2-D FIR FILTER DESIGN: WHAT SHAPE IS THE BEST ?

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ABSTRACT

In this paper, we compare minimax designs of 2-D FIR filters whose passbands and stopbands have four shapes: rectangular, diamond, circle and cos-oval. The experiments show that, when all the other design specifications (areas of passband and stopband, passband error bound) coincide, the cos-oval shape is usually better for narrow passbands. For such passbands, the cos-oval shape is very similar to the circular shape, but gives better stopband attenations. Hence, it constitutes an interesting alternative for practical designs.

1. INTRODUCTION

If we want to design a (univariate) lowpass filter using an optimization method, we need to specify only two frequencies that define the response, namely the edges of the passband and stopband. In the 2-D case, the situation changes radically: the edges are curves that can be defined in various ways. Theoretically, there are an infinity of passband or stopband shapes; additionally, each shape is characterized by one or several parameters, that typically take values over a whole interval. In some applications, the shape of the passband is strictly imposed (like e.g. for filters used before subsampling on a certain lattice [2]). In others, the shape is more vaguely defined and small variations around a basic shape are allowed.

Let us assume that we want to design a lowpass 2-D linear-phase FIR filter

$$H(z_1,z_2) = \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} h_{k_1,k_2} z_1^{-n_1} z_2^{-n_2}. \quad (1)$$

The condition $h_{-k_1,-k_2} = h_{k_1,k_2}$ ensures the linear phase property; actually, the filter is zero-phase, without loss of generality. On the unit circle, we use the notation $H(\omega)$ as a shortcut for $H(e^{j\omega_1}, e^{j\omega_2})$, understanding that $\omega = (\omega_1, \omega_2) \in [-\pi,\pi]^2$. The passband and stopband regions of the filter (1) are denoted by $\mathscr{D}_p$ and $\mathscr{D}_s$, respectively. We assume that we have a method for providing the optimal minimax filter, i.e. the solution to the problem

$$\begin{array}{rll} \gamma_s^\star = & \min\limits_{\gamma_s, H} & \gamma_s \\ & \text{subject to} & |H(\omega)| \le \gamma_s, \quad \forall \omega \in \mathscr{D}_s \\ & & |1 - H(\omega)| \le \gamma_p, \quad \forall \omega \in \mathscr{D}_p \end{array} \quad (2)$$

where γ_p is a given passband error bound. (Variations of this problem could be considered, where a linear combination of γ_p and γ_s is optimized, but they are not significant to the present discussion.)

The problem we study here is the influence of the shape of the frequency regions $\mathscr{D}_p$ and $\mathscr{D}_s$ on the value $\gamma_s^\star$. For a fair comparison, we assume that the *areas* of the passband and stopband are fixed, and denoted by A_p and A_s, respectively. So, our problem is: given A_p, A_s and γ_p (and the orders n_1, n_2), for what passband $\mathscr{D}_p$ of area A_p and stopband $\mathscr{D}_s$ of area A_s, do we obtain the smallest value $\gamma_s^\star$?

Since we are not aware of a possibility to determine the exact solution (it looks impossible to parameterize all shapes, or at least a significant class of shapes), our study will be purely experimental. Although we consider only a small number of regular shapes, the conclusions are somewhat surprising.

2. METHODOLOGY

We describe here the setup of the experimental study. The passband and stopband regions are delimited by four types of basic curves; the corresponding shapes are rectangular (square, actually), diamond, circle and cos-oval (this latter name is coined here and it might not be the most appropriate, but at least it is short). Here follows a rigorous description of these shapes.

Rectangular. A square passband is described by

$$\mathscr{D}_{p1} = \{\omega \in [-\pi,\pi]^2 \mid |\omega_1| \le c_1,\ |\omega_2| \le c_1\}, \quad (3)$$

where c_1 is a given frequency. A square stopband is the complementary of a passband defined by (3), namely

$$\mathscr{D}_{s1} = \{\omega \in [-\pi,\pi]^2 \mid |\omega_1| \ge c_1' \text{ or } |\omega_2| \ge c_1'\}. \quad (4)$$

Since for all shapes the stopband is defined by complementarity, in the sequel we give only the definitions of the passbands.

