DISCRETE TIME OBSERVATION OF ALMOST PERIODICALLY CORRELATED PROCESSES AND JITTER PHENOMENA

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ABSTRACT

The covariance kernel of an almost periodically correlated process $\{X(t) : t \in \mathbb{R}\}$, also called almost cyclostationary process, admits a Fourier-Bohr decomposition: $\mathrm{cov}[X(t), X(t+\tau)] \sim \sum_\lambda a(\lambda,\tau)\,\mathrm{e}^{i\lambda t}$.

This paper deals with the estimation of the spectral covariance $a(\lambda,\tau)$ from a discrete time observation of the process $\{X(t) : t \in \mathbb{R}\}$, whenever jitter and delay phenomena are present in conjunction with periodic sampling.

1. INTRODUCTION

Almost periodically correlated processes (also called almost cyclostationary processes) belong to the class of second order processes with periodicity (more precisely almost periodicity) properties for the covariance kernel. These processes have been subject to an intensive research for their applications among others in signal analysis (see [9, 5, 7]).

A spectral theory that is understandable and manageable, has been developed for this class of processes although they are not necessarily stationary : the shifted covariance kernel $(t,\tau) \mapsto \mathrm{cov}[X(t), X(t+\tau)]$ of such a process $\{X(t) : t \in \mathbb{R}\}$ admits a Fourier-Bohr decomposition [2]:

$$B(s,\tau) \sim \sum_{\lambda\in\Lambda} a(\lambda,\tau)\,\mathrm{e}^{i\lambda t}.$$

In the literature, the estimation of the spectral covariance $a(\lambda,\tau)$ has been studied whenever the process is observed along a continuous time interval $[0,T]$, as T tends to infinity [8]. For practical applications, it is more interesting to consider discrete time sampling. In this case, two kinds of specific issues arise. The well known spectrum aliasing (or folding) phenomenon appears with periodic discrete time sampling. Furthermore, the presence of perturbations in the timing which can be due to imperfections in the sampling mechanism introduce random timing jitter (see [1, 11, 12]) and delay (see [10]). This problem is less commonly studied.

For example, in signal processing the jitter and delay phenomena appear in passing from the continuous time to the discrete time by using real samplers, that is the analog-to-digital convertors. For more details on this subject the reader is referred to [11, 12, 10].

This work concerns with understanding the effect of the jitter and delay phenomena for APC processes. Futhermore we study the asymptotic behaviour of the empirical estimator $\widetilde{a}_n(\lambda,\tau)$ of the spectral covariance $a(\lambda,\tau)$ constructed in § 5.1 from a discrete time sampling of the process $X = \{X(t) : t \in \mathbb{R}\}$ whenever the sampling times $(t_k)_k$ have additive random perturbations due to delays, to noises or to indefinite locations. We state the asymptotic behavior of the estimator $\widetilde{a}_n(\lambda,\tau)$ whenever the sampling period nh tends to infinity, and for solving the spectrum aliasing problem we assume at the same time that the sampling step h goes towards 0.

For strictly stationary process and constant sampling step $h > 0$, Akaike [1, p.153-154] has pointed out that the effect of the timing perturbations on the power spectral distribution function of the time sampled data can be described as a *filter with an inner white noise source* (see also [11]). A similar phenomenon appears for almost periodically correlated process, the timing perturbations cause the limit of the estimator $\widetilde{a}_n(\lambda,\tau)$ to be $\widetilde{a}(\lambda,\tau)$ the spectral covariance of the process smoothed by the law of these perturbations. Nevertheless whenever the timing perturbations tends to 0 with adequate rate, the limit is the very spectral covariance $a(\lambda,\tau)$.

The asymptotic study is done under mixing conditions. These conditions allow control of the covariance between the values of the process at widely separated times, that are asymptotically independent in this case. These mixing conditions can be replaced by conditions on the cumulants of the process.

