refined prior tells that, say, $\mathscr{H}_0$ is becoming more likely ($\pi_{0B} > \pi_{0A}$), the unlucky broker should accordingly tip the balance toward the null hypothesis. The optimal way to do this is to change some of the decisions in favor of $\mathscr{H}_1$, based upon a suitable detection statistic of the available data set $\mathbf{x}$. The situation clearly reverses when $\pi_{0B} < \pi_{0A}$.

Also, the above Theorem and its Corollary imply the following. Whenever the functions $t_1(l_x)$ and $t_0(l_x)$ are invertible and strictly increasing, the required double check results in comparing $l_x(\mathbf{x})$ to a single threshold. Otherwise stated, *given* that the original decision is $\delta = 1$ (when $\pi_{0B} > \pi_{0A}$) or $\delta = 0$ (when $\pi_{0B} \leq \pi_{0A}$), the optimal test for S_B behaves as a (likelihood) single threshold test, based upon the available data set $\mathbf{x}$.

In the general case covered by the Theorem, instead, the double check may involve much more tricky log-likelihood comparisons: it requires checking whether $l_x(\mathbf{x})$ belongs to some arbitrarily shaped subset of the real line, not necessarily simply connected. This may in fact amount to compare $l_x(\mathbf{x})$ to multiple thresholds.

Before ending this section, we report the analytical expressions for system performances evaluation. The Bayes risk r_B can be evaluated by resorting to eq. (10). In [1] we show that when $\pi_{0B} > \pi_{0A}$, we have

$$\begin{aligned} P_{d,B} &= P_d - \int_{t_1(l_x)<\gamma_B} P_{dy}(\gamma - l_x) f_{L_x}(l_x;\mathscr{H}_1)\, dl_x, \\ P_{f,B} &= P_f - \int_{t_1(l_x)<\gamma_B} P_{fy}(\gamma - l_x) f_{L_x}(l_x;\mathscr{H}_0)\, dl_x, \end{aligned} \tag{13}$$

where $f_{L_x}(l_x;\mathscr{H}_1)$ is the pdf of the random variable L_x under the alternative hypothesis. In the opposite case of $\pi_{0B} \leq \pi_{0A}$, we get

$$\begin{aligned} P_{d,B} &= P_d + \int_{t_0(l_x)\geq\gamma_B} [1 - P_{dy}(\gamma - l_x)] f_{L_x}(l_x;\mathscr{H}_1)\, dl_x, \\ P_{f,B} &= P_f + \int_{t_0(l_x)\geq\gamma_B} [1 - P_{fy}(\gamma - l_x)] f_{L_x}(l_x;\mathscr{H}_0)\, dl_x. \end{aligned}$$

3. EXAMPLES

In this section, the above theoretical framework is applied to two specific decision problems. We start by considering a classical Gaussian shift-in-mean hypothesis test

$$\begin{aligned} \mathscr{H}_0 &: \quad X_i, Y_j \sim \mathscr{N}(0, \sigma^2), \\ \mathscr{H}_1 &: \quad X_i, Y_j \sim \mathscr{N}(\mu, \sigma^2). \end{aligned} \tag{14}$$

The upper plot in Fig. 2 shows the Bayes risk as a function of the refined prior π_{0B}, when $\pi_{0A} = 0.3$, for three different systems. We consider the optimal Bayes detector having access to the full data set $(\mathbf{x}, \mathbf{y})$ and the detector S_B that exploits the pair $(\mathbf{x}, \delta)$. For comparison purposes, we also show the risk pertaining to a detector which always retains the original decision, ignoring thus the availability of a refined prior. The error probabilities of this latter would clearly coincide with P_f and $1 - P_d$, yielding a linear behavior with π_{0B}. We can observe that the curves get in contact when the refined priors are equal to those used in the original test. Furthermore, a precise ordering relationship exists: S_A uniformly outperforms S_B, which in turn uniformly outperforms the "blind" system which ignores the refined prior availability.

