

## REDUCING NOISE SENSITIVITY IN F-ESPRIT USING WEIGHTING MATRICES

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### ABSTRACT

In this paper we explore the possibility of using a weighting matrix to improve the accuracy of F-ESPRIT - a recently developed frequency estimation algorithm. In F-ESPRIT, the estimation is based on frequency domain data, and the algorithm is developed to enable a frequency selective estimation. The estimation is based on forming a data matrix which for i.i.d. measurement noise, is perturbed by a non-diagonal noise matrix. Since F-ESPRIT is a subspace based algorithm, the non-diagonal perturbation matrix will for low Signal to Noise Ratios (SNR) reduce the estimation accuracy. In this paper, we derive an exact expression for the perturbation matrix, and evaluate a weighting strategy for improving the performance. Empirical results show a large improvement in the low SNR case.

### 1. INTRODUCTION AND PROBLEM FORMULATION

Frequency estimation is a well studied topic due to its vast number of applications - it occurs in such different areas as sonar and radar applications, speech analysis and MR spectroscopy [1]. Consequently, a large number of estimators have been suggested, and subspace based estimators such as MUSIC [2], and ESPRIT [3] have been recognized to provide very accurate estimates. In [4], a new frequency domain subspace algorithm was presented which enables a frequency selective estimation. Such a feature can be useful to incorporate prior knowledge regarding the location of the frequencies and to reduce the influence of unmodelled spectral disturbances. The algorithm has been analyzed in [5, 6, 7] where it is referred to as F-ESPRIT - a label which we also will use here. Additionally, F-ESPRIT has been evaluated empirically in several applications e.g., [8, 9].

The frequency selective property of the F-ESPRIT algorithm is achieved by deriving a parametric relation between the signal parameters and a subset of the frequency domain data, calculated using the Discrete Fourier transform (DFT). This relation will, for the case when the original signal contains white noise, be perturbed by a non-white noise process. In this paper we explore the possibility of including weighting matrices in the estimation of the signal parameters. The

weighting matrices are calculated by deriving an exact expression of the noise perturbation, and empirical studies show a promising performance.

In the frequency estimation problem we are considering, data is modelled as a sum of vector valued damped complex sinusoids buried in additive noise

$$\mathbf{y}(t) = \sum_{k=1}^n \beta_k e^{\lambda_k t} + \mathbf{v}(t), \quad (1)$$

where  $\beta_k \in \mathbb{C}^m$  is the unknown complex gain, and  $\lambda_k = \gamma_k + i\omega_k \in \mathbb{C}$  contains the damping,  $\gamma_k$ , and frequency,  $\omega_k$ , parameters. The objective is to retrieve  $\beta_k$ ,  $\gamma_k$  and  $\omega_k$  from a measured data set  $\{\mathbf{y}(t), t = 0, \dots, N-1\}$ , which is perturbed by an i.i.d. noise process  $\mathbf{v}(t)$ . To make the model unique, the signal parameters are constrained as  $\omega_k \in (-\pi, \pi]$ ,  $\omega_k \neq \omega_l$  for  $k \neq l$  and  $\beta_k \neq \mathbf{0}$ .

The main focus will be on the estimation of the non-linear  $\lambda_k$ -parameters and, once the  $\lambda_k$ -parameters are estimated and assumed known, the  $\beta_k$ -parameters can be recovered using linear regression.

#### 1.1. Time domain State Space Model

The signal model in (1) can be written in the form of a time domain state space model, i.e. a scheme in which  $\mathbf{y}(t)$  recursively can be computed as