2. UAPC PROCESSES

First, recall some definitions and properties about almost periodically correlated processes. More precisely, from Gladyshev [6]

Definition 1. *A zero-mean real-valued process $X = \{X(s) : s \in \mathbb{S}\}$ is uniformly almost periodically correlated (UAPC) whenever $\mathrm{E}\big[X(s)^2\big] < \infty$, for any $s \in \mathbb{S}$, and the shifted covariance kernel $B(s,\tau) = \mathrm{cov}\big[X(s), X(s+\tau)\big] = \mathrm{E}\big[X(s)X(s+\tau)\big]$ is almost periodic in s uniformly in τ. Here $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$.*

The *spectral covariance* $a(\lambda,\tau)$ is defined by :
i) whenever $\mathbb{S} = \mathbb{Z}$, for $\lambda \in [0, 2\pi) \sim \mathbb{R}/2\pi\mathbb{Z}$, $\tau \in \mathbb{Z}$ and $t \in \mathbb{Z}$,

$$a(\lambda,\tau) \triangleq \lim_{n\to\infty} \frac{1}{n} \sum_{k=t}^{n+t} B(k,\tau)\,\mathrm{e}^{-i\lambda k}$$

ii) whenever $\mathbb{S} = \mathbb{R}$, for $\lambda \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$a(\lambda,\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_t^{T+t} B(s,\tau)\,\mathrm{e}^{-i\lambda s}\,\mathrm{d}s.$$

The frequency set of the process X, defined by $\Lambda \triangleq \{\lambda : a(\lambda,\tau) \neq 0 \text{ for some } \tau\}$, is at most countable.

Examples

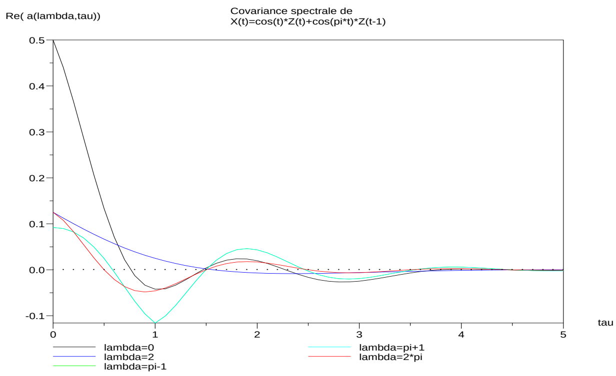
1) A very simple example of such a non-stationary process is provided by the so-called *amplitude modulated signal* $X(t) = X_1(t)\cos t + X_2(t)\cos(\pi t)$, $t \in \mathbb{R}$, where $\{X_i(t), t \in \mathbb{R}\}$, $i = 1,2$, are two independent zero-mean and stationary processes $\{X_i(t), t \in \mathbb{R}\}$, $i = 1,2$, with uniformly continuous covariance functions $r_i(\cdot)$, $i = 1,2$, respectively. Then the amplitude modulated signal $X(t) \triangleq X_1(t)\cos t + X_2(t)\cos(\pi t)$, $t \in \mathbb{R}$, is an UAPC process. Moreover $\Lambda = \{-2\pi, -2, 0, 2, 2\pi\}$, $a(0,\tau) = \frac{1}{2}r_1(\tau)\cos\tau + \frac{1}{2}r_2(\tau)\cos(\pi\tau)$, $a(2,\tau) = \overline{a(-2,\tau)} = \frac{1}{4}r_1(\tau)\,\mathrm{e}^{i\tau}$, and $a(2\pi,\tau) = \overline{a(-2\pi,\tau)} = \frac{1}{4}r_2(\tau)\,\mathrm{e}^{i\pi\tau}$.

2) Another example is given by the process $X(t) \triangleq Z(t)\cos t$, $t \in \mathbb{R}$, where Z is an Ornstein Uhlenbeck process. Then

$$\begin{aligned} B(s,\tau) &= \mathrm{e}^{-|\tau|}\cos s\cos(s+\tau) \\ &= a(-2,\tau)\,\mathrm{e}^{-i2s} + a(0,\tau) + a(2,\tau)\,\mathrm{e}^{i2s} \end{aligned}$$

with $\Lambda = \{-2,0,2\}$, $a(0,\tau) = \frac{1}{2}\mathrm{e}^{-|\tau|}\cos\tau$, $a(2,\tau) = \overline{a(-2,\tau)} = \frac{1}{4}\mathrm{e}^{-|\tau|}\mathrm{e}^{i\tau}$. This process is UAPC and more precisely it is *periodically correlated* (PC), that is the function $s \mapsto \mathrm{E}\left[X(s)X(s+\tau)\right]$ is periodic for any $\tau \in \mathbb{R}$, see [6, 7].

3) Now let $X(t) \triangleq Z(t)\cos t + Z(t-1)\cos(\pi t)$, $t \in \mathbb{R}$, where Z is an Orstein Uhlenbeck process. Then the process X is UAPC with $\Lambda = \{-2\pi, -1-\pi, 1-\pi, -2, 0, 2, \pi-1, \pi+1, 2\pi\}$. We illustrate the spectral covariances by the following graphic.