Diamond:

$$\mathscr{D}_{p2} = \{\omega \in [-\pi,\pi]^2 \mid |\omega_1| + |\omega_2| \le c_2\}. \quad (5)$$

Circle:

$$\mathscr{D}_{p3} = \{\omega \in [-\pi,\pi]^2 \mid |\omega_1|^2 + |\omega_2|^2 \le c_3^2\}. \quad (6)$$

Cos-oval.

$$\mathscr{D}_{p4} = \{\omega \in [-\pi,\pi]^2 \mid \cos\omega_1 + \cos\omega_2 \ge c_4\}. \quad (7)$$

Each of the above passbands is defined by a single parameter, taking values in an interval: $c_1 \in [0,\pi]$, $c_2 \in [0,2\pi]$, $c_3 \in [0,\sqrt{2}\pi]$, $c_4 \in [-2,2]$. If the area of a passband is given,

The author is on leave from Department of Automatic Control and Computers, "Politehnica" University of Bucharest. This work is supported by the Academy of Finland, project No. 213462, Finnish Centre of Excellence program (2006 - 2011).

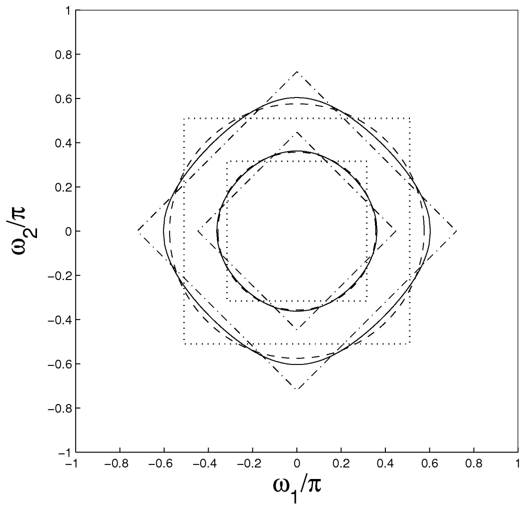

Figure 1: Passbands (interior curves) and stopbands (exterior curves) edges of four lowpass filters. The shapes are: rectangular (dotted line), diamond (dash-dot), circle (dashed), cos-oval (solid).

then it is easy to compute the value of the corresponding parameter, using either an explicit formula or (in the cos-oval case) a simple program for solving nonlinear equations, e.g. based on bisection. For convenience, we normalize to 1 all the areas, i.e. we divide the actual area to $4\pi^2$.

We assume that the passband and the stopband regions are delimited by the same type of curves, with different parameters. So, we consider only four types of filter specifications. An example is given in Figure 1. The passband (normalized) area is $A_p = 0.1$ and the stopband area is $A_s = 0.74$. Since the reader is familiar with square, diamond and circle passbands, we comment only on the cos-oval shape. This shape was introduced in [3], as the simplest example of a curve defined by positive trigonometric polynomials; this type of curve is appropriated to an optimization method providing practically optimal minimax FIR filters (with linear phase). More examples are available in the cited paper, for various values of the parameter c_4 from (7). For small A_p (and thus c_4 near 2), the shape is nearly circular, as it can be noticed for the passband from Figure 1, almost indistinguishable from the circle of the same area. As A_p grows (and c_4 decreases), the shape resembles more and more a diamond with rounded corners; for $A_p = 0.5$ (and $c_4 = 0$), it is a diamond.

For given passband and stopband regions (defined by one of the four basic shapes and the areas A_p and A_s) and passband error bound γ_p, the solution to problem (2) is computed as follows.

