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# TWO-DIMENSIONAL SAMPLING AND REPRESENTATION OF FOLDED SURFACES EMBEDDED IN HIGHER DIMENSIONAL MANIFOLDS

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## ABSTRACT

The general problem of sampling and flattening of folded surfaces for the purpose of their two-dimensional representation and analysis as images is addressed. We present a method and algorithm based on extension of the classical results of Gehring and Väisälä regarding the existence of quasi-conformal and quasi-isometric mappings between Riemannian manifolds. Proper surface sampling, based on maximal curvature is first discussed. We then develop the algorithm for mapping of this surface triangulation into the corresponding flat triangulated representation. The proposed algorithm is basically local and, therefore, suitable for extensively folded surfaces such as encountered in medical imaging. The theory and algorithm guarantee minimal metric, angular and area distortion. Yet, it is relatively simple, robust and computationally efficient, since it does not require computational derivatives. In this paper we present the sampling and flattening only, without complementing them by proper interpolation. We demonstrate the algorithm using medical and synthetic data.

## 1. INTRODUCTION

Two-dimensional representation by flattening of three-dimensional object scans is a fundamental step, required in various medical and other volumetric imaging applications. For example, it is often required to present three-dimensional MRI scans of the brain cortex as flat two-dimensional images. In the latter case it is possible, for example, to better observe and follow up developments of neural activity within the folds. Flattening of three-dimensional scans is in particular important in the case of CT virtual colonoscopy; a non-invasive, rapidly advancing, imaging procedure capable of determining the presence of colon pathologies such as small polyps [7]. However, because of the extensive folding of the colon, rendering of the 3D data for detection of pathologies requires the implementation of cylindrical or planar map projections [8]. In order to map such data in a meaningful manner, so that diagnosis will be accurate, it is essential that the geometric distortion, in terms of angles and lengths, caused by the representation, will be minimal. Due to these medical and also other applications such as face recognition, this problem has attracted a great of attention in the last few years.

In a recent study of Haker et al., a method for mapping a 3D-surface onto a flat surface in a conformal manner was presented [7], [8]. This method is basically a variational method. The method is essentially done by solving Dirichlet problem for the Laplace-Beltrami operator $\triangle u = 0$ on a given surface $\Sigma$, with boundary conditions on $\partial\Sigma$.

In [5] and a series of consequent papers, Gu et al. suggest to use holomorphic 1-forms in order to compute global conformal structure of a smooth surface of arbitrary genus and arbitrary number of boundary components. As such, this method can be applied to tissue or facial unfolding. Yet, implementation of this method is extremely time consuming.

In [10] Hurdal et al. suggest to build such a conformal map using circle packing. This relies on the ability to approximate conformal structure on surfaces by circle packings. The authors use this method for MRI brain images and conformally map them to the three possible models of geometry in dimension 2 (i.e. the 2-sphere, the Euclidian plane and the hyperbolic plane).

In all the above-mentioned methods the outcome is in fact not a conformal map but a quasi-conformal map. This fact implies that all the geometric measures are kept up to some bounded distortion. Yet, sometimes the resultant distortion may render the data to become unacceptable. This is due to the fact that, in general, a surface is only locally conformal to the plane so, if one tries to flatten a surface in some manner of globality, the best result is obtained by means of a quasi-conformal flattening. Given this fact, it would be desirable to have a precise bound on the distortion caused by the flattening. However, none of the mentioned methods is capable of yielding such a bound.

In this paper we propose an approach to minimal distorting representation of 3D-surfaces. The method adopted in this study is derived from theoretical results obtained by Gehring and Väisälä in the early 1960's [1]. They were studying the existence of quasi-conformal mappings between Riemannian manifolds. A simple algorithm, that can be easily implemented, was derived by the authors out of Gehring and Väisälä results ([3]). The basic advantages of the proposed method, insofar as implementation is concerned, are its robustness and speed. An additional advantage is that it is possible to guarantee that the distortion does not exceed a predefined bound, which can be as small as desired with respect to the amount of localization one is willing to accommodate. The proposed algorithm is most suitable for cases where the surface is complex (high and non-constant curvature) such as colon wrapping. It is important to stress that our algorithm, based on the study of Gehring and Väisälä is the only one that yields the desired bound on distortion.