Real parts of the spectral covariances
for $X(t) = Z(t)\cos t + Z(t-1)\cos(\pi t)$.

3. DISTURBED SAMPLING TIMES

For fixed $h > 0$, we assume that the periodic sampling scheme kh, $k \in \mathbb{Z}$ should be disturbed, Thus the continuous time process $X = \{X(t) : t \in \mathbb{R}\}$ is observed at discrete times $t_k = kh + U_k$, $k \in \mathbb{Z}$, where $\{U_k : k \in \mathbb{Z}\}$ is a family of independent random variables with the same law ν on $\mathbb{R}$ and which are independent on X. The random variables U_k, $k \in \mathbb{Z}$, can be considered as random errors, jitter or delay.

Furthermore, the shift time $\tau \in \mathbb{R}$ is not always a multiple of h, we approximate τ by the nearest multiple of h. This introduces the deterministic deviation $\tau - k_\tau h$, where $k_\tau = k_\tau(h)$ is the nearest integer to the ratio τ/h, for instance $\frac{\tau}{h} - \frac{1}{2} \le k_\tau < \frac{\tau}{h} + \frac{1}{2}$.

Then $U_{k,\tau} \triangleq U_{k+k_\tau} + k_\tau h - \tau$ is the timing perturbation for the deterministic time $kh+\tau$. For fixed τ, the random variables $U_{k,\tau}$, $k \in \mathbb{Z}$, has the same law ν_τ : $\nu_\tau(A) = \nu(A - k_\tau h + \tau)$ for any Borelian set A. Denote by μ_τ the common law of the random vectors $(U_k, U_{k,\tau})$, $k \in \mathbb{Z}$. If $k_\tau = 0$, then the law μ_τ is concentrated on the line $u_2 = u_1 - \tau$, and $\mu_\tau(A_1 \times A_2) = \nu\left(A_1 \cap (A_2+\tau)\right)$ for all Borelian sets A_1 and A_2. Otherwise, if $k_\tau \neq 0$ then $\mu_\tau = \nu \otimes \nu_\tau$ since the random variables $U_{k,\tau}$ and U_k are independent for any k.

Moreover for any $\tau \in \mathbb{R}$, the law ν_τ weakly converges to ν as h tends to 0. On the other hand, whenever $\tau \neq 0$ the law μ_τ weakly converges to $\nu \otimes \nu$. The law μ_0 does not depend on h, $\mu_\tau(A \times A^*) = \nu(A \cap A^*)$.

Effect of the timing perturbations

Since the timing perturbations are independent on the zero-mean UAPC process $X = \{X(t) : t \in \mathbb{R}\}$, the observed process $\{X(kh+U_k) : k \in \mathbb{Z}\}$ is UAPC, and

$$\begin{aligned} &\widetilde{a}_h\left(\lambda h, \frac{\tau}{h}\right) \\ &\quad \triangleq \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{E}\left[X\left(kh+U_k\right)X\left(kh+\tau+U_{k,\tau}\right)\right]\times\mathrm{e}^{-i\lambda kh} \\ &\quad = \begin{cases} \displaystyle\iint_{\mathbb{R}^2} a_{h,u}\left(\lambda h, k_\tau + \frac{u_2-u_1}{h}\right)\nu(\mathrm{d}u_1)\nu(\mathrm{d}u_2) \\ \qquad\qquad\qquad\qquad \text{if} \quad k_\tau \neq 0 \\ \displaystyle\int_{\mathbb{R}} a_{h,u}(\lambda h, 0)\,\nu(\mathrm{d}u) \qquad \text{if} \quad k_\tau = 0, \end{cases} \end{aligned}$$

for $h > 0$ fixed. Here

$$a_{h,u}\left(\lambda, \frac{\tau}{h}\right) \triangleq \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{E}\left[X(kh+u)X(kh+\tau+u)\right]\times\mathrm{e}^{-i\lambda kh}.$$

Whenever, for any $\lambda \in \Lambda$, the function $\tau \mapsto a(\lambda,\tau)$ is Lebesgue integrable on $\mathbb{R}$, the spectral density function $f(\lambda,\cdot)$ exists and fulfills

$$a(\lambda,\tau) = \int_{\mathbb{R}} f(\lambda,\alpha)\,\mathrm{e}^{i\alpha\tau}\,\mathrm{d}\alpha \qquad \tau \in \mathbb{R}.$$

