HYPERCOMPLEX ALGEBRAS IN DIGITAL SIGNAL PROCESSING: BENEFITS AND DRAWBACKS

Daniel Alfsmann, Heinz G. Göckler*, Stephen J. Sangwine†, and Todd A. Ell‡*

*Digital Signal Processing Group
Ruhr-Universität Bochum
44780 Bochum, Germany
email: alfsmann@nt.rub.de

†Dept. Electron. Syst. Engineering
University of Essex, Wivenhoe Park
Colchester CO4 3SQ, UK
email: S.Sangwine@IEEE.org

‡Goodrich Sensor Systems,
Burnsville, MN 55306, USA
email: T.Ell@IEEE.org

ABSTRACT

This tutorial contribution presents a short historical introduction and a survey of hypercomplex algebras in conjunction with some beneficial applications in predominantly time-based digital signal processing. Potential advantages and shortcomings of hypercomplex digital signal processing are discussed.

1. INTRODUCTION

For a long time, mathematicians have studied various (hypercomplex) algebras as a self-contained discipline. Most recently, physicists and engineers have gradually grasped the underlying theories for a variety of applications in signal and image processing, signal and system representation and analysis, computer graphics, etc. This tutorial paper presents an introduction to hypercomplex algebras needed for hypercomplex digital signal processing (HCDSP), and discusses the potentials and weaknesses of HCDSP. Note that pure geometric signal processing is beyond the scope of this paper.

A basic scientific motivation for the investigation of hypercomplex algebras is to extend complex (digital) signal processing (CDSP) to HCDSP. We start our presentation with a short historical survey of the advent of complex numbers and the discovery of hypercomplex algebras referring to [1].

Already during the Renaissance it had been recognised in Italy that the real algebra ($\mathbb{R}$) is *algebraically not closed under exponentiation*, when CARDANO (1501-76) and his competitor FONTANA-TARTAGLIA (1499-1557) were looking for a general solution of 3rd order equations, nowadays known as Cardanic formulae. To overcome this limitation, CARDANO in conjunction with BOMBELLI (1526-72) first introduced *complex numbers* ($\mathbb{C}$). GAUSS (1777-1855) proposed the illustrative complex plane to represent complex numbers in Cartesian and polar coordinates, respectively. EULER (1707-83) contributed a great multitude of widely used complex-valued functional relationships. Moreover, GAUSS proved the fundamental theorem of algebra encompassing the result that the complex algebra ($\mathbb{C}$) is *algebraically closed*.

Another track leads to *integral* or *functional transforms*, as introduced by LAPLACE (1749-1827), FOURIER (1768-1830) and LAURENT (1813-54), respectively: Real or complex functions of one or more independent parameters (time, location, etc.) are mapped onto a complex variable domain, the *frequency* or *spectral* domain. It is commonplace that a *signal spectrum* typically gives much more insight into the nature and properties of a signal than the original signal.

A first step beyond complex algebra ($\mathbb{C}$) was made by HAMILTON (1805-65) discovering the four-dimensional (4-D) *quaternions* ($\mathbb{H}$) in 1843. Soon after HAMILTON's publication of the quaternion algebra, his student GRAVES and later CAYLEY (1821-95) introduced a first kind of 8-D hypercomplex algebras, the *Octaves/Octonions* ($\mathbb{O}$) or *Cayley numbers*, respectively. Moreover, HAMILTON himself introduced still another 8-D hypercomplex algebra [2], known as complexified quaternions or biquaternions. A kind of generalisation to the n-D case was presented by CLIFFORD (1845-79) emphasising, however, geometrical viewpoints and n-D rotation. Nevertheless, CLIFFORD algebras include the former $\mathbb{R}-,\mathbb{C}-,\mathbb{H}-$algebras and the hyperbolic algebras being investigated in more detail only most recently [3, 4, 5].