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_{t_0} \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t), \end{aligned} \quad (2)$$

using the matrix definitions

$$\mathbf{A} = \text{diag}[e^{\lambda_1} \ e^{\lambda_2} \ \dots \ e^{\lambda_n}] \in \mathbb{C}^{n \times n} \quad (3)$$

$$\mathbf{C} = [\beta_1 \ \beta_2 \ \dots \ \beta_n] \in \mathbb{C}^{m \times n} \quad (4)$$

$$\mathbf{x}_{t_0} = [1 \ 1 \ \dots \ 1]^T \in \mathbb{R}^n. \quad (5)$$

The vector  $\mathbf{x}(t)$  is called the state vector, and  $\mathbf{A}$  is the state transition matrix. The triple  $(\mathbf{A}, \mathbf{C}, \mathbf{x}_{t_0})$  is called a state space realization of  $\mathbf{y}(t)$  and is not unique. By a change of variables  $\mathbf{x}(t) = \mathbf{T}\bar{\mathbf{x}}(t)$ , where  $\mathbf{T}$  is a non-singular matrix, a new state space realization of  $\mathbf{y}(t)$  is formed by the triple

$$(\bar{\mathbf{A}}, \bar{\mathbf{C}}, \bar{\mathbf{x}}_{t_0}) \triangleq (\mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \mathbf{C}\mathbf{T}, \mathbf{T}^{-1}\mathbf{x}_{t_0}). \quad (6)$$

Note that the state transition matrices for the different realizations,  $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ , are similar matrices and hence share the same set of eigenvalues. From (3) we see that the eigenvalues of  $\mathbf{A}$  are equal to  $e^{\lambda_k}$ , which implies that the  $\lambda_k$  parameters can be retrieved from the eigenvalues of the state transition matrix regardless of the realization.

## 1.2. Frequency Domain State Space Model

The desired frequency selective feature is obtained by transforming the state space model in (2) to the frequency domain using the DFT. Let the  $N$ -point DFT of the state vector time series,  $\mathbf{x}(t)$ , for  $k = 0, \dots, N-1$  be given by

$$\mathbf{x}_k^N = \text{DFT}\{\mathbf{x}(t)\}_k \triangleq \sum_{t=0}^{N-1} \mathbf{x}(t) W_N^{-kt}, \quad (7)$$

where  $W_N = e^{j\frac{2\pi}{N}}$ , and similarly define

$$\mathbf{y}_k^N \triangleq \text{DFT}\{\mathbf{y}(t)\}_k, \quad \mathbf{v}_k^N \triangleq \text{DFT}\{\mathbf{v}(t)\}_k. \quad (8)$$

The DFT of the time-shifted state vector is derived in [4] as

$$\begin{aligned} \text{DFT}\{\mathbf{x}(t+1)\}_k &= W_N^k \mathbf{x}_k^N - (\mathbf{I} - \mathbf{A}^N) \mathbf{x}_{t_0} W_N^k \\ &= W_N^k \mathbf{x}_k^N - \mathbf{B} W_N^k, \end{aligned} \quad (9)$$

where

$$\mathbf{B} \triangleq (\mathbf{I} - \mathbf{A}^N) \mathbf{x}_{t_0}. \quad (10)$$

A frequency domain representative model of (2) can now be written as

$$\begin{aligned} W_N^k \mathbf{x}_k^N &= \mathbf{A} \mathbf{x}_k^N + \mathbf{B} W_N^k \\ \mathbf{y}_k^N &= \mathbf{C} \mathbf{x}_k^N + \mathbf{v}_k^N. \end{aligned} \quad (11)$$

Note that if any modes in the state vector are  $N$ -periodic, then the corresponding rows in the  $\mathbf{B}$  vector are zero.

## 2. ESTIMATION ALGORITHM

In (11), we have derived a relation between frequency domain data and the state transition matrix  $\mathbf{A}$ . In this section we will see how state space theory and subspace based methods can be applied to estimate the signal parameters of interest from frequency domain data,  $\mathbf{y}_k^N$ .

### 2.1. The observability matrix and its shift-invariance property

In linear systems theory, the observability matrix is used to deduce whether a change in the state vector is observable in the output signal. It is defined as

$$\mathcal{O}_s = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{s-1} \end{bmatrix}, \quad (12)$$

and when the number of block rows,  $s$ , is chosen larger than  $n$ , the signal model order,  $\mathcal{O}_s$  is referred to as the extended observability matrix. Note that if  $s \geq n$  the block Vandermonde structure of  $\mathcal{O}_s$  and the constraints on  $\omega_k$  in the problem formulation implies that the rank of  $\mathcal{O}_s$  is equal to  $n$ , the number of complex sinusoids.

