SUBSAMPLING FOR APC STOCHASTIC PROCESSES

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ABSTRACT

We investigate the problem of consistency of subsampling procedure for nonstationary almost periodically correlated (APC) processes. It is shown that an appropriately normalized estimator of covariance has a consistent subsampling version provided that some mild regularity conditions are fulfilled. The result allows to improve the tests related to APC models, and to get more exact confidence intervals for such models. At the end of the paper we discuss open questions related to the optimal choice of the subsampling window b and the speed of convergence of the procedure considered.

1. INTRODUCTION

The nonstationary and seasonal behavior is quite common for many random phenomena observed in time. There are many applied problems ranging from climatology to financial data and telecommunication that need nonstationary models in continuous time with some cyclical or seasonal behavior [7]. Since early 1990 the seminal papers [5] and [6] have opened up a possibility of making a statistical inference for almost periodically correlated continuous-time stochastic processes. The natural focus of the above mentioned research was to get a consistent and asymptotically normal estimator of the mean and the covariance function and also of spectral density functions. It is now well known that the fundamental limiting properties of estimators for APC continuous time models are established. However, a major deficiency of limiting normal law is due to the fact that the variance of the limiting normal law was simply intractable in practical applications.

This paper opens up a new perspective on inference for nonstationary continuous-time stochastic processes. The general idea is to consider a subsampling technique as presented in the book [9]. Subsampling can be proved to provide the asymptotically valid quantiles for estimating parameters and tests in nonstationary models under very mild regularity conditions. This property of subsampling is called *consistency*. Once consistency is established, one can compute the finite-sample confidence intervals and critical values for tests from subsampling distributions and not from asymptotic distributions. It is very important to note here, that the use of asymptotic distributions for dependent data can lead to dangerous mistakes. Even for simple models such as AR(1) time series it was shown [4] that the convergence to the asymptotic normal law of the estimate of the parameter ϕ_1 is very slow and hard to justify for the data of the length of 200 observations. Therefore, there is a strong motivation to have a closer look at finite-sample distributions of the estimators. This paper provides the fundamental result on that and poses open questions leading to further research. Similar idea of using subsampling to the inference problem to time series is also presented in the paper [8].

2. APC PROCESSES

For the convenience of the reader we recall some basic facts from the theory of nonstationary, almost periodically correlated processes.

The zero mean, real valued process $\{X(t)\,;t\in\mathbb{R}\}$ is *almost periodically correlated* (APC) when its shifted covariance $B(t,\tau)=cov\{X(t),X(t+\tau)\}=E\{X(t)X(t+\tau)\}$ is uniformly almost periodic in t with respect to τ in $\mathbb{R}$.

Then the shifted covariance kernel admits a Fourier-Bohr decomposition:

$$B(t,\tau)\sim\sum_{\lambda\in\Lambda}a(\lambda,\tau)\,e^{i\lambda t}.$$

The *spectral covariance* $a(\lambda,\tau)$ is defined by :

$$a(\lambda,\tau)=\lim_{T\to\infty}\frac{1}{T}\int_t^{T+t}B(s,\tau)\,e^{-i\lambda s}\,ds.$$

The frequency set of the process X, defined by $\Lambda=\{\lambda: a(\lambda,\tau)\neq 0$ for some $\tau\}$, is at most countable.

Assume now that the sample $\{X(t)\,;t\in[0,n]\}$ of the APC process was observed on the interval $[0,n]$. It is known (see [6]) that the following estimator of the parameter $a(\lambda,\tau)$ defined as

$$\widehat{a}_n(\lambda,\tau)=\frac{1}{n}\int_{|\tau|}^{n-|\tau|}X(s+\tau)X(s)e^{-i\lambda s}\,ds$$

for $-n/2\le\tau\le n/2$, and $\widehat{a}_n(\lambda,\tau)=0$ otherwise, is consistent and asymptotically normal with the usual $\sqrt{T}$ speed of convergence provided some α-mixing (strong mixing) condition is fulfilled. See [2, 6] for more details.

We finish this section by providing the definition of α-mixing process, quite useful in subsequent considerations.

Definition 1 ([3]) *The stochastic process is called α-mixing if $\alpha_X(s)\to 0$ for $s\to\infty$, where*

$$\alpha_X(s)=\sup|Prob(A\cap B)-Prob(A)Prob(B)|$$

where the supremum is taken aver all $t>0$ and all the $A\in\mathscr{F}_X(-\infty,t)$ and $B\in\mathscr{F}_X(t+s,\infty)$, and $\mathscr{F}_X(t_1,t_2)$ stands for the σ-algebra generated by $\{X(t)\,;t_1\le t\le t_2\}$.