Notice that for $\lambda \notin \Lambda$ we have $a(\lambda,\cdot) \equiv 0$ and $f(\lambda,\cdot) \equiv 0$. If in addition the function $\alpha \mapsto \sum_{\lambda\in\Lambda}\left|f(\lambda,\alpha)\right|$ is integrable on $\mathbb{R}$, that is, if the process X is strongly harmonizable [3], then we can readily state that

$$a_h\left(\lambda h, \frac{\tau}{h}\right) = \int_{\mathbb{R}}\sum_{j\in\mathbb{Z}} f\left(\lambda + \frac{2j\pi}{h}, \alpha\right)\mathrm{e}^{i\alpha\tau}\,\mathrm{d}\alpha$$

$$\begin{aligned} \widetilde{a}_h\left(\lambda h, \frac{\tau}{h}\right) &= \int_{\mathbb{R}}\sum_{j\in\mathbb{Z}} f\left(\lambda + \frac{2j\pi}{h}, \alpha\right)\varphi\left(\lambda - \alpha + \frac{2j\pi}{h}\right) \\ &\quad \times\varphi(\alpha)\,\mathrm{e}^{i\alpha k_\tau h}\,\mathrm{d}\alpha \qquad \text{if} \quad k_\tau \neq 0 \end{aligned}$$

and

$$\begin{aligned} \widetilde{a}_h\left(\lambda h, \frac{\tau}{h}\right) &= \left(\int_{\mathbb{R}}\sum_{j\in\mathbb{Z}} f\left(\lambda + \frac{2j\pi}{h}, \alpha\right)\mathrm{d}\alpha\right)\varphi\left(\lambda + \frac{2j\pi}{h}\right) \\ &\quad \text{if} \quad k_\tau = 0, \end{aligned}$$

where $\varphi(\cdot)$ is the common characteristic function of the random vectors U_k, $k \in \mathbb{Z}$, $\varphi(\alpha) = \mathrm{E}\left[\mathrm{e}^{i\alpha U_k}\right]$.

Whenever the process X is stationary then all the spectral density functions $f(\lambda,\cdot)$, $\lambda \in \mathbb{R}$ except for $\lambda = 0$, are identically null [3], and we find Akaike's results [1, p.153] as a particular case of the previous equalities.

This effect is quite different to the one due to noise for the values of the observed process. For instance assume that we observe $Y_k = X(kh) + Z_k$, $k \in \mathbb{Z}$, where the noise $\{Z_k : k \in \mathbb{Z}\}$ is a zero-mean stationary process with covariance $r_k = \mathrm{E}\left[Z_0 Z_k\right]$, $k \in \mathbb{Z}$, and which is independent of the process $\left\{X(kh) : k \in \mathbb{Z}\right\}$. Then the process $\{Y_k : k \in \mathbb{Z}\}$ is UAPC with spectral covariance

$$a_Y(\lambda,k) = \begin{cases} a_h(\lambda,k) & \text{if} \quad \lambda \neq 0 \\ a_h(0,k) + r_k & \text{if} \quad \lambda = 0. \end{cases}$$

This last relation has been applied in signal theory for the detection in a noisy environment of an UAPC signal with some known spectral information, see [5].

4. HYPOTHESES

4.1 Hypotheses on the timing perturbations

From now on, for any integer n we assume that we observe the continuous time process X at discrete times $kh_n + U_{n,k}$, $\quad 0 \le k \le n$, where for simplicity $h_n = n^{-\delta}$ with $0 < \delta < 1$. The family $\left\{U_{n,k} : 0 \le k \le n\right\}$ is formed by independent random variables with the same law on $\mathbb{R}$ denoted by ν_n, and which are independent on the process X.

Notice that $|k_{n,\tau} h_n - \tau| \le h_n/2$. On the other hand, since $h_n \to 0$ as $n \to \infty$, for any $\tau \neq 0$ there exists $n_\tau \in \mathbb{N}$ such that for any $n \ge n_\tau$ we have $k_{n,\tau} \neq 0$.

In this work we need the forthcoming conditions.

(PB) . $\mathrm{supp}(\nu_n) \subset K$ where K is some compact subset of $\mathbb{R}$

(PW$_0$) . The sequence of laws $(\nu_n)_n$ weakly converges to a probability measure ν on $\mathbb{R}$, that is, for any $f \in \mathscr{C}(K)$,

$$\lim_{n\to\infty} \int_K f(u)\, \nu_n(\mathrm{d}u) = \int_K f(u)\, \nu(\mathrm{d}u),$$

where $\mathscr{C}(K)$ denotes the space of the continuous functions $f : K \to \mathbb{R}$.