Utilizing the structure of the observability matrix, it is possible to compute the state-transition matrix using a technique known as shift-invariance estimation [10]. The name shift-invariance originates from the fact that the state transition matrix is related to the observability matrix by constructing up- and down shifted sub-matrices of the latter. An equation describing the relation is written as

$$\mathbf{J}_1 \mathcal{O}_s \mathbf{A} = \mathbf{J}_2 \mathcal{O}_s, \quad (13)$$

where

$$\begin{aligned} \mathbf{J}_1 &= [\mathbf{I}_{(s-1)m} \quad \mathbf{0}_{(s-1)m \times m}] \\ \mathbf{J}_2 &= [\mathbf{0}_{(s-1)m \times m} \quad \mathbf{I}_{(s-1)m}]. \end{aligned}$$

The retrieval of  $\mathbf{A}$  from (13) is possible if  $\text{rank}(\mathbf{J}_1 \mathcal{O}_s) = n$ , which is guaranteed if  $s$  is chosen as  $s \geq n+1$ . However, in the presence of noise,  $\mathbf{v}(t)$ , the estimation accuracy is improved by choosing  $s$  larger than  $n+1$ , and a state transition matrix is determined as the least-squares solution of

$$\min_{\mathbf{A}} \|\mathbf{J}_1 \hat{\mathcal{O}}_s \mathbf{A} - \mathbf{J}_2 \hat{\mathcal{O}}_s\|_F^2, \quad (14)$$

where  $\hat{\mathcal{O}}_s$  is the, from data, estimated extended observability matrix. The state-transition matrix can also be determined from Equation (13) using the method of Total-Least-Squares [11].

### 2.2. Subspace Based Estimation

It is easy to verify that if  $\mathcal{O}_s$  is the observability matrix of the realization  $(\mathbf{A}, \mathbf{C}, \mathbf{x}_{t_0})$ , then the observability matrix corresponding to the realization  $(\hat{\mathbf{A}}, \hat{\mathbf{C}}, \hat{\mathbf{x}}_{t_0})$  from (6) is given by

$$\mathbf{Z}_s = \mathcal{O}_s \mathbf{T}. \quad (15)$$

The relation above implies that the range space of  $\mathbf{Z}_s$  equals that of  $\mathcal{O}_s$ , and is a property of the signal  $\mathbf{y}(t)$ , usually denoted the signal subspace. In this section we describe how a subspace based approach can be used to find an estimate of  $\mathbf{Z}_s$ .

First we phase shift the vectors  $\mathbf{y}_k$  and  $\mathbf{v}_k$  to construct a relation to the observability matrix using (11)

$$\mathbf{Y}_k = \mathcal{O}_s \mathbf{x}_k + \mathbf{\Gamma}_s \mathbf{u}_k + \mathbf{V}_k \quad (16)$$

where

$$\mathbf{Y}_k \triangleq [\mathbf{y}_k^T W_N^k \mathbf{y}_k^T W_N^{2k} \mathbf{y}_k^T \dots W_N^{(s-1)k} \mathbf{y}_k^T]^T \quad (17)$$

$$\mathbf{u}_k \triangleq [W_N^k W_N^{2k} W_N^{3k} \dots W_N^{sk}]^T \quad (18)$$

$$\mathbf{V}_k \triangleq [\mathbf{v}_k^T W_N^k \mathbf{v}_k^T W_N^{2k} \mathbf{v}_k^T \dots W_N^{(s-1)k} \mathbf{v}_k^T]^T, \quad (19)$$

and

$$\Gamma_s \triangleq \begin{bmatrix} \mathbf{0} & & & & & \\ \mathbf{CB} & \mathbf{0} & & & & \\ \mathbf{CAB} & \ddots & \ddots & \ddots & & \\ \vdots & & & & \ddots & \\ \mathbf{CA}^{s-2}\mathbf{B} & \mathbf{CA}^{s-3}\mathbf{B} & \dots & \mathbf{CB} & \mathbf{0} & \end{bmatrix}. \quad (20)$$

