ORTHOGONAL PROJECTIONS DERIVED FROM LOCALIZATION OPERATORS

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ABSTRACT

Projection operators derived from Gabor multipliers are suggested as a time-frequency localization tool. We give a description of the numerical realization of such projection operators and investigate the dependence of their localization properties on the parameters used in the definition. We also provide a comparison with iterated localization operators.

1. INTRODUCTION

Time-frequency concentration is a topic that has been repeatedly treated by various authors during the last decades. The observation, going back to Paley-Wiener, that no non-zero signal can be simultaneously band- and time-limited raises the desire for a precise definition of the notion time-frequency concentration. Various approaches have been suggested. Slepian and Pollak [7] investigated operators consisting of consecutive time- and bandlimiting steps. The eigenvectors of the resulting operators are the famous prolate-spheroidal wave functions, which can be interpreted as being optimally concentrated in time-frequency in a certain sense. However, this approach is rather restrictive concerning the shape of the area to which the resulting operators are concentrated. Daubechies [1] suggested a time-frequency concentration procedure by means of restricting reproducing formulas to certain areas of the time-frequency plane. This approach is much more general but only leads to analytic results for very special cases including the condition that the generating vector for the coherent system is Gaussian. Ramanathan/Topiwala [6] and Hlawatsch/Kozek [4] use the Weyl-correspondence for the definition of the time-frequency support of a function. Their result can be roughly summarized by saying that the time-frequency support of a signal corresponds to the effective support of its spectrogram with respect to a reasonable window. In all of the cited work the analysis of the eigenfunctions and eigenvalues corresponding to the suggested time-frequency operators play a crucial role. Here, we suggest to derive families of eigenfunctions from a class of time-frequency concentration operators called Gabor multipliers. Gabor multipliers can be seen as a generalization of Daubechies' approach: they are derived from Gabor frames by masking in the coefficient-space. As the redundancy of the system used for analysis can be drastically reduced, this approach allows for numerically very efficient methods. On the other hand, the freedom in the choice of the time- and frequency-shift parameters introduces a certain ambiguity. We will start from a Gabor multiplier given by a certain mask and define a projection operator by means of the salient eigenfunctions of the Gabor multiplier. We show how the number of eigenfunctions concentrated inside the originally masked area can be estimated. This number turns out to be relatively independent of the redundancy of the Gabor system used in the definition of the Gabor multiplier. Finally we demonstrate the time-frequency concentration achieved by the resulting projection operators in comparison with iterated Gabor multipliers.

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2. GABOR MULTIPLIERS

Time-frequency shift operators are crucial elements in Gabor analysis. M_ω and T_x denote frequency-shift by ω and time-shift by x, respectively, of a function g, i.e., $M_\omega T_x g(t) = e^{2\pi i t\omega} g(t-x)$ for $(x,\omega) \in \mathbb{R}^{2d}$. $M_\omega T_x$ is a **time-frequency shift** operator. Writing $\lambda = (x, w)$, $M_\omega T_x$ is often denoted by $\pi(\lambda)$ in the sequel. A set of functions f_k in $\mathbf{L}^2(\mathbb{R})$ is called a frame, if there exist constants $A, B > 0$, so that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \; for \; all \; f \in \mathbf{L}^2(\mathbb{R}). \quad (1)$$

The frame bounds A and B are the infimum and supremum, respectively, of the eigenvalues of the *frame operator* S, defined as

$$Sf = \sum_k \langle f, f_k\rangle f_k.$$

If $A = B$ the frame is called *tight*. Here we consider the special case $f_k = g_{m,n} = M_{mb}T_{na}g$ of Gabor frames. The coefficients of a Gabor frame, given by $[\langle f, g_{m,n}\rangle]_{m,n}$ correspond to the samples of the STFT on the product lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$.