- For rectangular, diamond and cos-oval shape, the method from [3] is used. This method, based on a parameterization of trigonometric polynomials that are positive on a frequency domain, transforms minimax FIR filter design into a semidefinite programming (SDP) problem and gives practically optimal solutions.
- For circular shape, the problem (2) is discretized on a set of frequency points that is the union of a grid containing 40×40 points covering $[0,\pi]^2$ and of 50 points on each of the passband and stopband edges (which are quarters of a circle, located in the first quadrant). A linear programming problem (LP) is obtained; the first LP approach appeared in [5]. Other methods could be used, like those based on multiple exchange [4] or iterative reweighting [1], that might be faster but cannot give better results. Such discretized formulations do not give the optimal solution; the actual passband and stopband errors are larger than γ_p and $\gamma_s^\star$, respectively. To compensate this effect, we replace γ_p from (2) with $\alpha\gamma_p$, for several values of α ranging from 0.96 to 0.99. For each solution thus obtained, we measure the actual passband and stopband errors with respect to the ideal response, on a finer 200×200 grid. We take as solution of (2) the maximum actual stopband error γ_s that corresponds to the actual passband error that is nearest from γ_p. Although with this method we compute only an approximation of the solution, this approximation can serve well for comparisons between the circular and the other shapes.

The SDP and LP problems corresponding to (2) have been solved with the library SeDuMi [6].

3. RESULTS

The values $\gamma_s^\star$ computed as described in the previous section are used for comparing the four basic passband and stopband shapes: rectangle, diamond, circle and cos-oval. Since the design problem (2) has three parameters (besides the orders n_1, n_2), i.e. the areas A_p, A_s and the passband error bound γ_p, several ways of reporting the results are possible. Most experiments have been performed with $n_1 = n_2 = 7$, i.e. with FIR filters of support 15×15; in what follows, these are the orders of the filters, unless otherwise specified.

Figure 2 shows the computed optimal stopband error for $\gamma_p = 0.05$ and passband and stopband areas chosen as follows. For each passband area A_p in the range 0.04 : 0.02 : 0.30, we have solved (2) for several values of the stopband area A_s. For each A_p, we report in the figure only the results for a single value of A_s, precisely the one for which the best $\gamma_s^\star$ (among the four filters with the basic band shapes) is nearest from $0.01 = -40$ dB. This is like we would want to design a filter with fixed γ_p and A_p, with stopband error γ_s approximately equal to -40 dB and a transition band as narrow as possible (not caring about the shape of the best filter). The first pairs (A_p, A_s), starting from the left, are $(0.04, 0.84)$, $(0.06, 0.81)$, $(0.08, 0.78)$, $(0.10, 0.74)$. We remark that the differences between the values $\gamma_s^\star$ (for the same A_p, but different shapes) are quit large; hence, the small errors made in evaluating $\gamma_s^\star$ for circular shape are not significant.

The same procedure for choosing the areas was used for generating Figure 3, this time with $\gamma_p = 0.01$. Since the passband ripple is smaller, the transition band must be larger in order to obtain the same stopband attenuation (of around 40 dB). Indeed, now the first pairs (A_p, A_s) are $(0.04, 0.81)$, $(0.06, 0.76)$, $(0.08, 0.73)$, $(0.10, 0.70)$ (i.e. the stopband areas are smaller than those for Figure 2).

Figures 2 and 3 suggest that the cos-oval shape is the best for narrow passbands, while the circular shape becomes better as the passband widens. Typically, these two shapes are better than the diamond; the rectangle gives (not surprisingly) almost always the worst results. The cos-oval shape has not been considered until [3], and there mainly because of its simplicity (in the context of curves generated with

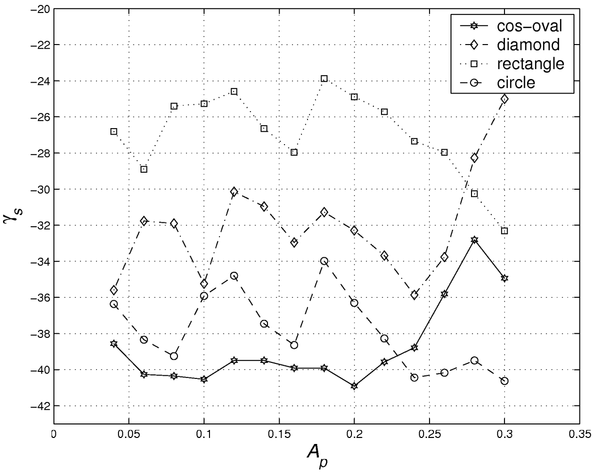

Figure 2: Optimal stopband errors (in dB) for $\gamma_p = 0.05$ and various passband and stopband areas.