Using the phase shifted vectors in Equation (16), all DFT data points can be related to the observability matrix. Consequently, we can focus the estimation on a frequency interval by selecting a subset of the DFT data. Let the size of this subset be denoted by  $M$ , and we can form the matrix relation

$$\mathbf{Y} = \mathcal{O}_s \mathbf{X} + \Gamma_s \mathbf{U} + \mathbf{V}, \quad (21)$$

where

$$\mathbf{Y} \triangleq [\mathbf{Y}_{k_1} \ \mathbf{Y}_{k_2} \ \dots \ \mathbf{Y}_{k_M}] \quad (22)$$

$$\mathbf{V} \triangleq [\mathbf{V}_{k_1} \ \mathbf{V}_{k_2} \ \dots \ \mathbf{V}_{k_M}] \quad (23)$$

$$\mathbf{U} \triangleq [\mathbf{u}_{k_1} \ \mathbf{u}_{k_2} \ \dots \ \mathbf{u}_{k_M}] \quad (24)$$

$$\mathbf{X} \triangleq [\mathbf{x}_{k_1} \ \mathbf{x}_{k_2} \ \dots \ \mathbf{x}_{k_M}]. \quad (25)$$

The second step is to remove the influence of the  $\Gamma_s \mathbf{U}$  term, which can be accomplished using a projection matrix  $\Pi^\perp$  projecting onto the nullspace of  $\mathbf{U}$

$$\Pi^\perp \triangleq \mathbf{I} - \mathbf{U}^H (\mathbf{U} \mathbf{U}^H)^{-1} \mathbf{U}. \quad (26)$$

Post multiplying  $\mathbf{Y}$  with  $\Pi^\perp$  results in

$$\mathbf{Y} \Pi^\perp = \mathcal{O}_s \mathbf{X} \Pi^\perp + \mathbf{V} \Pi^\perp. \quad (27)$$

Finally, we estimate a basis of the range space of  $\mathcal{O}_s$  using the Singular Value Decomposition (SVD) of  $\mathbf{Y} \Pi^\perp$  and partitioning it as

$$\mathbf{Y} \Pi^\perp = \begin{bmatrix} \hat{\mathbf{Z}}_s & \hat{\mathbf{Z}}_n \end{bmatrix} \begin{bmatrix} \hat{\Sigma}_s & \mathbf{0} \\ \mathbf{0} & \hat{\Sigma}_n \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}}_s^H \\ \hat{\mathbf{V}}_n^H \end{bmatrix}, \quad (28)$$

where  $\hat{\mathbf{Z}}_s$  contains the  $n$  principal left singular vectors and  $\hat{\Sigma}_s$  the corresponding singular values. By viewing  $\hat{\mathbf{Z}}_s$  as an estimated observability matrix for some realization of  $\mathbf{y}(t)$ , the shift-invariance techniques described in Equation (13) can be used to compute the corresponding state transition matrix estimate,  $\hat{\mathbf{A}}$ . The signal parameters are then recovered from the eigenvalues of  $\hat{\mathbf{A}}$ .

### 3. NOISE PERTURBATION MATRIX

The SVD on  $\mathbf{Y} \Pi^\perp$  basically implies an eigenvalue decomposition of  $\mathbf{Y} \Pi^\perp (\mathbf{Y} \Pi^\perp)^H$  and the expected value of this matrix is

$$\mathbb{E} \left\{ \mathbf{Y} \Pi^\perp \mathbf{Y}^H \right\} = \mathcal{O}_s \mathbf{X} \Pi^\perp \mathbf{X}^H \mathcal{O}_s^H + \mathbb{E} \left\{ \mathbf{V} \Pi^\perp \mathbf{V}^H \right\}. \quad (29)$$

The last term in (29) is not a matrix proportional to the identity matrix, and will hence perturb the eigenvectors of  $\mathcal{O}_s \mathbf{X} \Pi^\perp \mathbf{X}^H \mathcal{O}_s^H$ . Therefore, the estimation accuracy can be improved using pre-whitening techniques.