It is a commonplace in mathematics that some or all of the following fundamental properties of $\mathbb{R}$ and $\mathbb{C}$ may vanish in hypercomplex algebras of higher dimension: *Commutativity*, *associativity* and *multiplicative inverse*. If an algebra is not a *division algebra*, some elements may lack a multiplicative inverse and *divisors of zero* exist with the consequence that the product of two non-zero numbers may vanish . This was thoroughly investigated by WEIERSTRASS (1815-97) [6].

This tutorial presents basic material for the understanding of time-based digital signal processing applying hypercomplex algebra. To this end, we recall the mainstream use of established CDSP under $\mathbb{C}$ (section 2). In section 3, we first give our reasons for transcending $\mathbb{C}$ to higher dimension or other algebras, we present the necessary definitions and representations of hypercomplex algebras for HCDSP and, finally, discuss benefits and drawbacks of HCDSP in view of a variety of suitable applications. In conclusion, in section 4 open issues are presented for future research.

2. ESTABLISHED USE OF COMPLEX ALGEBRA IN SIGNAL PROCESSING (CDSP)

Almost all physical signals and (equidistantly) sampled versions thereof are real-valued: $s(t), s(k) \in \mathbb{R}$. Nevertheless, a widely used family of versatile and illustrative integral transforms, LAPLACE-, z-Transform and some varieties of FOURIER Transform (FT) [7], provide a one-to-one mapping to a physically supported complex frequency domain under $\mathbb{C}$. For instance, the FT discloses the spectral content of a signal:

$$s(t),\, s(k) \in \mathbb{R} \xleftrightarrow{\text{FT}} S(j\omega),\, S(e^{j\Omega}) \in \mathbb{C}, \tag{1}$$

where any signal is likewise uniquely represented by its *spectrum* along the frequency "axis" $s := j\omega = j2\pi f$ or $z := e^{j\Omega}$, $\Omega = 2\pi f / f_S$, respectively (sampling rate: $f_S = 1/T$).

By interchanging time and frequency in (1), time-frequency *duality* [7] suggests the existence of complex signals that possess real (in general complex) spectra. The *analytic signal*, the most important class of complex signals, has found a variety of applications predominantly in CDSP: *i*) Efficient digital baseband processing of narrow-band bandpass signals with reduced sampling rate [8] (e.g. homodyne transceiver and single sideband amplitude modulation), *ii*) spectrally compacted data transmission by combining sequential binary data to higher level complex symbols (Quadrature Amplitude Modulation, Orthogonal Frequency Division Multiplex, etc.), *iii*) most efficient processing of discrete orthogonal transforms, such as DFT and FFT, *iv*) twofold system parallelisation for sample rate reduction by two, and *v*) efficient baseband simulation techniques based on the complex envelope of the analytic bandpass signal.

The imaginary part of an analytic signal is given by the HILBERT Transform (HT) of its real part. In general, the HT is implemented by means of a linear and time-invariant (LTI) digital system [8], for instance, as an FIR system with (zero-phase) impulse response $h_{\mathrm{HT}}(k) = 2/(k\pi)$ for odd k and 0 elsewhere:

$$\hat{s}(k) = h_{\mathrm{HT}}(k) * s(k) \stackrel{\mathrm{FT}}{\longleftrightarrow} \hat{S}(e^{j\Omega}) = -j\mathrm{sgn}(\sin\Omega)\cdot S(e^{j\Omega}), \quad (2)$$

where $(*)$ denotes convolution, yielding the digital analytic signal:

$$s_+(k) = s(k) + j\hat{s}(k) \stackrel{\mathrm{FT}}{\longleftrightarrow} S_+(e^{j\Omega}) = S(e^{j\Omega})[1+\mathrm{sgn}(\sin\Omega)]. \quad (3)$$

As a result, the overall spectral bandwidth of any analytic signal $s_+(k)$ comprises just half the width of the original real signal $s(k)$, allowing for most of the aforementioned applications exploiting the potential of unconstrained frequency shifting of analytic signals.