Definition 1 (Gabor multiplier) *Assume that* $g \in \mathbf{L}^2$*, i.e. is square summable, that* $\Lambda \subset \mathbb{R}^{2d}$ *a TF-lattice (a discrete subgroup of* $\mathbb{R}^{2d}$*) and* $\mathbf{m} = (m(\lambda))_{\lambda\in\Lambda}$ *a bounded, real-valued sequence in* $\ell^\infty(\Lambda)$*. Then the* **Gabor multiplier** *associated* *to* (g, Λ) *with symbol* $\mathbf{m}$ *is given as*

$$G_{\mathbf{m}}(f) = G_{g,\Lambda,\mathbf{m}}(f) = \sum_{\lambda \in \Lambda} \mathbf{m}(\lambda) \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g.$$

The subscripts g and Λ will be omitted if they are not crucial for the discussion. Note that $f \mapsto \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$ (up to a normalizing factor $\|g\|^{-2}$) is the orthogonal projection onto the one-dimensional spaces generated by $\pi(\lambda) g$. Gabor multipliers have been described in [3] and investigated to some detail in [2].

3. EIGENFUNCTIONS AND EIGENVALUES

It is a natural approach to try to describe the behavior of (self-adjoint) Gabor multipliers through their eigenvectors, which represent an orthonormal basis for the range of the operators. In this section, we assume for simplicity that a tight Gabor frame is used. This assumption makes the eigenanalysis of Gabor multipliers more comprehensible.

Recall that for tight Gabor systems, a constant multiplier symbol generates a multiple of the identity operator. Whenever the multiplier $\mathbf{m}$ takes values between 0 and c, the eigenvalues of the resulting Gabor multiplier lie in the interval $[0, c]$, see [3], provided that $A = 1 = B$.

Another motivation for studying the eigen-behavior of Gabor multipliers is the application to time-varying filtering tasks. Often a $0/1$ multiplier, i.e. a characteristic function corresponding to a (usually bounded) region in the time-frequency plane, will be a first choice for a time-varying filtering task. However, as the Gabor system under inspection is redundant, the process of filtering by means of the corresponding Gabor multiplier cannot be a projection and in particular it is not idempotent. It can even be shown that for multiplier sequences with compact support repeated application of the operator to any given signal even yields a series of resulting signals converging to zero. This can been seen by noting that in this case all eigenvalues of $\mathbf{G_m}$ must be less than 1, as $\|\mathbf{G_m}\| \leq 1$. With projection operators we can achieve better concentration inside a given masking region than with conventional methods like the LSE (least square error) filter and with less computational effort than the iteration filter suggested by Qian and Chen in [5]. For a $0/1$ multiplier, for instance, we expect the eigenvectors corresponding to the eigenvalues close to 1 to be well concentrated inside the mask. Hence a projection onto the eigenvectors corresponding to eigenvalues above a certain threshold ρ should generate a good result, i.e. a signal which is nicely concentrated inside the region of interest and close to 0 outside. Numerical examples confirm the applicability of this approach.

We give a precise mathematical description of the situation next: Let a tight Gabor system (g, Λ) with $g \in S_0$ be given. Denote by M_R a masking region, centered around $\lambda = 0$ without restriction of generality[1], with $M_R \subseteq B_R(0)$. Let the finite-dimensional approximation of the frame operator S be given as

$$S_R f = \sum_{\lambda \in M_R \cap \Lambda} \langle f, g_\lambda \rangle g_\lambda, \tag{2}$$

which is a compact, self-adjoint operator. Let φ_k^R be the eigenvectors of S_R corresponding to its eigenvalues α_k^R. We fix a threshold $\rho > 0$ and define the following subspaces of $\mathbf{L}^2(\mathbb{R}^d)$: $H_R = \text{range}(S_R)$, $E_R = \text{span}\{\varphi_k^R | k \in \mathcal{I}_R^e\}$, where $\mathcal{I}_R^e = \{k : |\alpha_k^R| \geq \rho\}$. Note that due to the spectral theorem for compact, self-adjoint operators, $H_R = E_R \oplus R_R$ and with $\mathcal{I}_R = \mathcal{I}_R^e \cup \complement\mathcal{I}_R^e$, we can write the finite-dimensional approximation of the frame operator as $S_R = \sum_{k \in \mathcal{I}_R} \alpha_k^R \langle f, \varphi_k^R \rangle \varphi_k^R$. We now obtain the following two statements which have been proved in [2].