(PW$_\eta$) . The condition (PW$_0$) is satisfied and for $\eta > 0$

$$\lim_{n\to\infty} n^\eta \iint_{\mathbb{R}^2} a(\lambda, \tau + u_2 - u_1)\, \mathrm{e}^{i\lambda u_1} \times \Big(\mu_{n,\tau}(\mathrm{d}u_1, \mathrm{d}u_2) - \mu_\tau(\mathrm{d}u_1, \mathrm{d}u_2)\Big) = 0,$$

where $\mu_\tau \stackrel{\Delta}{=} \nu \otimes \nu$ for $\tau \neq 0$, and μ_0 is defined by $\mu_0(A \times A^*) \stackrel{\Delta}{=} \nu(A \cap A^*)$ for all Borelian subsets A and A^* of $\mathbb{R}$.

Remarks

1) Note that the sampling step h_n tends to 0 and the sampling period nh_n tends to infinity as n goes to infinity.

2) Whenever the sequence of laws $(\nu_n)_{n\in\mathbb{N}}$ converges in total variation to ν, then the condition (PW$_0$) is fulfilled. However a sequence of zero-mean gaussian laws whose variances converge to 0, weakly converges to Dirac measure at 0, but does not converge in total variation.

3) In the conditions (PW$_0$), (PB) and (PW$_\eta$), the case when $\nu = \delta_{\{0\}}$ is the Dirac measure at 0, corresponds to the convergence of the timing perturbations towards 0. However, it is not the only case of interest, some jitter or delay may not diminish asymptotically, for instance see [10, 12].

4) Assume that $\nu = \delta_{\{0\}}$ and $\mathrm{supp}(\nu_n) \subset [-\beta_n, \beta_n]$, with $\lim_{n\to\infty} \beta_n = 0$. Then (PW$_0$) is satisfied. In addition, under the condition (H$_2$) (see below § 4.2) we have for sufficiently large n,

$$\left| \iint_{\mathbb{R}^2} a(\lambda, \tau + u_2 - u_1)\, \mu_{n,\tau}(\mathrm{d}u_1, \mathrm{d}u_2) - a(\lambda,\tau) \right| \le c\,(\beta_n + h_n)^{\kappa_2}.$$

Thus whenever $\lim_{n\to\infty} n^\eta \beta_n^{\kappa_2} = \lim_{n\to\infty} n^\eta h_n^{\kappa_2} = 0$, the condition (PW$_\eta$) is fulfilled. In particular, this is the case whenever $\beta_n = h_n = n^{-\delta}$ with $\delta\kappa_2 > \eta$.

5) Instead of the condition (PW$_\eta$) with the spectral covariance $a(\lambda,\tau)$ we could consider the following more general condition

(PW$^*_\eta$) . For any $f \in \mathscr{C}_b(\mathbb{R}^2)$

$$\lim_{n\to\infty} n^\eta \iint_{\mathbb{R}^2} f(u_1,u_2) \Big(\mu_{n,\tau}(\mathrm{d}u_1, \mathrm{d}u_2) - \mu_\tau(\mathrm{d}u_1, \mathrm{d}u_2)\Big) = 0.$$

However this condition is much more stringent than (PW$_\eta$). In fact, for the example exposed in the remark 6, we can state that the condition (PW$^*_\eta$) is satisfied for any function which satisfies the regularity condition (H$_2$) with exponent κ_2 such that $\lim_{n\to\infty} n^\eta \beta_n^{\kappa_2} = \lim_{n\to\infty} n^\eta h_n^{\kappa_2} = 0$. But there is no reason that the condition (PW$^*_\eta$) is satisfied by any function f in $\mathscr{C}_b(\mathbb{R}^2)$.

4.2 Hypotheses on the process

All along this work we also use the following conditions on the zero-mean process $X = \{X(t) : t \in \mathbb{R}\}$.

(LB) . Almost every trajectory of the process X is locally bounded.

(PP$_2$) . The function $s \mapsto \mathrm{E}\left[X(s)X(s+\tau)\right]$ is almost periodic uniformly with respect to τ. That is the process X is UAPC.