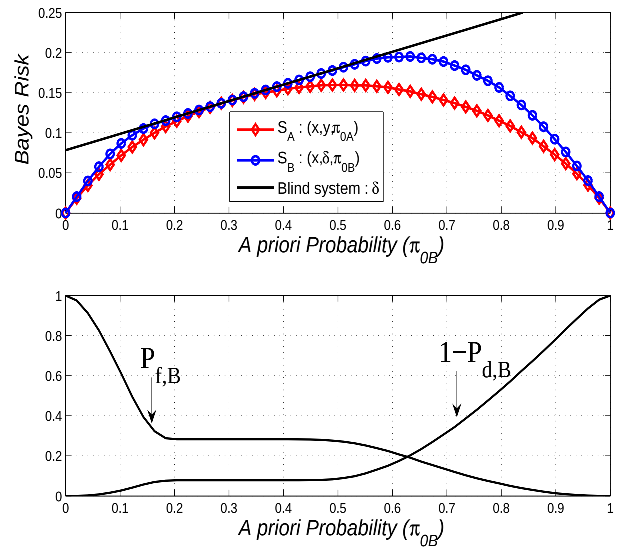

Figure 2: Bayes risk (upper plot) and probabilities $P_{f,B}$, $1 - P_{d,B}$ (lower plot) versus a-priori probability π_{0B} for the Gaussian shift-in-mean problem with $N_x = N_y = 2$, $\sigma = 1$, $\mu = 1$ and $\pi_{0A} = 0.3$.

The lower plot in Fig. 2 depicts the behavior of the probabilities $P_{f,B}$, $1 - P_{d,B}$ as functions of the refined prior π_{0B} and allows us to describe more in detail what happens in terms of Bayes risk. In the considered example, the original a priori probability is $\pi_{0A} = 0.3$, giving $\gamma = -0.85$. If the priors selected by the broker (entity S_B) coincide with these initial values, the best that one can do is to retain the original decision, ignoring the surviving data. Accordingly, $P_{f,B} = P_f$, $P_{d,B} = P_d$, and the curves in the upper plot of Fig. 2 get in contact just for $\pi_{0A} = \pi_{0B} = 0.3$, when the priors are actually not refined at all.

Now, let us choose $\pi_{0B} \neq 0.3$, but sufficiently close to that. Assume, for instance, $\pi_{0B} = 0.4$, implying $\gamma_b = -0.41$. In this case we have $\pi_{0B} > \pi_{0A}$, so we must compare (see the Theorem in Sect. 2) $t_1(l_x)$ to γ_B. It can be shown that the function $t_1(l_x)$, plotted in Fig. 3, crosses the threshold γ_B when $l_x = -4.6$, and we can compute the unlucky broker's detection probability thanks to the first of eqs. (13). By numerically integration, we find that the integral term in the first of eqs. (13) is much smaller ($\approx 10^{-7}$) than the detection probability P_d (0.92 in the example) and, accordingly, $1 - P_{d,B}$ is almost equal to $1 - P_d$. Similarly, we have $P_{f,B} \approx P_f$ (0.28 in the example). This basically means that, in a neighborhood of π_{0A}, the original decision is very often retained ($\delta_B \approx \delta$). Looking at Fig. 2, this immediately explains the similarity between the performance of S_B and that of the system which always retains the decision δ, in the range where π_{0B} is sufficiently close to π_{0A}.

Figure 4 shows, instead, what happens when $\pi_{0B} = 0.7$, implying $\gamma_B = 0.85$. The intersection between $t_1(l_x)$ and γ_B occurs when $l_x = 0.4$. The integral term in the first of eqs. (13) is now comparable (≈ 0.27) to P_d, and $1 - P_{d,B}$ grows as shown in Fig. 2. In a similar manner, we find that the integral term in the second of eqs. (13) is comparable (≈ 0.16) to P_f, and $P_{f,B}$ decreases as shown in the lower plot of Fig. 2.