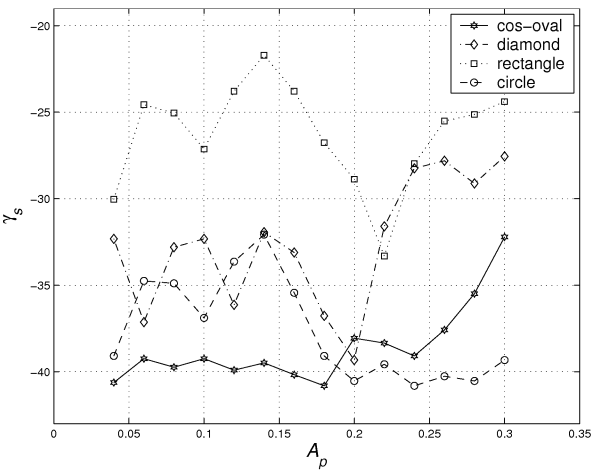

Figure 3: Optimal stopband errors (in dB) for $\gamma_p = 0.01$.

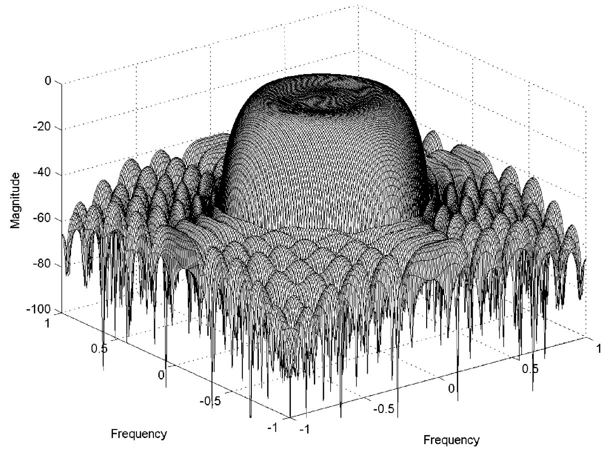

Figure 4: Frequency response of 15×15 FIR filter with cos-oval passband and stopband.

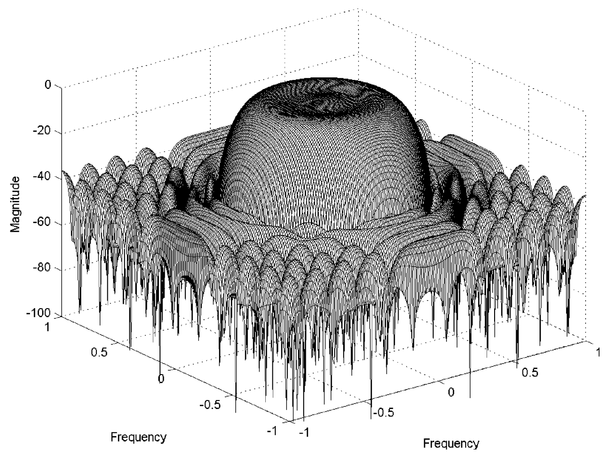

Figure 5: Frequency response of 15×15 FIR filter with circular passband and stopband.

trigonometric polynomials). We give here only one example of design in which the cos-oval shape is clearly better than the much more popular circular one. For $A_p = 0.1, A_s = 0.74$, i.e. the values used in Figure 1, and $\gamma_p = 0.05$, the frequency responses of the filters obtained by solving (2) are shown in Figure 4, for cos-oval shape, and Figure 5 for circular shape. The stopband attenuations are -40.54 dB and -35.92 dB, respectively. With a practically insignificant change of shape (see again Figure 1), we gain more than 4 dB attenuation; otherwise, the frequency responses are quite similar.