The paper is organized as follows: In Section 2 we provide some theoretical background to the fundamental work of Gehring and Väisälä and also regarding the sampling and reconstruction problem. In the subsequent section we describe our algorithm for surface flattening, based on their ideas. In Section 4 we present some experimental results obtained by means of this scheme and in Section 5 we summarize the paper and discuss possible future studies.

## 2. THEORETICAL BACKGROUND

**Definition 1** Let $D \subset \mathbb{R}^3$ be a domain. A homeomorphism $f : D \to \mathbb{R}^3$ is called a *quasi-isometry* (or a *bi-Lipschitz mapping*), iff $\frac{1}{C}|p_1 - p_2| \le |f(p_1) - f(p_2)| < C|p_1 - p_2|$ , for all $p_1, p_2 \in D$; $1 \le C < \infty$.
$C(f) = \min\{C \,|\, f \text{ being a quasi}-\text{isometry}\}$ is called the minimal distortion of $f$ (in $D$). (The distances considered are the induced intrinsic distances on the surfaces.)

Any quasi-isometry is a quasi-conformal mapping (see, e.g. [16]), while not every quasi-conformal mapping is a quasi-isometry). The definition of quasi-conformality closely resembles that of quasi-isometry, where distances are replaced by angles. For quasi-conformal mappings, the role of $C(f)$ is played by $K(f)$ – the *maximal dilatation* of $f$.

**Definition 2** Let $S \subset \mathbb{R}^3$ be a connected surface. $S$ is called *admissible* iff for any $p \in S$, there exists a quasi-isometry $i_p$ such that for any $\varepsilon > 0$ there exists a neighbourhood $U_p \subset \mathbb{R}^3$ of $p$, such that $i_p : U_p \to \mathbb{R}^3$ and $i_p(S \cap U_p) = D_p \subset \mathbb{R}^2$, where $D_p$ is a domain and such that $C(i_p)$ satisfies:
$(i)$ $\sup_{p \in S} C(i_p) < \infty$ and $(ii)$ $\sup_{p \in S} C(i_p) < 1 + \varepsilon$ .

Let $S$ be a surface, $\vec{n}$ be a fixed unitary vector, and $p \in S$, such that there exists a neighbourhood $V \subset S$, such that $V \simeq D^2$, where $D^2 = \{x \in \mathbb{R}^2 \,\big|\, ||x|| \le 1\}$. Moreover, suppose that for any $q_1, q_2 \in S$, the acute angle $\measuredangle(q_1q_2, \vec{n}) \ge \alpha$. We refer to the last condition as *the Geometric Condition* or *Gehring Condition*.

Then, for any $x \in V$ exists a unique representation of the form: $x = q_x + u\vec{n}$, where $q_x$ lies on the plane through $p$ which is orthogonal to $\vec{n}$ and $u \in \mathbb{R}$. We define: $Pr(x) = q_x$. (Note that $\vec{n}$ need not be the normal vector to $S$ at $p$.)

We have that for any $p_1, p_2 \in S$, and any $a \in \mathbb{R}_+$, the following inequalities hold: $\frac{a}{A}|p_1 - p_2| \le |Pr(p_1) - Pr(p_2)| \le A|p_1 - p_2|$, where $A = \frac{1}{2}[(a \csc\alpha)^2 + 2a + 1]^2 + \frac{1}{2}[(a\csc\alpha)^2 - 2a + 1]^2$. In particular for $a = 1$ we get that $C(f) \le \cot\alpha + 1$ and $K(f) \le \left(\left(\frac{1}{2}(\cot\alpha)^2 + 4\right)^{\frac{1}{2}} + \frac{1}{2}\cot\alpha\right)^{\frac{3}{2}} \le (\cot\alpha + 1)^{\frac{3}{2}}$.

**Theorem 1 ([3])** The quasi-isometric projection, $Pr$, produces minimal distortion: $C(f) \le \cot\alpha + 1$ and maximal dilatation $K(f) \le (\cot\alpha + 1)^{\frac{3}{2}}$.

From the above discussion we conclude that $S \subset \mathbb{R}^3$ is an admissible surface if for any $p \in S$ there exists $\vec{n}_p$ such that, for any $q_1, q_2 \in U_p$ close enough to $p$, the acute angle $\measuredangle(q_1q_2, \vec{n}_p) \ge \alpha$. In particular, any smooth surface in $S \in \mathbb{R}^3$ is admissible.