We can conclude that for small variations of the priors around the original values, the unlucky broker changes the

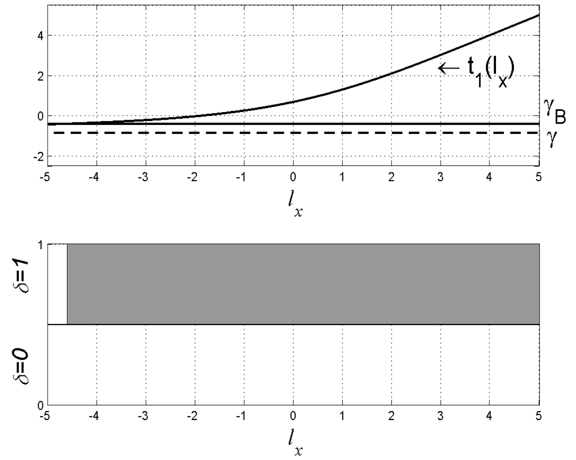

Figure 3: Function $t_1(l_x)$ (upper plot) and final decisions in the plane (l_x, δ) (lower plot) when $\gamma_B = -0.41$. This refers to the Gaussian example.

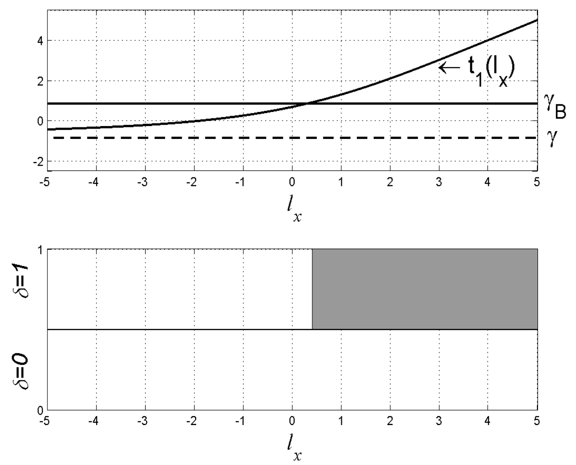

Figure 4: Function $t_1(l_x)$ (upper plot) and final decisions in the plane (l_x, δ) (lower plot) when $\gamma_B = 0.85$. This refers to the Gaussian example.

original decision with probability much smaller than the false alarm and detection probabilities of the original decisions. Hence, in this range, the Bayes risk of the broker is almost linear. A physical interpretation is that in this interval the unlucky broker essentially exploits only the information provided by the original decision.

From the upper plot of Fig. 4 we see that the function $t_1(l_x)$ for the case study we are considering is strictly increasing, as formally proved in [1]. The same result holds for the function $t_0(l_x)$. Thus, we are in the setting of the Corollary in Sect. 2, and we can state that the optimal solution to the unlucky broker problem is provided by a detector that compares the log-likelihood to a single threshold. This is what one usually expects by a detector.

The lower plot in Fig. 4 is to be interpreted as follows. In the white regions the final decision is in favor of $\mathscr{H}_0$, while in the grey region a decision in favor of $\mathscr{H}_1$ is taken. We see that if the original decision (given on the vertical axis) is

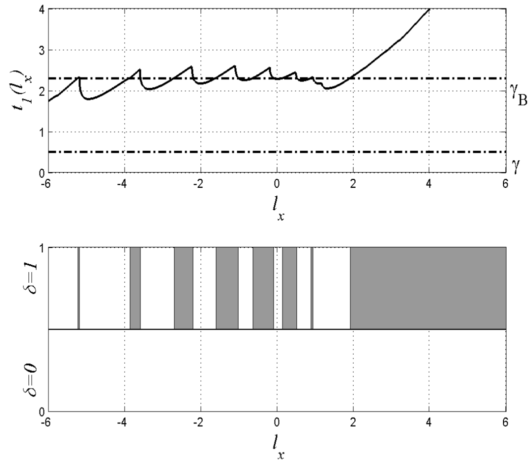

Figure 5: Function $t_1(l_x)$ (upper plot) and final decisions in the plane (l_x, δ) (lower plot), for the example involving the balanced mixture of Gaussians; values of the parameters are detailed in the main text. As for the previous figures, the grey (resp. white) regions in the lower plot mean that $\delta_B = 1$ (resp. $\delta_B = 0$).