3. HYPERCOMPLEX ALGEBRAS IN DIGITAL SIGNAL PROCESSING (HCDSP)

3.1 Motivation

Exploiting the potential of CDSP to advantage, as outlined in section 2, the following questions related to HCDSP are motivated: What are the benefits of HCDSP w.r.t. the above collection of applications and beyond? What is the impact on HCDSP, if the underlying hypercomplex algebra lacks *commutativity*, *associativity*, and/or contains *divisors of zero*? Hence, at least the following basic system properties, well understood for real and complex DSP [8], call for thorough investigation in the case of HCDSP: *i*) *LTI property* tightly related to *convolution*, *ii*) existence of *hypercomplex spectral transforms* [9], similar to *z*- and FOURIER Transform, their impact on convolution theorem and the availability of some kind of analytic signal [10, 11, 12], *iii*) overall expenditure of system implementation, and potential benefits of *divisors of zero* for reduced computational burden [13, 14], and *iv*) potential advantages of multidimensional [11, 15, 16] and multirate HCDSP.

Furthermore, there are three basic motivations for the application of HCDSP: *i*) The holistic, compact processing of vector-valued signals that are a function of one or more independent parameters (e.g. time, location, physical quantities). Here, the dimension of the algebra must be chosen in compliance with the dimension of the signal vector. This means that each vector-sample is treated as a whole rather than treating its components separately. Classically, the reason for this is that the sample as a whole conveys information (direction in vector space) that is lost if the components of the sample are processed independently. *ii*) Hypercomplex digital processing of real or complex signals, respectively, with applications, for instance, in digital filtering [17, 18]. *iii*) Concurrent processing of a couple of independent signals in a hypercomplex system, where the dimension of the algebra must be consistent with the number of signals to be processed. For applications in digital filter banks see [19, 20, 21].

3.2 Definitions and representations

We define a general hypercomplex algebra $\mathbb{A}$ [22]

$$\boldsymbol{a} = a_1 + a_2\boldsymbol{i}_2 + \ldots + a_n\boldsymbol{i}_n \in \mathbb{A}, \quad a_1,\ldots,a_n \in \mathbb{K} \quad (4)$$

as an n-D $\mathbb{K}$-vector space over the field $\mathbb{K} = \mathbb{R},\mathbb{C}$ with an associated multiplication rule, comprising the unit element 1 ($1\cdot\boldsymbol{a} = \boldsymbol{a}\cdot 1 = \boldsymbol{a}$). The *multiplication table* defines every relation between the *imaginary units* $\boldsymbol{i}_2,\ldots,\boldsymbol{i}_n$ and implies the specific properties of the algebra. For instance, the *quaternions* are a 4-D $\mathbb{R}$-algebra

$$\boldsymbol{q} = q_1 + q_2\boldsymbol{i} + q_3\boldsymbol{j} + q_4\boldsymbol{k} \in \mathbb{H}, \quad q_1,q_2,q_3,q_4 \in \mathbb{R}, \quad (5)$$

with the multiplication table:

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},\ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j},\ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1. \quad (6)$$

From (6) it becomes clear that quaternions are not *commutative*: In general $\boldsymbol{pq} \neq \boldsymbol{qp}$ for $\boldsymbol{p},\boldsymbol{q} \in \mathbb{H}$. Nevertheless, they are *associative*: $\boldsymbol{p}(\boldsymbol{qr}) = (\boldsymbol{pq})\boldsymbol{r}$. Addition of two hypercomplex numbers $\boldsymbol{a},\boldsymbol{b} \in \mathbb{A}$ is always performed componentwise:

$$\boldsymbol{a}+\boldsymbol{b} = a_1 + b_1 + \sum_{\nu=2}^{n}(a_\nu + b_\nu)\boldsymbol{i}_\nu, \quad (7)$$

whereas multiplication is generally *distributive* over addition. Obviously, a hypercomplex addition (7) consists of n $\mathbb{K}$-valued additions. A hypercomplex multiplication demands n^2 $\mathbb{K}$-valued multiplications. For instance, the multiplication of two complex numbers, $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$, ($\mathbb{C}$ is considered as a 2-D hypercomplex algebra with $\mathbb{K} = \mathbb{R}$ and $\boldsymbol{i}_2^2 = \mathrm{i}^2 = -1$) according to:

$$z_1z_2 = x_1x_2 - y_1y_2 + \mathrm{i}\,(x_1y_2 + x_2y_1)\,, \quad (8)$$

requires $n = 2$ real additions and $n^2 = 4$ real multiplications. Of course, these operations have to be considered when evaluating the computational load of HCDSP.