To derive an exact expression for  $\mathbb{E} \left\{ \mathbf{V} \Pi^\perp \mathbf{V}^H \right\}$  we start with the factorization  $\mathbf{V} = \mathbf{G} \mathbf{H}$ , where  $\mathbf{G}$  and  $\mathbf{H}$  are defined as

$$\mathbf{G} = \begin{bmatrix} \text{diag}(\mathbf{v}_{k_1}) & \dots & \text{diag}(\mathbf{v}_{k_M}) \\ W_N^{k_1} \text{diag}(\mathbf{v}_{k_1}) & \dots & W_N^{k_M} \text{diag}(\mathbf{v}_{k_M}) \\ \vdots & & \vdots \\ W_N^{k_1(s-1)} \text{diag}(\mathbf{v}_{k_1}) & \dots & W_N^{k_M(s-1)} \text{diag}(\mathbf{v}_{k_M}) \end{bmatrix}, \quad (30)$$

and

$$\mathbf{H} = \begin{bmatrix} \mathbf{1}_m & & & \\ & \mathbf{1}_m & & \\ & & \ddots & \\ & & & \mathbf{1}_m \end{bmatrix}, \quad (31)$$

where  $\mathbf{1}_m$  is a column vector of length  $m$  containing ones. The  $\mathbf{G}$  and  $\mathbf{H}$  matrices can be written in a more condensed form

$$\begin{aligned} \mathbf{G} &= (\mathbf{W} \otimes \mathbf{I}_m) \mathbf{\Lambda}_D \\ \mathbf{H} &= \mathbf{I}_M \otimes \mathbf{1}_m, \end{aligned} \quad (32)$$

using the matrices

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ W_N^{k_1} & W_N^{k_2} & \dots & W_N^{k_M} \\ \vdots & \vdots & & \vdots \\ W_N^{k_1(s-1)} & W_N^{k_2(s-1)} & \dots & W_N^{k_M(s-1)} \end{bmatrix}, \quad (33)$$

and

$$\mathbf{\Lambda}_D = \begin{bmatrix} \text{diag}(\mathbf{v}_{k_1}) & & & \\ & \text{diag}(\mathbf{v}_{k_2}) & & \\ & & \ddots & \\ & & & \text{diag}(\mathbf{v}_{k_M}) \end{bmatrix}, \quad (34)$$

which results in the following expression for  $\mathbf{V}$

$$\mathbf{V} = (\mathbf{W} \otimes \mathbf{I}_m) \mathbf{\Lambda}_D (\mathbf{I}_M \otimes \mathbf{1}_m). \quad (35)$$

The expected value  $\mathbb{E} \left\{ \mathbf{V} \Pi^\perp \mathbf{V}^H \right\}$  can now be written as

$$\mathbb{E} \left\{ \mathbf{V} \Pi^\perp \mathbf{V}^H \right\} = (\mathbf{W} \otimes \mathbf{I}_m) \mathbb{E} \left\{ \cdot \right\} (\mathbf{W} \otimes \mathbf{I}_m)^H, \quad (36)$$

where

$$\begin{aligned} \mathbb{E} \left\{ \cdot \right\} &= \mathbb{E} \left\{ \mathbf{\Lambda}_D (\mathbf{I}_M \otimes \mathbf{1}_m) \Pi^\perp (\mathbf{I}_M \otimes \mathbf{1}_m)^H \mathbf{\Lambda}_D^H \right\} \\ &= \mathbb{E} \left\{ \mathbf{\Lambda}_D \Pi^\perp \otimes (\mathbf{1}_m \mathbf{1}_m^T) \mathbf{\Lambda}_D^H \right\} \\ &= \sigma_v^2 \mathbf{\Lambda} \Pi^\perp \otimes \mathbf{I}_m, \end{aligned} \quad (37)$$

and  $\Lambda_{\Pi^\perp}$  is a diagonal matrix whose diagonal elements are identical to those in  $\Pi^\perp$ .