(PP$_4$) . The function $s \mapsto \mathrm{E}\big[X(s)X(s+\tau_1)X(s+\tau_2)X(s+\tau_3)\big]$ is almost periodic uniformly with respect to (τ_1,τ_2,τ_3).

(S). $\sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-2} < \infty$, where $\Lambda \stackrel{\Delta}{=} \big\{\lambda \in \mathbb{R}, a(\lambda,\tau) \neq 0 \quad \text{for some } \tau \in \mathbb{R}\big\}$.

(H$_2$) . There are $\kappa_2, \eta, c > 0$ such that for $|s_i - t_i| \le \eta$, $i = 1,2$

$$\Big|\mathrm{E}\left[X(s_1)X(s_2)\right] - \mathrm{E}\left[X(t_1)X(t_2)\right]\Big| \le c\,\Big(|s_1 - t_1| + |s_2 - t_2|\Big)^{\kappa_2}.$$

(H$_4$) . There are $\kappa_4, \eta, c > 0$ such that for $\|\mathbf{s} - \mathbf{t}\| \le \eta$

$$\Big|\mathrm{E}\left[X(s_1)X(s_2)X(s_3)X(s_4)\right] - \mathrm{E}\left[X(t_1)X(t_2)X(t_3)X(t_4)\right]\Big| \le c\,\Big(\|\mathbf{s} - \mathbf{t}\|\Big)^{\kappa_4},$$

where $\mathbf{s} = (s_1, s_2, s_3, s_4)$, $\mathbf{t} = (t_1, t_2, t_3, t_4)$, and $\|\mathbf{s}\| = |s_1| + |s_2| + |s_3| + |s_4|$.

The process X is said to be *strongly mixing* whenever $\lim_{t\to\infty} \alpha(t) = 0$ where the mixing coefficient is defined by

$$\alpha(t) \triangleq \sup\big\{|\mathrm{P}[A\cap B] - \mathrm{P}[A]\,\mathrm{P}[B]|\big\}$$

where the supremum is taken over all $s > 0$ and all the $A \in \sigma^{s}_{-\infty}(X)$ and $B \in \sigma^{\infty}_{s+t}(X)$, and $\sigma^{s_2}_{s_1}(X)$ stands for the $\sigma-$fields generated by $\{X(s) : s_1 \le s \le s_2\}$.

Remarks

1) Condition (LB) is technical. The almost periodicity conditions (PP_2) and (PP_4), and the Hölderian regularity ones (H_2) and (H_4) are satisfied by the processes defined in the examples 1, 2 and 3. The regularities conditions (H_2) and (H_4) are used for the study of the rate of convergence of the estimators.

2) The condition (S) is satisfied by any periodically correlated process and also by the UAPC processes defined in the examples 1, 2 and 3. This condition is well known in the theory of almost periodically correlated processes [5]. This condition is an identifiability condition. In fact it implies that the frequency set Λ of the process has no cluster point.

3) The mixing coefficient function is an indicator of the asymptotic independence of the process, i.e. of the asymptotic independence between $\big\{X(u) : u \le s\big\}$ and $\big\{X(u) : u \ge s+t\big\}$ as $t \to \infty$ (see [4] and references therein).

In order to justify the construction of our estimators and some Lebesgue integrals which will appear in the paper, from now on, we consider only a measurable version of the process X. Such a version always exists for any mean square continuous process, and in particular for any UAPC process.

5. MAIN RESULTS

5.1 Estimators

For continuous time observations of an UAPC process X, Hurd and Leśkow [8] establish the consistency and the asymptotic normality of the following continuous time estimator of $a(\lambda,\tau)$

$$\widehat{a}_T(\lambda,\tau) \triangleq \frac{1}{T}\int_0^T X(t)X(t+\tau)\,\mathrm{e}^{-i\lambda t}\,\mathrm{d}t.$$

From now on, we assume that we have observed a sampling $\big\{X(kh_n + U_{n,k}) : 0 \le k \le n\big\}$ and we study the following discrete time version of $\widehat{a}_T(\lambda,\tau)$

$$\widetilde{a}_n(\lambda,\tau) \triangleq \frac{1}{n}\sum_{k=1}^{n} \widetilde{b}_{n,k}(\lambda,\tau)$$

where $\widetilde{b}_{n,k}(\lambda,\tau) \triangleq$

$$X\big(kh_n + U_{n,k}\big)X\big((k+k_{n,\tau})h_n + U_{n,k+k_{n,\tau}}\big)\,\mathrm{e}^{-i\lambda k h_n},$$

whenever $0 \le k \le n$ and $0 \le k + k_{n,\tau} \le n$, and $\widetilde{b}_{n,k}(\lambda,\tau) \triangleq 0$, otherwise.