$\delta = 0$, no matter what l_x is, the final decision will be $\delta_B = 0$. On the other hand, a decision $\delta = 1$ is retained only if l_x is large enough. Finally, we can observe that if the two thresholds approach each other, then the original decisions are always retained. Summarizing, when the functions $t_1(l_x)$ and $t_0(l_x)$ are invertible and strictly increasing, the optimal detector solving the unlucky broker problem *works like a log-likelihood threshold test.*

For the observation model (14), proving when the functions $t_1(l_x)$ and $t_0(l_x)$ defined in (6) are strictly monotone is by no means trivial. Nonetheless, aside from mathematical difficulties, the reader might guess that the monotone property may hold very in general. Instead, the unlucky broker problem does not lead, in general, to a simple single-threshold detector. Usually, the behavior is more complex, since typically the functions $t_1(l_x)$ and $t_0(l_x)$ defined in (6) are not monotone. As an example, consider the following detection problem.

$$\begin{aligned} \mathscr{H}_0 &: \; X_i \text{ and } Y_i \sim \mathscr{N}(0, \sigma_0^2), \\ \mathscr{H}_1 &: \; X_i \text{ and } Y_i \sim \tfrac{1}{m} \textstyle\sum_{i=1}^m \mathscr{N}(\mu_i, \sigma_1^2), \end{aligned}$$

where the variables X_i's and Y_i's are zero-mean Gaussian with variance σ_0^2 under the null hypothesis $\mathscr{H}_0$, while under the alternative hypothesis they are identically distributed as a balanced mixture of m Gaussian random variables with mean values $\mu_1, \mu_2, \ldots, \mu_m$, and common variance σ_1^2.

To get insights about the detector structure, let us consider the above problem for $N_x = N_y = 1$, $m = 20$, $\sigma_0^2 = 2$, $\sigma_1^2 = 0.2$, $\gamma = 0.5$, and the mean values μ_i's selected as equally spaced points in the range $[-9, 9]$. In the upper plot of Fig. 5, obtained numerically, the function $t_1(l_x)$ is shown and we see that it is no longer monotone.

This non-monotone behavior has a strong impact on the decision rule, as it can be appreciated by looking at the lower plot in Fig. 5. The figure shows the final decisions taken by the detector, with $\pi_{0B} > \pi_{0A}$. The decisions are plotted on the "plane" (l_x, δ): in the white regions the final decisions are for $\mathscr{H}_0$, while in the grey regions a final decision for $\mathscr{H}_1$ is made. As predicted by the theory, since $\pi_{0B} > \pi_{0A}$, all the decisions $\delta = 0$ are retained. However, the rule for retaining the decisions $\delta = 1$ is markedly different from that observed in the previous case. Indeed, the grey region is no longer simply connected, meaning that the detector structure does *not* simply involve the comparison of the log-likelihood ratio l_x with a single threshold.

In general, therefore, the detection regions have a complicated shape and the unlucky broker task cannot be reduced to a single-threshold comparison. The behavior of $t_1(\cdot)$ and $t_0(\cdot)$ that rules the optimal detector, indeed, implies multiply-connected and irregularly-shaped optimal decision regions.

4. CONCLUSIONS

This paper introduces a novel topic in detection theory — called the unlucky broker problem— with a broad range of potential applications in different fields (sensor networks, economics, medicine). We have a statistical test between two hypotheses, exploiting a certain data set $(\mathbf{x}, \mathbf{y})$ and the knowledge of the a priori probabilities of the hypotheses. This leads to a decision δ obeying the Bayes optimality criterion. Afterward, the priors used in the test are revealed to be unreliable and a refined pair is made available. At this point, however, vector $\mathbf{y}$ is lost and we can rely only on the surviving data set $\mathbf{x}$, the original decision δ, and the refined version of the priors. Due to the statistical dependence between δ and $\mathbf{x}$, the problem is markedly different from the tandem decision systems studied in the literature. The main results for the optimal Bayes test can be so summarized.