To confirm the above findings, we have run the test for a large set of pairs (A_p, A_s), with $\gamma_p = 0.05$. A map of the best shapes is shown in Figure 6. Only the cos-oval and the circle are the best, over the whole range of passband/stopband areas. The figure contains also the (approximate) level-curves on which the stopband error $\gamma_s^\star$ (of the best filter) has the values -20, -40 and -60 dB. The figure shows one more trend: for a given passband area, the cos-oval is best for large stopband areas (and thus narrow transition bands). As the

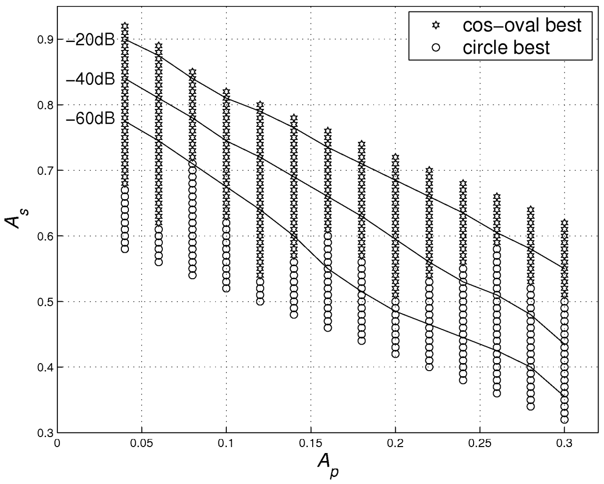

Figure 6: Map of best shapes, for $\gamma_p = 0.05$.

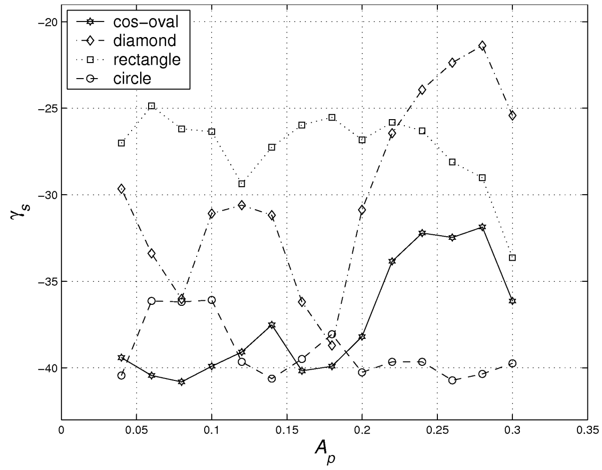

Figure 7: Optimal stopband errors (in dB) for $\gamma_p = 0.05$, $n_1 = n_2 = 5$.

stopband shrinks (and the transition band widens), the circle becomes better and stays so. For narrow passbands, the cos-oval is better up to large attenuations (more than 60 dB), while for wide passbands, the circle becomes better for attenuations only slightly greater than 20 dB.

Less extensive tests for smaller degrees confirm the above trends. For example, experiments similar to those leading to Figures 2 and 3, keeping $\gamma_p = 0.05$ but lowering the degree, produce Figure 7 for $n_1 = n_2 = 5$ and Figure 8 for $n_1 = n_2 = 6$. The cos-oval and circle have now similar values for several passband areas, but there are still many passband areas for which the stopband attenuations are neatly different.

4. CONCLUSION

The experiments reported in this paper suggest that 2-D linear-phase FIR filters whose passbands and stopbands are delimited by cos-oval curves (7) may be used instead of filters with circular bands, as they give better stopband attenuation for similar design specifications. The cos-oval shape,

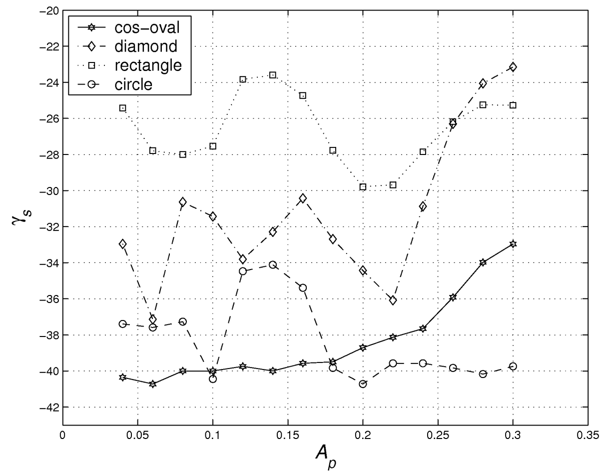

Figure 8: Optimal stopband errors (in dB) for $\gamma_p = 0.05$, $n_1 = n_2 = 6$.

not considered until now in 2-D filter design, is clearly better for relatively narrow passbands and transition bands.

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