5.2 Smoothed spectral covariance

Now we can define the *smoothed spectral covariance of the process X* by:

$$\begin{aligned}
&\widetilde{a}(\lambda,\tau)\\
&\triangleq \lim_{n\to\infty}\iint_{\mathbb{R}^2} a_{h_n,u}\Big(\lambda h_n, \frac{\tau+u_2-u_1}{h_n}\Big)\mu_{n,\tau}(\mathrm{d}u_1,\mathrm{d}u_2)\\
&= \begin{cases} \iint_{\mathbb{R}^2} a\big(\lambda, \tau+u_2-u_1\big)\,\mathrm{e}^{i\lambda u_1}\,\nu(\mathrm{d}u_1)\nu(\mathrm{d}u_2) & \text{if} \quad \tau \neq 0\\ a(\lambda,0)\varphi_\nu(\lambda) & \text{if} \quad \tau = 0.\end{cases}
\end{aligned}$$

Notice that if $\nu = \delta_{\{0\}}$, that is, there is no timing perturbation, then $\widetilde{a}(\lambda,\tau) = a(\lambda,\tau)$.

5.3 Consistency

We can establish that $\widetilde{a}_n(\lambda,\tau)$ is an asymptotically unbiased estimator of the smoothed spectral covariance $\widetilde{a}(\lambda,\tau)$. This is a straightfoward consequence of the asymptotic unbiasedness of the continous time estimator $\widehat{a}_T(\lambda,\tau)$, see [8]. Next under mixing conditions we control the behaviour of the variance of the estimator $\widetilde{a}_n(\lambda,\tau)$ as $n \to \infty$ and we deduce its convergence. Hence we obtain the mean square convergence of the estimator $\widetilde{a}_n(\lambda,\tau)$

Theorem 2. *Assume that the process X and the timing perturbations satisfy the following conditions*

1. (LB) and (PP_2);

2. there exists $\eta > 0$ such that $\sup_t \mathrm{E}\big[|X(t)|^{4+\eta}\big] < \infty$ and $\alpha(t) = o(1)$ as $t \to \infty$;

3. (PB) and (PW_0).

Then for any λ and any τ we have the following convergence in quadratic mean (q.m.)

$$\lim_{n\to\infty} \widetilde{a}_n(\lambda,\tau) = \widetilde{a}(\lambda,\tau) \qquad q.m. \quad .$$

5.4 Asymptotic normality

The goal of this section is the statement of the asymptotic normality of $\sqrt{nh_n}\big(\widetilde{a}_n(\lambda,\tau) - \widetilde{a}(\lambda,\tau)\big)$.

In the general case, the spectral covariance $a(\lambda,\tau)$ is a complex number, and its estimator $\widetilde{a}_n(\lambda,\tau)$ is a complex-valued random variable. As usual in such a situation, for the statement of the asymptotic normality of the estimator we consider the vectorial versions of $a(\lambda,\tau)$ and $\widetilde{a}(\lambda,\tau)$. For simplicity we will keep the same notations for the complex numbers and their vectorial forms. Since the process X is real-valued, the vectorial form of the spectral covariance is written as

$$a(\lambda,\tau) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{E}\big[X(s)X(s+\tau)\big]S(\lambda k h_n)\mathrm{d}s,$$

and

$$\widetilde{a}_n(\lambda,\tau) \triangleq \frac{1}{n}\sum_{k=1}^{n} \widetilde{b}_{n,k}(\lambda,\tau)$$

where $\widetilde{b}_{n,k}(\lambda,\tau) \triangleq X(kh_n + U_{n,k})X(kh_n + \tau + U_{n,k,\tau})S(\lambda k h_n)$, for $0 \le k \le n$ and $0 \le k + k_{n,\tau} \le n$, and $\widetilde{b}_{n,k}(\lambda,\tau) = (0,0)^{\dagger}$ otherwise, and where $S(\lambda k h_n)) \triangleq \big(\cos(\lambda k h_n), \sin(\lambda k h_n)\big)^{\dagger}$.

Here $(\cdot,\cdot)^\dagger$ denotes the transpose column vector of the row vector $(\cdot,\cdot)$.