3. SUBSAMPLING FOR APC PROCESSES

The estimator $\widehat{a}_n(\lambda,\tau)$ is defined on the complete sample $\{X(t)\,;\,t\in[0,n]\}$ coming from the APC process X, and we denote

$$\widehat{a}_n = \widehat{a}_n\{X(t)\,;\,t\in[0,n]\} = \widehat{a}_n(\lambda,\tau).$$

We will now apply the construction of the subsampling introduced in [9]. To this end fix $h>0$, and for $b>0$ and $t\geq 0$ define $Y_{b,t}=\{X(u)\,;\,u\in E_{b,h,t}\}$ where $E_{b,h,t}=\{u\in\mathbb{R}: th\leq u\leq th+b\}$.

In the formula for the subset $E_{b,h,t}$ one can observe that the parameter h is the *overlap factor* when t varies. We are getting the minimal overlap when $h=b$. Since we are in the continuous time case, the maximal overlap is obtained as $h\to 0$. In the following $b=b_n$ goes to infinity with n, $0<b\leq n$, and it is called the *subsampling size*.

The *subsampling version* $\widetilde{a}_{b,t}=\widetilde{a}_{b,t}(\lambda,\tau)$ of the estimator $\widehat{a}_n(\lambda,\tau)$ generated by the data $Y_{b,t}$ is defined as

$$\widetilde{a}_{b,t} = \frac{1}{b}\int_{|\tau|}^{b-|\tau|} X(th+u+\tau)X(th+u)e^{-i\lambda u}du$$

for $|\tau|\leq b/2$ and $\widetilde{a}_{b,t}=\widetilde{a}_{b,t}(\lambda,\tau)=0$ otherwise. From the property of the almost periodic functions we can easily see that

$$\lim_{b\to\infty} E\{\widetilde{a}_{b,t}\} = a(\lambda,\tau)e^{i\lambda th}$$

for all λ, τ, t and h. Thus we need to modify the subsampling version $\widetilde{a}_{b,t}$ to get an asymptotically unbiased estimator of $a(\lambda,\tau)$.

Definition 2 *The modified subsampling version $\widehat{a}_{b,t}$ of the estimator $\widehat{a}_n(\lambda,\tau)$ is defined by*

$$\widehat{a}_{b,t} = \widetilde{a}_{b,t}e^{-i\lambda th}$$

for $|\tau|\leq b/2$ and $\widehat{a}_{b,t}=\widehat{a}_{b,t}(\lambda,\tau)=0$ otherwise.

Notice that the spectral covariance $a(\lambda,\tau)$ and the estimators $\widehat{a}_n$ and $\widehat{a}_{b,t}$ are complex-valued, so we can see them as two-dimensional valued : real part and imaginary part. Furthermore in the following we consider the partial order in $\mathbb{C}\sim\mathbb{R}^2$ defined by : $x\preceq y$ means that $x_1\leq y_1$ and $x_2\leq y_2$, for $x=x_1+ix_2$ and $y=y_1+iy_2$. Then we can define the empirical process of $\sqrt{b}\{\widehat{a}_{b,t}-\widehat{a}_n\}$.

Definition 3 *The empirical process $L_{n,b}(x)$ induced by the subsampling procedure and the sample $\{X(t):t\in[0,n]\}$ is defined as*

$$L_{n,b}(x) = \frac{1}{q}\sum_{t=0}^{q-1}\mathbf{1}\big\{\sqrt{b}\{\widehat{a}_{b,t}-\widehat{a}_n\}\preceq x\big\}, \tag{1}$$

where $q=q_n=\lfloor\frac{n-b}{h}\rfloor+1$ is the number of intervals $E_{b,h,t}$ contained in $[0,n]$.

Denote the distributions $J_n=\mathscr{L}\big\{\sqrt{n}\{\widehat{a}_n-a(\lambda,\tau)\}\big\}$ and $J_{b,t}=\mathscr{L}\big\{\sqrt{b}\{\widehat{a}_{b,t}-a(\lambda,\tau)\}\big\}$ and the corresponding cumulative distribution functions

$$J_n(x) = P\big\{\sqrt{n}\{\widehat{a}_n-a(\lambda,\tau)\}\preceq x\big\}$$

$$J_{b,t}(x) = P\big\{\sqrt{b}\{\widehat{a}_{b,t}-a(\lambda,\tau)\}\preceq x\big\}.$$

Denote by J the limit distribution of the statistic $\sqrt{n}\{\widehat{a}_n-a(\lambda,\tau)\}$, that is, $J_n(x)\longrightarrow J(x)$ at any point of continuity x of J. It is known that J is two-dimensional normal, and it can be degenerate [2].

4. SUBSAMPLING CONSISTENCY

We assume the following conditions:

A1 $b/n\to 0$ while both $n,b\to\infty$. The overlap parameter h is either constant or $h\to\infty$ so that $h/b\to a$ for some $a\in[0,1]$.

A2 $\sup_t E\left\{|X(t)|^{4+\delta}\right\}<\infty$ for some $\delta>0$, the fourth moment is almost periodic in the following sense : the function $v\mapsto \mathrm{cov}\big\{X(u+v+\tau)X(u+v),X(v+\tau)X(v)\big\}$ is almost periodic for each u. Moreover the mixing coefficient satisfies $\int_0^\infty \alpha_X(t)^{\delta/(4+\delta)}dt<\infty$.