- When there is no refinement in the a priori probabilities, namely the threshold used by the original detector is equal to that exploited by the unlucky broker, the best that we can do is to retain the original decision; the available data set is useless. Indeed, S_A exploits the full data set and, for a fixed pair of a-priori probabilities, it guarantees the best Bayes risk level.
- When there is an effective refinement, some decisions can be safely retained, but others require a deeper analysis. As one might expect, we find that the sufficient statistic for the final decision is the pair $(\delta, L_x(\mathbf{x}))$, where $L_x(\mathbf{x})$ is the log-likelihood of vector $\mathbf{x}$: both the original decision δ, and the data $\mathbf{x}$ influence the final decision δ_B, with data $\mathbf{x}$ playing their role only through the related log-likelihood $L_x(\mathbf{x})$. Any reader familiar with detection theory would say: "So what? That $\mathbf{x}$ played its role only through its likelihood $L_x(\mathbf{x})$ comes with no surprise at all."
  That reader would be right. But things must be proved because intuition is not always infallible (this, in particular, applies to the present authors' intuition). One contribution is to rigorously prove the above result.
- One's intuition perhaps might suggest that the final detection structure amounts comparing $L_x(\mathbf{x})$ to a suitable threshold level — what is commonly called a single threshold test.

  This, however, is quite far from the truth. In general, we show that, when the prior probability of $\mathscr{H}_0$ increases ($\pi_{0B} > \pi_{0A}$), the unlucky broker should always retain a decision $\delta = 0$, while the decisions in favor of $\mathscr{H}_1$ are changed only if $L_x(\mathbf{x})$ belongs to some subset of the real axis having (in general) a complicated structure, which is characterized in the paper. Similar considerations apply to the case $\pi_{0B} < \pi_{0A}$. Therefore, the found solution to the Bayesian version of the unlucky broker problem does not amount to a simple threshold test, and this is the main result of the paper.

The present work can be extended in many directions. These include (i) the extension to further detection stages (multiple successive refinements); (ii) addressing the case where surviving data are noisy; (iii) the study of simpler single-threshold (hence non optimal) detection schemes.

REFERENCES

[1] S. Marano, V. Matta, and F. Mazzarella, "Refining decisions after losing data: The unlucky broker problem," *IEEE Trans. Signal Processing*, vol. 58, no. 4, pp. 1980–1990, April 2010.

[2] R. R. Tenney and J. Sandell, N. R., "Detection with distributed sensors," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 17, pp. 501–510, July 1981.

[3] L. K. Ekchian, "Optimal design of distributed detection networks," Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, Mass., 1982.

[4] L. K. Ekchian and R. R. Tenney, "Detection networks," in *21th IEEE Conference on Decision and Control*, Orlando, 1982, pp. 686–691.

[5] J. D. Papastavrou, "Distributed detection with selective communications," Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, Mass., 1986.

[6] R. Viswanathan, S. Thomopoulus, and R. Tumuluri, "Optimal serial distributed decision fusion," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 24, pp. 366–376 510, July 1988.

[7] J. Pothiawala, "Analysis of a two-sensor tandem distributed detection network," Master thesis, Massachusetts Institute of Technology, Cambridge, Mass., 1989.

[8] J. D. Papastavrou and M. Athans, "Distributed detection by a large team of sensors in tandem," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 3, pp. 639–653, July 1992.

[9] H. L. Van Trees, *Detection, Estimation, and Modulation Theory. Part I.* New York: John Wiley & Sons, Inc., 1968 (reprinted, 2001).

[10] H. V. Poor, *An Introduction to Signal Detection and Estimation*. New York: Springer-Verlag, 1988.

[11] S. M. Kay, *Fundamentals of Statistical Signal Processing, Volume II: Detection Theory*. Englewood Cliffs, New Jersey: Prentice Hall, 1998.