Then from conditions (PP2) and (PP4) we can define

$$B^*(\lambda_1,\lambda_2,\tau_1,\tau_2) \triangleq \lim_{n\to\infty} nh_n \mathrm{Cov}\Big[\widetilde{a}_n(\lambda_1,\tau_1),\widetilde{a}_n(\lambda_2,\tau_2)\Big]$$
$$= \iiiint_{\mathbb{R}^4} S_1(\lambda_1 u_1) B(\lambda_1,\lambda_2,\tau_1-u_1+u_1^*,\tau_2-u_2+u_2^*)$$
$$\times S_1(\lambda_2 u_2)^\dagger \mu_{\tau_1}(\mathrm{d}u_1,\mathrm{d}u_1^*)\mu_{\tau_2}(\mathrm{d}u_2,\mathrm{d}u_2^*)$$

where

$$B(\lambda_1,\lambda_2,\tau_1,\tau_2) \triangleq \frac{1}{2}\int_{\mathbb{R}}\Big(b_c\big(\lambda_1-\lambda_2,\tau_1,t,t+\tau_2\big)\ \mathrm{S}_1(\lambda_2 t)+$$
$$+b_s\big(\lambda_1-\lambda_2,\tau_1,t,t+\tau_2\big)\ \mathrm{S}_2(\lambda_2 t)+b_c\big(\lambda_1+\lambda_2,\tau_1,t,t+\tau_2\big)$$
$$\times\ \mathrm{S}_3(\lambda_2 t)+b_s\big(\lambda_1+\lambda_2,\tau_1,t,t+\tau_2\big)\ \mathrm{S}_4(\lambda_2 t)\Big)\,\mathrm{d}t$$

with

$$b_c\big(\lambda,u,v,w\big) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{cov}\big[X(s)X(s+u),X(s+v)X(s+w)\big]\cos(\lambda s)\,\mathrm{d}s$$

$$b_s\big(\lambda,u,v,w\big) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{cov}\big[X(s)X(s+u),X(s+v)X(s+w)\big]\sin(\lambda s)\,\mathrm{d}s$$

$$\mathrm{S}_1(\theta) \triangleq \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathrm{S}_2(\theta) \triangleq \begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}$$

$$\mathrm{S}_3(\theta) \triangleq \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

$$\mathrm{S}_4(\theta) \triangleq \begin{bmatrix} -\sin(\theta) & \cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}.$$

Then we state the asymptotic normality of the estimator $\widetilde{a}(\lambda,\tau)$.

Theorem 3 (Asymptotic normality)**.**
Assume that the process X and the timing perturbations satisfy the following conditions

1. (H_2), (PP_2), (H_4) and (PP_4);

2. there is $\eta>0$ such that

$$\sup_t \mathrm{E}\big[|X(t)|^{4+\eta}\big]<\infty \qquad \textit{and} \qquad \alpha(\cdot)\in L^{\frac{\eta}{4+\eta}}(\mathbb{R}^+);$$

3. $\alpha(t)=o(t^{-4})$ as $t\to\infty$;

4. (PW_0), (PB).

Then for any λ and any τ we have

$$\sqrt{nh_n}\Big(\widetilde{a}_n(\lambda,\tau)-\mathrm{E}\big[\widetilde{a}_n(\lambda,\tau)\big]\Big)\xrightarrow{\mathscr{L}}\mathscr{N}_2\big(0,B^*(\lambda,\tau)\big),$$

where $B^*(\lambda,\tau)=B\big(\lambda,\lambda,\tau,\tau\big)$.

From Slusky lemma we readily deduce the following corollary.

Corollary 4. *In addition of the hypotheses of Theorem 3, assume that the conditions (S) and ($PW_{(1-\delta)/2}$) are fulfilled and $h_n=n^{-\delta}$, for some δ such that* $\max\big\{\frac{1}{3},\frac{1}{1+2\kappa_2}\big\}<\delta<1$. *Then for any τ and any λ in $\mathbb{R}$ we have*

$$\sqrt{nh_n}\Big(\widetilde{a}_n(\lambda,\tau)-\widetilde{a}(\lambda,\tau)\Big)\xrightarrow{\mathscr{L}}\mathscr{N}_2\big(0,B^*(\lambda,\tau)\big).$$

6. CONCLUSION

In this paper we have studied the effect of jitter and time delay in the discrete sampling of an almost periodically correlated process. Moreover we have defined an estimator of the spectral covariances of the process constructed from such a sampling. Then, we have established the consistency and the asymptotic normality of this estimator.

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