Every associative algebra can be represented by an *isomorphic* $\mathbb{K}$-*valued* $n\times n$ matrix algebra. For instance, the matrix

$$\mathbf{Q} = \begin{bmatrix} q_1 & -q_2 & -q_3 & q_4 \\ q_2 & q_1 & -q_4 & -q_3 \\ q_3 & q_4 & q_1 & q_2 \\ -q_4 & q_3 & -q_2 & q_1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \quad (9)$$

is completely equivalent to the quaternion (5), and all operations and properties can likewise be validated with both representations (e.g. that $\mathbf{Q}$ is not commutative). However, (9) is highly redundant and therefore computationally inefficient compared to the direct calculation derived from the algebra's multiplication table (6).

An LTI system based on a hypercomplex algebra can always be decomposed either into $\mathbb{K}$-valued basic operations, as in (8), or into $\mathbb{K}$-valued subsystems. For the latter, a MIMO (Multiple Input Multiple Output) representation is useful, where the components $x_\nu(k), y_\nu(k) \in \mathbb{K}$ of the hypercomplex input and output signals $\boldsymbol{x}(k), \boldsymbol{y}(k) \in \mathbb{A}$ are combined to the respective input and output vectors $\mathbf{x}(k)$, $\mathbf{y}(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}$. To this end, a transfer matrix $\mathbf{H}(z)$ can be derived from the multiplication rule of the underlying algebra by replacing each real multiplication $(\cdot)$ with the convolution operator $(*)$; cf. (8) for complex systems. For the quaternion example, matrix representation results in the general 4×4 MIMO transfer matrix

$$\mathbf{H}(z) = \begin{bmatrix} H_{\mathrm{r}}(z) & -H_{\mathrm{i}}(z) & -H_{\mathrm{j}}(z) & -H_{\mathrm{k}}(z) \\ H_{\mathrm{i}}(z) & H_{\mathrm{r}}(z) & -H_{\mathrm{k}}(z) & H_{j}(z) \\ H_{\mathrm{j}}(z) & H_{\mathrm{k}}(z) & H_{\mathrm{r}}(z) & -H_{\mathrm{i}}(z) \\ H_{\mathrm{k}}(z) & -H_{\mathrm{j}}(z) & H_{\mathrm{i}}(z) & H_{\mathrm{r}}(z) \end{bmatrix} \quad (10)$$

that is composed of only four different real subsystems $H_{\mathrm{r}}(z)$, $H_{\mathrm{i}}(z)$, $H_{\mathrm{j}}(z)$, $H_{\mathrm{k}}(z)$.

3.3 Classes of hypercomplex algebras and their advantages

3.3.1 Division algebras

Due to the great variety of possible vector spaces and particularly their associated multiplication tables, one has to choose an appropriate algebra for a specific application. However, the need for beneficial properties explicitly narrows the number of alternatives. Firstly, the most important distinction of algebra classes is whether or not an algebra $\mathbb{A}$ is a *division algebra*. If this is the case, every non-zero element $\boldsymbol{a} \in \mathbb{A}$ has a unique inverse element $\boldsymbol{a}^{-1} \in \mathbb{A}$. In 1877, FROBENIUS stated his famous theorem that there exist only three associative division algebras $\mathbb{A}$ over the real numbers: The real ($\mathbb{R}$) and complex numbers ($\mathbb{C}$), and the quaternions ($\mathbb{H}$) [23]. Since the latter are not commutative regarding multiplication (6), the only commutative and associative division algebras (or fields) are generally limited to dimension $n \leq 2$ ($\mathbb{R}$, $\mathbb{C}$) [22]. This means that we generally have to renounce the familiar properties of $\mathbb{R}$ and $\mathbb{C}$ if we call for higher algebra dimensions. For instance, if we lower the requirements regarding the associativity of multiplication, resulting in the so-called *alternative* property $(\boldsymbol{aa})\boldsymbol{b} = \boldsymbol{a}(\boldsymbol{ab})$ and $\boldsymbol{a}(\boldsymbol{bb}) = (\boldsymbol{ab})\boldsymbol{b}$, respectively, we can utilise a division algebra even for $n = 8$: The octonions $\mathbb{O}$. However, as ZORN has detected in 1931, there is no real *alternative* division algebra other than $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ [24]. Therefore, these algebras have an exceptional position compared with any other possible algebra [22]. They have in common that they can be generated with the *doubling procedure* developed by