Naturally, the existence of faithful quasi-conformal/quasi-isometric representations for sampled surface strongly depends on the quality of the sampling. In this context, we present here the following basic versions of the sampling and reconstruction theorems proven in [2].

**Theorem 2 ([2])** Given a $\mathcal{C}^2$ surface $\Sigma$, with absolute principal curvatures bounded by some bound $K_\Sigma$, there exists a sampling scheme of the surface $\Sigma$, with a proper density $\mathcal{D}$, corresponding to the maximum absolute curvature $K_\Sigma$, i.e. $\mathcal{D} = \mathcal{D}(K_\Sigma)$.

**Theorem 3 ([2])** If $\Sigma$ is not a $\mathcal{C}^2$ surface, then there exists a smoothing reproducing kernel $\mathcal{H}_\Sigma$, for which $\mathcal{H}_\Sigma * \Sigma$ is of class $\mathcal{C}^2$. The smooth surface can be represented by a sampling scheme of density $\mathcal{D}$, according to Theorem 2.

## 3. THE ALGORITHM

In this section we present our algorithm, we developed in order to apply Gehring - Väisälä's theorem ([3]) for quasi-isometrically flattening of a 3D-surface. Extension of this approach to flattening of images and surfaces embedded in higher dimensional manifolds, incorporating image attributes, such as color and texture, are addressed elsewhere ([18]).

Assume that the surface is equipped with some nonuniform set of sampling points and corresponding triangulation $T$. Let $N_p$ stand for the vector normal to the surface at a point $p$ on the surface.

A triangle $\Delta$, of the triangulation must be chosen. We project a patch of the surface quasi-isometrically onto the plane included in $\Delta$. This patch is called the patch of $\Delta$, and it consists of at least one triangle, i.e. $\Delta$ itself. There are two possibilities to chose $\Delta$, one is in a random manner and the other is based on curvature considerations. We will refer to both ways later. For the moment, assume $\Delta$ was somehow chosen. After $\Delta$ is (trivially) projected onto itself we move to its neighbors. Suppose $\Delta'$ is a neighbor of $\Delta$ having edges $e_1$, $e_2$, $e_3$, where $e_1$ is the edge common to both $\Delta$ and $\Delta'$.

We consider $\Delta'$ to be *Gehring compatible w.r.t* $\Delta$, if the maximal angle between $e_2$ or $e_3$ and $N_\Delta$ (the normal vector to $\Delta$), is greater then a predefined measure suited to the desired predefined maximal allowed distortion, i.e. $\max\{\measuredangle(e_2, N_\Delta), \measuredangle(e_3, N_\Delta)\} \ge \alpha$. We project $\Delta'$ *orthogonally* onto the plane included in $\Delta$ and insert it to the patch of $\Delta$, iff it is Gehring compatible w.r.t $\Delta$.

We keep adding triangles to the patch of $\Delta$ moving from an added triangle to its neighbors (of course) while avoiding repetitions, till no triangles can be added.

If by this time all triangles are added to the patch, we have concluded. Otherwise, we choose a new triangle that has not been projected yet, to be the starting triangle of a new patch. A pseudocode for this procedure can be easily written.

We conclude this section with the following remarks: One should keep in mind that the above given algorithm, as for any other flattening method, is local. Indeed, in a sense the (proposed) algorithm gives a measure of "globality" of this intrinsically local process. Our algorithm is most suitable for highly folded surfaces, because of its intrinsic locality on the one hand and computational simplicity, on the other.

Furthermore, as stated earlier, one could may choose the starting triangles in two ways. If one takes into account the curvature at vertices of the triangles then choosing those triangles having minimal curvature at their vertices will reduce the number of patches in the final flat presentation of the surface. Experimental results show that the reduction in the number of patches is of approximately 25 percent w.r.t. the version where triangles are randomly chosen.

Another important comment refers to the robustness of the algorithm proposed herein. The outcome of the algorithm does not depend on the triangulation of the surfaces. It depends only on the surface geometry and in particular its curvature, given that the triangulation well represents the surface, as far as its sampling is concerned. Surface sampling in particular and of high dimensional signals/manifolds, in particular, is a major subject of study, and various results exist on this matter (e.g. [1], [9]). As far as our algorithm is concerned, we assume the nonuniform sampling is faithful to the geometry of the surface, i.e. to surface curvature ([2]). This is analogous to determining the density of nonuniform sampling of one-dimensional signal according to its derivative ([18]).