A3 The set of frequencies Λ corresponding to the covariance representation has the following separability property

$$\sum_{\lambda\in\Lambda\setminus\{0\}}\frac{1}{\lambda^2}<\infty.$$

Then we can state the subsampling consistency.

Theorem 1 *Let X be APC and zero-mean. Assume that (A1), (A2) and (A3) are fulfilled. Then $L_{b,t_n}(x)\to J(x)$ in probability at any point of continuity x of $J(x)$ for any sequence $\{t_n\}$ which converges to infinity as $n\to\infty$.*

Proof . We will first prove that $J_{b,t_n}(x)\to J(x)$ and then we will deduce that $L_{n,b}(x)\to J(x)$.

To show that $J_{b,t_n}(x)\to J(x)$, observe

$$\begin{aligned}\sqrt{b}_n\big\{\hat{a}_{b_n,t_n}(\lambda,\tau)-a(\lambda,\tau)\big\} &= \sqrt{b}_n\big\{\hat{a}_{b_n,t_n}(\lambda,\tau)-E\{\hat{a}_{b_n,t_n}(\lambda,\tau)\}\big\}\\ &+\sqrt{b}_n\big\{E\{\hat{a}_{b_n,t_n}(\lambda,\tau)\}-a(\lambda,\tau)\big\}.\end{aligned}$$

The first term can be represented in the form

$$\begin{aligned}\sqrt{b}_n\big\{\hat{a}_{b_n,t_n}(\lambda,\tau)-E\{\hat{a}_{b_n,t_n}(\lambda,\tau)\}\big\} = \frac{1}{\sqrt{b_n}}\int_{t_nh}^{t_nh+b_n}Z(s,\tau)S(\lambda s)ds+\varepsilon_n\end{aligned}$$

where $Z(s,\tau)=\big\{X(s)X(s+\tau)-E\{X(s)X(s+\tau)\}\big\}$ and $S(\lambda t)=\big(\cos(\lambda t),-\sin(\lambda t)\big)$. Using Cramer-Wold device, the central limit theorem for α-mixing time series gives that the first term converges in law to the limiting distribution J. The second term is asymptotically negligible.

Now we prove that $L_{n,b}(x)\to J(x)$. Using ideas from Politis et al. [9], it suffices to show that

$$U_{n,b}(x) = \frac{1}{q_n}\sum_{t=0}^{q_n-1}\mathbf{1}\big\{\sqrt{b}\{\hat{a}_{b,t}-a(\lambda,\tau)\}\preceq x\big\}$$

also tends to $J(x)$. In order to study asymptotics of $U_{n,b}(x)$ one can consider the bias and the variance of $U_{n,b}$. It is easy to show that $E\{U_{n,b}\}\to J(x)$. The variance of $U_{n,b}(x)$ can be represented as

$$\begin{aligned}var\{U_n(x)\} &= \frac{1}{q_n^2}\sum_{j=0}^{q_n-1}var\{f(b_n,j)\}\\ &+\frac{2}{q_n^2}\sum_{0\leq j_1<j_2\leq q_n-1}cov\{f(b_n,j_1),f(b_n,j_2)\}\end{aligned}$$

where

$$f(b,j) = \mathbf{1}\big\{\sqrt{b_n}\{\hat{a}_{b_n,j} - a(\lambda,\tau)\} \preceq x\big\}.$$

It is obvious that the first term of $var\{U_n(x)\}$ tends to 0. The asymptotic negligibility of the second term follows from the α-mixing assumption and the rate of convergence of α_X to 0.

5. CONCLUSION

5.1 Confidence intervals

The consistency of subsampling procedure allows us to construct confidence intervals from finite samples of APC continuous time models. Consider for example the question of finding a confidence interval for the spectral covariance $a(\lambda,\tau)$. The asymptotic confidence interval will use the limiting law of $\sqrt{T}\{\widehat{a}_T(\lambda,\tau) - a(\lambda,\tau)\}$ for that purpose. In that case one will encounter a fundamental difficulty with the asymptotic variance which is not easily tractable (see [2] and [6]). Due to our result, we can calculate the quantiles $c_{n,b}(\alpha)$ of $L_{n,b}$ to get the confidence interval for $a(\lambda,\tau)$. Given the equivalence between confidence intervals and tests, we are able to test hypotheses of the type $\mathrm{H}_0 : a(\lambda,\tau) = 0$ against $\mathrm{H}_1 : a(\lambda,\tau) \neq 0$, to assess the significance of the spectral covariance.

5.2 Open problem

One of the open problems of the introduced subsampling procedure is the optimal choice of the block size b for a given sample size n. The papers of Politis et al. [9, 1] indicate an appropriate direction for some special cases. One may, for example, try to establish the Edgeworth expansion for the subsampling estimator and then try to obtain the optimal choice of b. The other way to proceed is to consider a calibration method presented in [9]. So far there is no known result for APC models. Finding ways of establishing an optimal choice for b is a topic of an ongoing research of the authors.

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