CAYLEY and DICKSON: A $2n$-D algebra $\mathbb{B}$ is constructed of two n-D subalgebras $\mathbb{A}$ by (possibly multiple) application of

$$\boldsymbol{b} = \boldsymbol{a}_1 + \boldsymbol{a}_2 \boldsymbol{i}_{n+1} \in \mathbb{B}, \quad \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{A} \tag{11}$$

and a new imaginary unit $\boldsymbol{i}_{n+1}^2 = -1$ for each iteration step, which anticommutes ($\boldsymbol{i}_\alpha \boldsymbol{i}_\beta = -\boldsymbol{i}_\beta \boldsymbol{i}_\alpha$) with the already existing imaginary units $\boldsymbol{i}_2, \ldots, \boldsymbol{i}_n$. For instance, with (11) the quaternions $\mathbb{B} = \mathbb{H}$ ($n = 2 \cdot 2$) can be generated from complex numbers ($n = 2$) with $\mathbb{A} = \mathbb{C}$, $\boldsymbol{i}_3^2 = -1$ and $\boldsymbol{i}_2 \boldsymbol{i}_3 = -\boldsymbol{i}_3 \boldsymbol{i}_2$. To comply with (5) and (6), the imaginary units have to be relabeled only: $\boldsymbol{i} = \boldsymbol{i}_2, \boldsymbol{j} = \boldsymbol{i}_3, \boldsymbol{k} = \boldsymbol{i}_4$.

Due to the exceptional position of the *quaternions* within hypercomplex algebras (the only associative real division algebra with $n > 2$), they have been investigated most thoroughly and most commonly been used in HCDSP. In particular, their usefulness in the n-D signal (image) processing has been shown [12, 15, 25], especially in connection with an intrisically quaternionic FT (QFT).

There are many possible definitions of a quaternionic FT, all stemming from the non-commutative nature of the algebra. Besides the possibility of placing the transform's exponential kernel on the left or right of the input function, the kernel itself can take multiple forms. Multiple kernel definitions are a result of the fact that, in general, $e^{q+r} \neq e^q e^r$, where $q, r \in \mathbb{H}$.

The first definition of a quaternionic FT was that of Ell [9, 26]. Sangwine's discrete version of this transform [27] has shown applicability by allowing generalizations of many image processing techniques dependent on FT of vector data. In particular, using a more recent hypercomplex FT definition [25], the validity of hypercomplex auto- and cross-correlation, and vector phase correlation have been demonstrated [28, 29, 30]. In [12] a hypercomplex FT was applied in the analysis of greyscale images because of its symmetry properties (which are similar to the symmetry properties of the complex FT of real signals). It was not until 2007 that suitable hypercomplex spectral convolution operator formulas were provided for use in vector image processing [31].

The QFT suffers from the same lacuna as the complex FT; the class of useful functions do not all have a QFT (e.g. unit step function). The Laplace transform broadens the class of transformable functions [9]. Moreover, the concept of the analytic signal (3) depending on a single (time) parameter has been extended to the quaternionic analytic signal, which is a function of two independent variables, by means of a QFT [11, 12, 32, 33].