The first two relevant facts imply that for any function obtained by localization through masking can also be approximated arbitrarily well by linear combinations of the corresponding eigenfunctions, while the second statement (Thm. 1) shows that the eigenspaces for large eigenvalues "exhaust" the time-frequency plane for expanding regions. More precisely we have the following results:

Proposition 1 *Fix* $R_0 > 0$. *For any* $\varepsilon > 0$, *there exists* R_1 *such that for all* $f \in H_{R_0}$

$$\|f - S_R f\|_2^2 < \varepsilon \|f\|_2^2 \text{ for all } R > R_1.$$

Theorem 1 *For any fixed* ρ, $0 < \rho < \|S\|$, *the union of* E_R *is dense in* $\mathbf{L}^2(\mathbb{R}^d)$, *i.e.*

$$\overline{\bigcup_{R>0} E_R} = \mathbf{L}^2(\mathbb{R}^d).$$

The above statements motivate an approach using eigenfunctions of Gabor multipliers for describing signals concentrated in certain regions of the time-frequency. The next section deals with the calculation and properties of the projection operators.

4. PROJECTION OPERATORS

We define the rank N-projection operator P_N with respect to a mask $\mathbf{m}$ and a Gabor frame $\mathcal{G}$ as

$$P_N^{\mathbf{m},\mathcal{G}} f = \sum_{k=1}^{N} \langle f, \varphi_k \rangle \varphi_k, \tag{3}$$

where $\varphi_k, k = 1, \ldots, N$ the eigenvectors corresponding to the N biggest eigenvalues of the Gabor multiplier generated by $\mathcal{G}$ and $\mathbf{m}$.

4.1. Calculating the eigenvectors

The calculation of the eigenspaces of Gabor multipliers can be realized with a numerical effort corresponding to the size of the masked area rather than the signal-length. Let the matrix G be defined as the $k \times n$ matrix having $\overline{g_{m,n}}$ as its

[1] All results in the sequel can be stated and proved analogously for any $\lambda \in \Lambda$ due to Λ-invariance of the essential properties of the operators.

$(m + nM)$-th row. The numerical realization relies on the fact that for rectangular matrices G with size $k \times n$, where k is the number of building blocks inside the masking area and n is the signal length, the eigenvectors of the frame operator $S = G^*G$ can be obtained from the eigenvectors of the Gramian matrix $\mathbf{\Gamma} = GG^*$. As we assume $k << n$ here, this is numerically a lot cheaper than calculating the eigenvectors of S directly. In fact, let $\varphi_j, j = 1, \dots, k$ be the eigenvectors of S such that $S\varphi_j = \alpha_j \varphi_j$. If $\mathbf{\Gamma}\omega_j = \alpha_j \omega_j$, then

$$\varphi_j = \frac{1}{\sqrt{\alpha_j}} \sum_{i=1}^{k} \omega_j(i) g_i.$$

Typically, the eigenvectors corresponding to eigenvalues close to 1 of a Gabor multiplier with a rectangular $0/1$ mask are concentrated inside the mask. In all subsequent examples, the signal length will be 144. Any numerical realization of practically interesting tasks requires the choice of systems with low redundancy. The choice of the lattice Λ and the choice of the window thus entail certain parameters of freedom whose influence on the behavior of the eigenvalues and eigenvectors of the resulting operators will be investigated in the sequel. It will be shown that the eigenspaces are relatively independent of these parameters, which is a desired feature.
In the present section we want to study the TF-localization behavior on a quantitative level. We are interested in the connection between eigenspaces generated by a certain number of eigenfunctions of Gabor multipliers obtained from different Gabor systems and the same symbol (mask). According to Theorem 1, in the limiting case the eigenspaces become independent of the lattice. Numerical evidence will be given for the following statement:

Proposition 2 *For given tight Gabor frames $\mathcal{G}_i$ and a multiplier $\mathbf{m}$ in $\ell^\infty(\lambda)$, consider the subspace E^i_N generated by the first N eigenvectors of the operator $G_{g_i,\Lambda_i,\mathbf{m}}$. Then for a fixed error ε, any function f_i in a N_1-dimensional subspace spanned by the first N_1 eigenfunctions of $G_{g_1,\Lambda_1,\mathbf{m}}$ can be represented up to the error ε as a linear combination of N_2 eigenfunctions of $G_{g_i,\Lambda_i,\mathbf{m}}, i = 2, \dots, k$, with $N_2 = N_1 + E(\varepsilon, N_1, m)$.*

We carry out the following experiment: 3 tight Gabor systems with lattice parameters $(a_i, b_i)_{i=1,\dots,3}$ are considered, the length of the signals is $n = 144$. The mask is cone-shaped, centered around 0 and restricted to the area 71×71.

Systems I - III

a	b	Redundancy	Lattice points inside the mask
9	3	5.3333	161
9	4	4	119
4	3	12	391

By the choice of the shape of the mask, the eigenvalues are strictly decreasing, and as a consequence the TF-concentration of the eigenfunctions is nicely centered around 0. Hence, the size of the area covered by the eigenspace spanned by eigenfunctions $\varphi_1, \dots \varphi_N$ grows in a concentric manner. Now, for all systems, the projections onto the first 20, 25 and 30, respectively, eigenfunctions were determined, denoted by P^i_N, where N is the dimension of the eigenspace, hence the rank of the operator, and i is the respective Gabor system from which the operator has been derived. The following measure for the error in approximation was then introduced:

$$err^i_N(j) = \frac{1}{M} \sum_{k=1}^{M} \| P^i_N \varphi^j_k - \varphi^j_k \|_2,$$

where N is the rank of the projection operator P^i_N, i is the system from which it is derived. $\varphi_k, k = 1 \dots M$ are the eigenvectors derived from the System $\mathcal{G}_j$ and M is the dimension of the eigenspace we wish to represent. In our experiment, M was chosen to be 20, and the approximation quality for $N = 20, 25, 30$ was investigated. The results can be found in Table 1. The indexes $i = 1, \dots 3$ run from top to bottom and $j = 1, \dots 3$ from left to right, i.e. in the first row and second column the approximation error of the first system approximating the second can be read.
Proposition 2 implies that the eigenspaces generated by eigenfunctions of Gabor multipliers corresponding to various different Gabor frames grow concentrically and cover each other as long as enough eigenfunctions are chosen from each system. The differences between the eigenspaces are small compared to the size of the region of time-frequency concentration.
The next section gives a criterion for the number of eigenvalues which yields an optimal concentration inside the masked region.

Table 1: **Results of Experiment**

$err^i_{20}(j)$	System1	System2	System3
System 1	0.0000	0.0247	0.0334
System 2	0.0246	0.0000	0.0108
System 3	0.0325	0.0104	0.0000

$err^i_{25}(j)$	System1	System2	System3
System 1	0.0000	0.0100	0.0126
System 2	0.0094	0.0000	0.0014
System 3	0.0096	0.0006	0.0000

$err^i_{30}(j)$	System1	System2	System3
System 1	0.0000	0.0042	0.0043
System 2	0.0049	0.0000	0.0000
System 3	0.0051	0.0000	0.0000

4.2. The number of eigenvectors concentrated inside a mask

Proposition 3 *Let a tight Gabor system $\mathcal{G} = (g, \Lambda)$, $\Lambda = \frac{n}{b}\mathbb{Z} \times \frac{n}{a}\mathbb{Z}$, i.e. with redundancy $\varrho = \frac{1}{ab}$ be given. Furthermore, assume a $0/1$ mask such that k sampling points of Λ*