## 4. EXPERIMENTAL RESULTS

We present some experimental results obtained by applying the algorithm presented in the previous section to flattening of the skull (Fig. 1), tooth (Fig. 2) and a lobe of the brain (Fig. 3). In each of the examples both the input surface and a flattened representation of some patch are shown. Distortion is computed according to the theory presented in Section 2. In addition, details regarding the resolution of the mesh and the number of patches are also available.

Since at this stage we did not address the problem of properly gluing of patches, in the following example of colon flattening, one can see the appearance of holes in the flattened presentation caused by artificially gluing neighboring patches to each other. We refer to this problem in the next section.

## 5. CONCLUDING REMARKS AND FUTURE STUDY

Sampling and flattening of folded surfaces embedded in higher dimensional Riemannian manifolds combines several important facets and problems encountered in image processing and analysis of surfaces. In our broader study [2], we deal with the issues of nonuniform smoothing and sampling by proper reproducing kernels. Here we assumed that a proper sampling and triangulation of the surfaces are given. the emphasis was therefore on quasi-conformal and quasi-isometric aspects of the mapping between Riemannian manifolds. While the theory is general and applicable to mapping from any higher to lower dimensional manifolds, here we presented a specific algorithm developed for the case of mapping from a three-dimensional to two-dimensional flat surface.

From the implementation results it is evident that this algorithm while being simple to program as well as efficient, also gives good flattening results and maintains small dilatations even in areas where curvature is large and good flattening is a challenging task. Moreover, since there is a simple way to ases the resulting dilatation, the algorithm was implemented in such a way that the user can set in advance an upper bound on the resulting dilatation.

An additional advantage of the presented algorithm resides in the fact that in contradiction to some of the related works, no use of derivatives is made. In consequence the algorithm does not suffer from typical drawbacks of derivative computations like robustness, etc.

Moreover, since no derivatives are employed, no smoothness assumption about the surface to be flattened are made, which makes the algorithm presented herein ideal for use in cases where smoothness is questionable (to say the least).

The algorithm may be of extremely practical use for applications where local yet good analysis is required such as medical imaging with the emphasis on flattened representation of the brain and the colon (virtual colonoscopy). Such a study is undertaken these days.

The main question for future study, remains that of passing from local to global in a more precise fashion, i.e. how can one glue two neighbouring patches while keeping fixed bounded dilatation. (In more technical terms, this amounts to actually *computing* the *holonomy map* of the surface – see [15].) Indeed, we may flatten the neighborhood of some vertex $u$ obtaining the flat image $I_u$ and the neighborhood of another vertex $v$ obtaining the image $I_v$ so that these two neighborhood have some intersection along the boundary yet, it will not be possible to adjust the resulting images to give one flat image $I_{u\cup v}$ of the union of these neighborhood which still keeping the quasi-isometric property. Here too, study is underway.

Evidently, as can be noted in Fig. 4, of the colon flattening example, one can have two neighbouring patches, with markedly different dilatations/distorsions,producing different lengths for the common boundary edges. Therefore, "cuts" and "holes" appear when applying a "naive gluing".

We conclude by remarking that while the application presented here is for $2D$-images of $3D$-surfaces, the results of Gehring and Väisälä are stated and proven for any dimension (and co-dimension). Therefore, implementations for higher dimensions are feasible.

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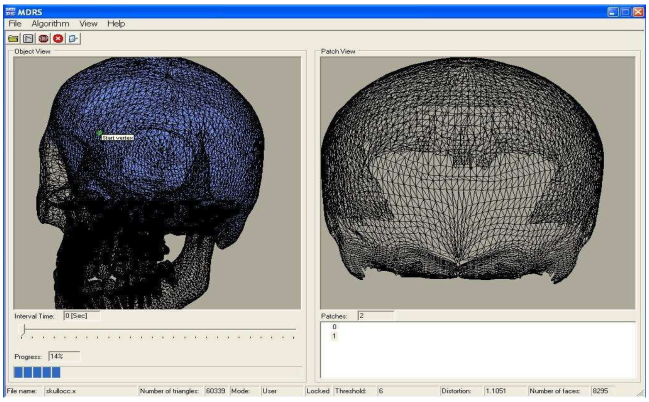


Figure 1: Skull flattening: The role of almost flat regions is accentuated. The resolution is of the whole skull 60,339 triangles. Here $\alpha = 10°$. The dilatation is 1.1763.