3.3.2 *Geometric algebras*

Geometric algebra is a branch of mathematics and computer science (especially robotics) concerned with the representation and manipulation of algebraic quantities representing points, lines and planes in 3-D space [34, 35]. Arbitrary rotation in 3-D space is efficiently described and performed using quaternions $\mathbb{H}$. Moreover, most advantageously, repeated rotation does not accumulate errors caused by angle quantisation. As a result, "integer" rotation is feasible.

Most common geometric algebras are examples of the CLIFFORD algebra class. They utilise a formulation law different from (11) and permit the so-called *generator* units both with $\boldsymbol{i}_{n+1}^2 = -1$ (a) and $\boldsymbol{i}_{n+1}^2 = +1$ (b). The number p of type-a generator units, and the number q of type-b generator units completely determine the particular CLIFFORD algebra over the field $\mathbb{K}$ with $\mathscr{C}\ell(p,q)(\mathbb{K})$. Generally, every CLIFFORD algebra is *associative*, but is not necessarily a division algebra. They include $\mathbb{C}$ and $\mathbb{H}$, but not $\mathbb{O}$, since the latter is not associative. In [11], a CLIFFORD FT is presented.

Differing from CLIFFORD's biquaternions [1], HAMILTON was the first to introduce a *biquaternion* algebra by replacing the real quantities of quaternions $\mathbb{H}$ according to (5) with complex parameters:

$$\boldsymbol{q} = q_1 + q_2 \boldsymbol{i} + q_3 \boldsymbol{j} + q_4 \boldsymbol{k}, \quad q_1, q_2, q_3, q_4 \in \mathbb{C}, \tag{12}$$

The biquaternion algebra is not a division algebra, but it is associative. The biquaternion FT has been defined [36] and some of its properties elucidated. Furthermore, biquaternions have been applied to vector-sensor array processing [15], but much work remains to be done on the application of biquaternions to signal processing.

3.3.3 *Commutative algebras*

If we forgo both the call for a division algebra and a geometric meaning according to sec. 3.3.2, we can utilise commutative and associative non-division algebras, which always exhibit *zero divisors* for $n > 2$. Mostly, commutative algebras are constructed by a modified CAYLEY-DICKSON rule (11) with commuting imaginary units. The first attempt to apply these algebras to signal processing was made by SCHÜTTE [17]: He proposed the *Reduced Biquaternions* defined by (12) with $q_3 \equiv q_4 \equiv 0$. Alternatively, they can be represented as *bicomplex numbers* [37]:

$$\boldsymbol{a} = a_1 + \boldsymbol{j} a_2 \in \mathbb{C} \otimes \mathbb{C}, a_1 = a_1^{'} + \mathrm{i} a_1^{''} \in \mathbb{C}, a_2 = a_2^{'} + \mathrm{i} a_2^{''} \in \mathbb{C}, \tag{13}$$

where $\mathrm{i}^2 = \boldsymbol{j}^2 = -1$, which belong to the family of *multicomplex numbers* [38]. *Hypercomplex Commutative Algebras* (HCA) [11, 39] (with CLIFFORD-like generation scheme) and *Tessarines* [14, 40, 41] are isomorphic to (13). The latter are based on the 2-D *hyperbolic numbers* [3, 4] $\mathbb{D}$ (*double numbers* [22], *split-complex numbers*). They can be defined like complex numbers

$$\boldsymbol{a} = a^{'} + \boldsymbol{j} a^{''} \in \mathbb{D}, \quad a^{'}, a^{''} \in \mathbb{R}, \quad \boldsymbol{j}^2 = +1, \quad \boldsymbol{j} \notin \mathbb{R}, \tag{14}$$

but with a different definition of the imaginary unit $\boldsymbol{j}$ [40, 4]. As a consequence, zero divisors exist already for 2-D numbers (14):

$$(1 - \boldsymbol{j})(1 + \boldsymbol{j}) = 1 - \boldsymbol{j}^2 = 1 - 1 = 0. \tag{15}$$

Hyperbolic numbers can be extended to any power-of-two n-D hyperbolic numbers [5]: e.g. $\mathbb{D} \otimes \mathbb{D}$ for $n = 4$. Moreover, these commutative algebras have the benefit that they retain their properties throughout all dimensions n.