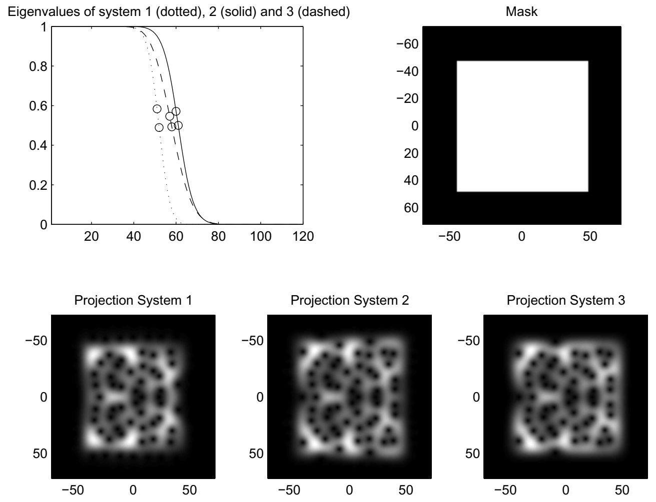

Figure 1: *Number of eigenvectors inside a mask*

fall inside the mask. The number K of eigenvectors concentrated in the masked region of the time-frequency plane can be estimated by $K = \frac{k}{\varrho}$. This number corresponds to the number of eigenvalues greater than or equal to 0.5.

Note that this numerical criterion is in accordance with a result by Hlawatsch and Kozek in [4].
Figure 1 shows the results of eigenanalysis of three different tight systems $\mathcal{G}_i, i = 1, \ldots, 3$. The systems have redundancy 1.5, 2 and 6, respectively. Again, the signal length n was 144, and the mask was a square of size 94×94, centered about $(0, 0)$. For the three systems, 77, 121 and 345 points, respectively, fall into the masked region, hence the number of eigenvectors can be approximated by $\frac{77}{1.5} = 51.333$, $\frac{121}{2} = 60.5$ and $\frac{345}{6} = 57.5$. The first plot shows the eigenvalues of the Gabor multipliers corresponding to the three systems. The critical values, corresponding to the numbers of eigenvectors given above rounded to the closest integers, are marked. Note that the systems' redundancies differ signif icantly, leading to different actual mask sizes. Still the distribution of eigenvalues is very similar.
The next plots show the mask and the short-time Fourier transforms of the projection of random signals onto the eigenvectors. These projections correspond to the projection operators of rank K, $K = 52, 60, 57$, for the systems $1, 2, 3$, respectively. The number of eigenvalues $\alpha_i \geq 0$ was 51, 60 and 56, respectively, for the three systems.

4.3. Comparison with iterated localization operators

We finally compare different methods that have been suggested for tackling time-varying filtering tasks. As mentioned before, in [5] Qian and Chen propose iterated localization operators in order to increase the concentration of the resulting signal inside a given masking region. Hence we compare their method with the proposed projections on bases of eigenvectors. Iterated localization operators result from applying a Gabor multiplier (with $0/1$ symbol) to a signal repeatedly. The improvement in concentration inside the masking region is obvious from observing the eigenvalue distribution. As we have $\mathbf{G_m} = U\Sigma U^*$, where U is a unitary operator and Σ denotes the diagonal matrix of eigenvalues, it follows that $\mathbf{G}_\mathbf{m}^k = U(\Sigma)^k U^*$, for k iterations of $\mathbf{G_m}$. However, this method requires high computational effort. The concentration of the projection onto the most salient eigenvectors of a Gabor multiplier as suggested by Proposition 3 has been investigated and the concentration achieved by projection onto salient eigenvectors was indeed comparable to the result achieved by the iteration method. Hence, in order to achieve good time-frequency concentration, either iterations of a low-redundancy Gabor multiplier must be performed or the eigenvectors of a concentration operator must be obtained. As described in Section 4.1, the eigenanalysis can be realized in a numerically efficient manner.

5. CONCLUSIONS AND FUTURE WORK

We suggested the usage of projection operators as an attractive alternative to conventional time-frequency localization methods. By means of analysis of the eigenvalues and eigenvectors of Gabor-multipliers, we investigated the achievable concentration in the time-frequency plane. With the aid of these results, we plan to develop methods tailored to filtering tasks in certain signal classes, such as music signals comprising contributions from different instruments. It will be the topic of future research to classify the necessary computational effort and the resulting improvement in more detail and with respect to specific time-varying filtering tasks.

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