Finally, by combining (36) and (37) we end up with

$$\begin{aligned} E \left\{ \mathbf{V} \Pi^\perp \mathbf{V}^H \right\} &= \sigma_v^2 (\mathbf{W} \otimes \mathbf{I}_m) (\Lambda_{\Pi^\perp} \otimes \mathbf{I}_m) (\mathbf{W} \otimes \mathbf{I}_m)^H \\ &= \sigma_v^2 (\mathbf{W} \Lambda_{\Pi^\perp} \mathbf{W}^H) \otimes \mathbf{I}_m. \end{aligned} \quad (38)$$

### 3.1. Noise matrix diagonalization

To reduce the impact of the noise matrix, we seek to make it proportional to the identity matrix it using a weighting matrix  $\mathbf{K}^{-1}$  satisfying

$$\mathbf{I} = \mathbf{K}^{-1} (\mathbf{W} \Lambda_{\Pi^\perp} \mathbf{W}^H) \otimes \mathbf{I}_m \mathbf{K}^{-H}, \quad (39)$$

or identically

$$\mathbf{K} \mathbf{K}^H = (\mathbf{W} \Lambda_{\Pi^\perp} \mathbf{W}^H) \otimes \mathbf{I}_m. \quad (40)$$

To obtain  $\mathbf{K}$ , the SVD is used to compute the factorization

$$(\mathbf{W} \Lambda_{\Pi^\perp} \mathbf{W}^H) \otimes \mathbf{I}_m = \mathbf{U}_W \Sigma_W \mathbf{U}_W^H, \quad (41)$$

which results in

$$\begin{aligned} \mathbf{K}^{-1} &= \Sigma_W^{-1/2} \mathbf{U}_W^H \\ \mathbf{K} &= \mathbf{U}_W \Sigma_W^{1/2}. \end{aligned} \quad (42)$$

Finally, the SVD of  $\mathbf{K}^{-1} \mathbf{Y} \Pi^\perp$  is partitioned into a signal and noise subspace

$$\mathbf{K}^{-1} \mathbf{Y} \Pi^\perp = \begin{bmatrix} \hat{\mathbf{Z}}'_s & \hat{\mathbf{Z}}'_n \end{bmatrix} \begin{bmatrix} \hat{\Sigma}'_s & \mathbf{0} \\ \mathbf{0} & \hat{\Sigma}'_n \end{bmatrix} \begin{bmatrix} \hat{\mathbf{E}}_s'^H \\ \hat{\mathbf{E}}_n'^H \end{bmatrix}, \quad (43)$$

and  $\mathbf{K} \hat{\mathbf{Z}}'_s$  is taken as an estimate of  $\mathcal{O}_s$ .

### 3.2. Regularized weighting matrix

In many scenarios, especially when the number of time domain data is high and the estimation is based on a narrow frequency interval, the matrix  $(\mathbf{W} \Lambda_{\Pi^\perp} \mathbf{W}^H) \otimes \mathbf{I}_m$  becomes ill-conditioned. To improve the results when  $(\mathbf{W} \Lambda_{\Pi^\perp} \mathbf{W}^H) \otimes \mathbf{I}_m$  is close to singular, the weighting matrix can be regularized by adding a multiple of the identity matrix to the  $\Sigma_W$  matrix, which results in the following weighting matrix pair

$$\begin{aligned} \mathbf{K}_{reg}^{-1} &= (\Sigma_W + \alpha \mathbf{I})^{-1/2} \mathbf{U}_W^H, \\ \mathbf{K}_{reg} &= \mathbf{U}_W (\Sigma_W + \alpha \mathbf{I})^{1/2}. \end{aligned} \quad (44)$$

How to optimally choose the regularization parameter  $\alpha$  is not yet investigated, but has empirically been tuned to be of the order  $10^{-10}$ .