The existence of zero divisors may cause difficulties (cf. sec. 3.4.2), but it also confers a major advantage: The potential of *orthogonal decomposition*, allowing for the representation of any number in terms of its *orthogonal components*, e.g.

$$\tilde{a}_1 = \frac{1}{2}\left(a^{'} + a^{''}\right) \in \mathbb{R}, \quad \tilde{a}_2 = \frac{1}{2}\left(a^{'} - a^{''}\right) \in \mathbb{R} \tag{16}$$

in case of 2-D hyperbolic numbers (14). Any operation can be carried out componentwise on the orthogonal components (16) and, hence, the computational load of multiplication/convolution is dramatically reduced. (Regular 4-D hypercomplex multiplication requires $4^2 = 16$ real multiplications, whereas multiplication employing orthogonal decomposition requires only 4 real multiplications!) The operation saving effect increases with algebra dimension n. In contrast to n-D hyperbolic algebras referring to (16), commutative algebras containing complex subalgebras can only be decomposed to $n/2$ orthogonal components [14]. Mathematically, the orthogonal decomposition is based on the fact that any commutative and associative algebra over $\mathbb{R}$ is isomorphic to a *direct sum* of $\mathbb{R}$ [6].

3.4 Problems

3.4.1 *Non-commutativity and non-associativity*

Many hypercomplex algebras lack either or both of commutativity and associativity. This is fundamental to the algebras and it follows from their geometric meaning distinguishing them from the commutative algebra classes. Resulting from this meaning, it follows that *the ordering of cascaded sections becomes crucial* and the convolution theorem, linking the time and spectral domains, does not hold in the common form [11, 41]. Apart from that, non-commutativity does not pose fundamental problems in HCDSP. In the case of quaternions, as an example, re-ordering of products is expressed by the generalized conjugate rule

$$\overline{\boldsymbol{pqr}} = \overline{\boldsymbol{r}}\,\overline{\boldsymbol{q}}\,\overline{\boldsymbol{p}}, \quad \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \in \mathbb{H}, \tag{17}$$

where the overbar represents a quaternion conjugate [22]. If we have two terms in an algebraic expression which we wish to re-order, we must apply the rule: $\boldsymbol{pq} = \overline{\overline{\boldsymbol{q}}\,\overline{\boldsymbol{p}}}$. This makes algebraic derivations more difficult. A good example of such a derivation is given in [42], where it is proved that the QFT is uniquely invertible.

Associativity is a more complex problem. An associative binary operation $\circ$ is one for which $(a \circ b) \circ c = a \circ (b \circ c)$ for all

a, b and c. I.e. the order in which the operations are carried out is significant (not the ordering of the operands). The quaternions, biquaternions and CLIFFORD algebras are associative, so the problem does not arise there, but many other hypercomplex algebras of dimension $n > 4$ lack associativity. This was first discovered when HAMILTON's ideas were extended to the 8-D octonions, where multiplication is not associative. Nevertheless, a digital system based on an (associative or non-associative) hypercomplex algebra always remains linear and time-invariant (LTI), if it is realised exclusively with (ideal) adders, constant multipliers and delays.

3.4.2 Existence of zero divisors

Although the existence of zero divisors allows for orthogonal decomposition (highly reducing computational burden), it also implies that the Euclidean norm $|\boldsymbol{a}| = \sqrt{\sum_{\nu=1}^{n} a_\nu^2}$ is not multiplicative: $|\boldsymbol{a}| \cdot |\boldsymbol{b}| \neq |\boldsymbol{ab}|$. This results from the underlying non-Euclidean geometric meaning (sec. 3.3.2), and severely complicates the definition of signal energy and the definition of lossless systems [20].