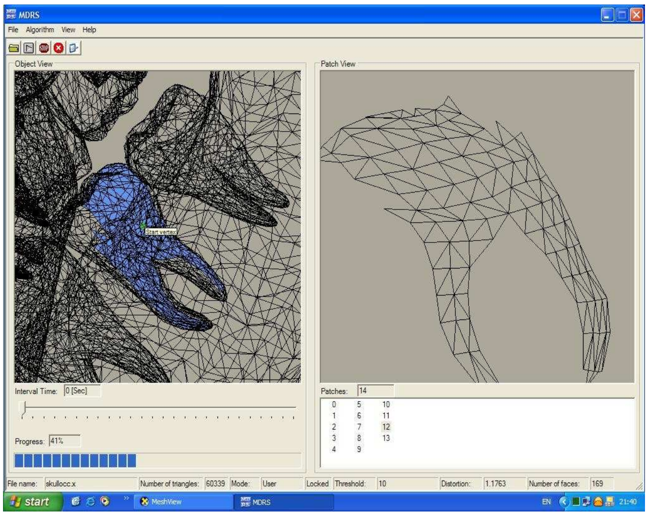


Figure 2: Tooth model: This represents part of skull model. Here again $\alpha = 10°$ and the dilatation is 1.1763.

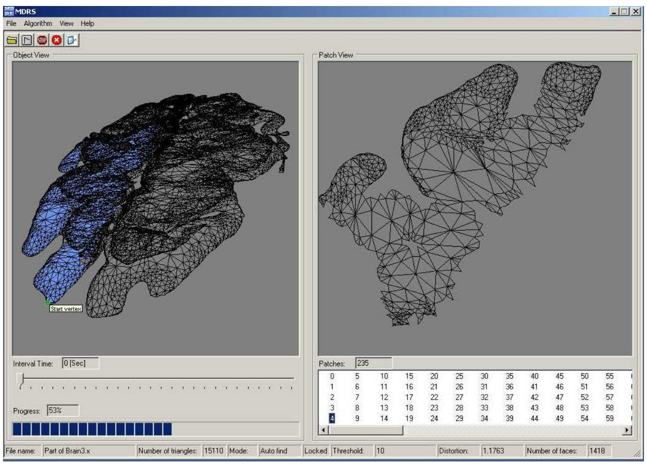
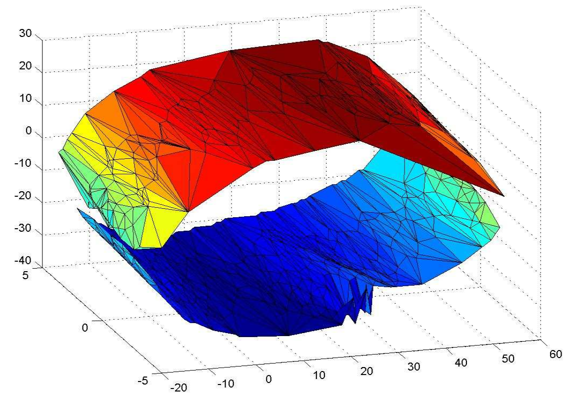
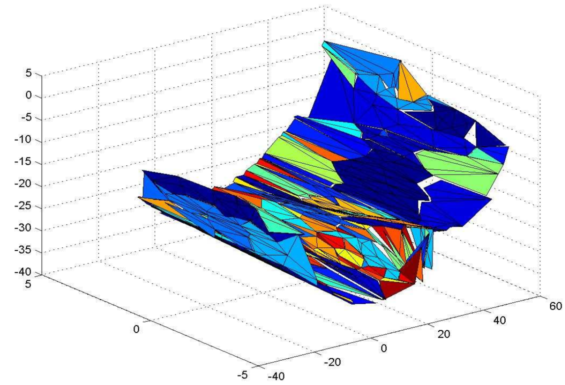


Figure 3: Cerebral Cortex Flattening: patch obtained in the representation of the parietal region. The resolution is of 15,110 triangle. Here $\alpha = 5°$, producing a dilatation of 1.0875.



$(a)$



(b)

Figure 4: Colon CT-Images: (a) Triangulated colon surface taken from 3 slices of human colon scan and (b) One half of the colon, after flattening. One is able to observe the holes caused by improper gluing of neighbouring patches. CT-data is in curtesy of Dr. Doron Fisher from Rambam Medical Center in Haifa.