## 4. PERFORMANCE STUDY

To examine the performance of the weighting technique several empirical studies have been conducted. In this section we present one such study which well presents the typical behavior of using the regularized weighting matrix. The test signal contains two undamped complex exponentials located at  $\omega_1 = 0.20$  and  $\omega_2 = 0.203$ . The data length is  $N = 256$ , of which a total of  $M = 51$  frequency domain data located between  $\omega_l = 0.1$  and  $\omega_u = 0.3$  are used in the estimation. The noise variance is varied between  $\sigma_v^2 = 1$  and  $\sigma_v^2 = 0.01$ , and for each setting the MSE of the different techniques is estimated using 1000 Monte Carlo simulations. In Fig 1 the noise free spectrum of the signal is presented, and in Fig. 2 the performance of F-ESPRIT is compared to that when a regularized weighting matrix with  $\alpha = 10^{-10}$  is employed. To avoid confusion, only the estimate of  $\omega_1$  is shown in Fig. 2. However, the performance is similar for both frequencies. In Fig. 2, also the Cramér Rao lower bound (CRLB) for  $\omega_1$  is displayed along with a curve denoted  $\text{Var}\{\omega_1\}$  which corresponds to an analytical variance expression for F-ESPRIT derived in [7]. As can be seen, using a regularized weighting matrix can significantly improve the performance of F-ESPRIT in the case of a low SNR.

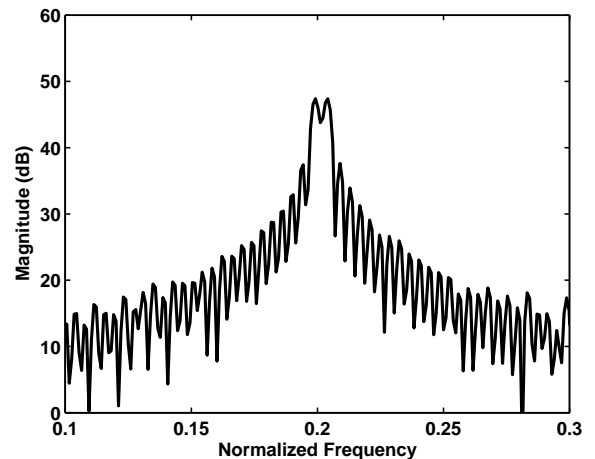
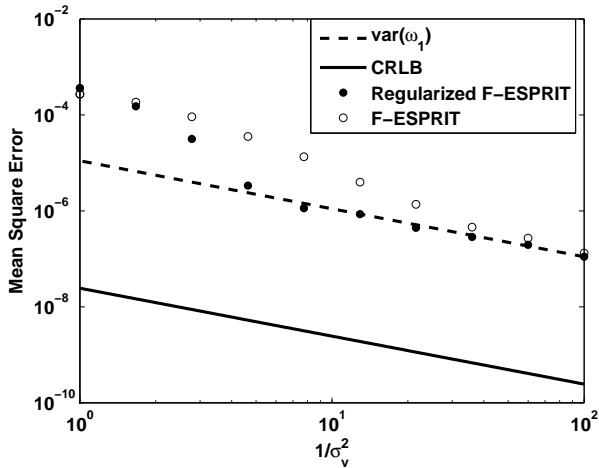


Fig. 1: Frequency spectrum of the test signal containing two undamped complex exponentials with frequencies located at  $\omega_1 = 0.20$  and  $\omega_2 = 0.203$ .

## 5. CONCLUSION

In this paper, we have derived a weighting matrix to improve the accuracy of F-ESPRIT, an algorithm enabling a frequency selective estimation of the parameters in a sinusoidal model. From empirical studies the suggested algorithm shows a promising performance, especially in the low SNR case and when the estimation is based on data from a narrow and densely



**Fig. 2:** Theoretical mean square error as a function of SNR and an empirical evaluation based on 1000 Monte Carlo simulations.

sampled frequency interval.

## 6. REFERENCES

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