3.4.3 Non-availability of hypercomplex transforms

While useful hypercomplex spectral transforms are known for signals depending on two variables [9], e. g. spatial variables in case of image processing, expedient intrinsically hypercomplex spectral transforms are claimed to be not available for signals depending on only one variable, e. g. on time in case of communication or audio signals. In [11, 32], the argument is based on the number of symmetries inherent in the signal, forming a hierarchy of spectral transforms. In fact, the QTFM toolbox [43] includes computation of one-dimensional quaternion FOURIER transforms, but so far the argument of [11, 32] has not been examined or refuted in print.

3.4.4 Non-availability of hypercomplex analytic signals

In the theory of complex analysis there is a four-way equivalence between a complex function being analytic, regular, continuously differentiable and conjugate harmonic. This four-way equivalence is exploited in the simplification of many theorems and the analysis of mathematical models used in signal processing. For example, the HT used to construct an analytic signal (3) is the boundary value of a conjugate harmonic function along a closed contour about the upper half plane of $\mathbb{R}^2$. Further, causality is the FOURIER dual to the one-sidedness we find in the frequency domain of analytic signals. But in the z-domain causal functions are free of poles outside the unit circle, hence they are 'analytic' there. It is therefore natural to look for the same expressive power in hypercomplex analytic signals. It is also desirable that hypercomplex analyticity be a generalization of complex analytic functions in such a manner that it includes the standard complex definitions.

Quaternions are an obvious starting point since they are the next non-trivial division algebra. HAMILTON, TAIT and JOLY developed the theory of functions of a quaternion variable, but they did not study the class of analytic quaternion-valued functions. In 1932 FUETER [44] published a definition of 'analytic' for quaternionic functions (see also [45, 46]). Later he introduced a definition of 'regular' quaternionic functions [47] modeled on an analog of the Cauchy-Riemann equations, which is used to define conjugate harmonic functions. Both of these definitions contain the theories of analytic and regular functions of a complex variable as special cases, however, not completely equivalently. Furthermore, FUETER's definition for regular quaternionic functions is restricted to linear polynomials. FUETER then concentrated on the study of regular quaternion functions and obtained results in terms of Cauchy's integral formula, which can be used in the definition of the HT. Later, NONO [48] developed the theory of hyperholomorphic functions, relating them to FUETER's regularity conditions; some new results were supplemented in [49]. Finally, Kocherlakota [50] addressed the question as to why quaternion analytic functions are not in general regular; there is no suitable quaternion exponential function which equals its own gradient.

One approach to overcome these issues is to study quaternion-valued functions of one or more reduced quaternions. Ell [9] studied quaternion-valued functions of 'ortho-complex' variables. This allowed for the definition of bi-analytic and bi-regular functions and functional compositions, which retained these properties. Likewise, quaternion-valued functions of a single pure quaternion variable $ix + jy + kz$, $x, y, z \in \mathbb{R}$, were investigated in [51].

An alternate approach is to limit the regions for the definition of the analytic extension to local subfields within the quaternion field. De LEO [52] generalized analyticity by defining a 'local' derivative operator that depends upon the four-dimensional point at which the derivative is to be made. This approach has the added benefit of being directly generalizable to octonionic functions of octonionic variables.

4. CONCLUSION

We have presented an overview of the main ideas of hypercomplex digital signal processing as an extension of the ideas of complex signal processing. The many possible hypercomplex algebras have been shown to have common features and in some cases common drawbacks (lack of commutativity, associativity, existence of divisors of zero) which, however, can even be used to advantage in certain cases. Nevertheless, the use of hypercomplex algebras in signal processing has developed great promise over the last ten years, and we believe that the actual difficulties of using hypercomplex algebras are not insurmountable. The next 5–10 years will show, whether this optimism is justified.

Future research in HCDSP will urge us to study the entire knowledge offered by the mathematics and computer science community more thoroughly, and to investigate in how far this knowledge is applicable to the classical signal processing tasks predominantly considered in this survey paper. This may lead us via the work of FELSBERG [10], BÜLOW [12, 32] and SOMMER [11] and beyond to the monogenic signal as a kind of generalised analytic signal, and to the RIESZ transform as a generalised HILBERT transform.

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