TWO-DIMENSIONAL AUTOREGRESSIVE MODELLING USING JOINT SECOND AND THIRD ORDER STATISTICS AND A WEIGHTING SCHEME

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ABSTRACT

The two-dimensional autoregressive modelling problem is attempted using a combination of the Yule-Walker system of equations and the Yule-Walker system of equations in the third-order statistical domain. A novel weighting scheme that relates the contribution of the two systems is proposed and some simulations results are provided to verify the improved estimations.

1. INTRODUCTION

Two-dimensional (2-D) autoregressive (AR) modelling has been used as one of the methods to characterise textured images [2][3]. Each texture is described by a different set of 2-D AR model coefficients. In the literature, two of the most commonly used methods for solving the AR model coefficients estimation problem are the Yule-Walker system of equations (YW) and the Yule-Walker system of equations in the third-order statistical domain (YWT). The YW method uses the second-order moment samples of the signals, which are sensitive to the external Gaussian noise. However, the estimated coefficients arisen from a number of realisations have lower variances [7][8]. The YWT methods employs the third-order moments, which result in that the external Gaussian noise can be eliminated, but the variances computed from a number of realisations are higher than the variances obtained from the YW method. In [1], the authors proposed a method which uses both the YW and YWT. The weighting matrix that related the contribution of the two systems was taken to be an identity matrix, i.e., both second-order and third-order moment samples contribute equally. From the simulations, it was found that the estimation results are not close to the original values for low SNR (signal-to-noise) systems. In this paper, a new weighting scheme is proposed, which results in improved AR model coefficients estimation in both low and high SNR systems.

2. TWO-DIMENSIONAL AUTOREGRESSIVE MODEL

Let us consider a digitised image x of size $M \times N$. Each pixel of x is characterised by its location $[m,n]$ and can be represented as $x[m,n]$, where $1 \leq m \leq M$, $1 \leq n \leq N$ and $x[m,n]$ is a positive intensity (gray level) associated with it. A two-dimensional $(2-D)$ autoregressive (AR) model is defined as [4]

$$x[m,n] = -\sum_{i=0}^{p_1}\sum_{j=0}^{p_2} a[i,j]x[m-i,n-j] + w[m,n], \qquad (1)$$

where $[i,j] \neq [0,0]$,
$a[i,j]$ is the AR model coefficient,
$w[m,n]$ is the input driving noise, and
$p_1 \times p_2$ is the order of the model.

The driving noise, $w[m,n]$, is assumed to be zero-mean, i.e., $E\{w[m,n]\} = 0$ and non-Gaussian. The AR model coefficient $a[0,0]$ is assumed to be 1 for scaling purposes, therefore we have $[(p_1+1)(p_2+1)-1]$ unknown coefficients to solve.

An external zero-mean Gaussian noise, $v[m,n]$, is added onto the system. Mathematically the new system can be written as

$$y[m,n] = x[m,n] + v[m,n]. \qquad (2)$$

The signal-to-noise ratio (SNR) of the system is calculated by

$$SNR = 10log_{10}\frac{\sigma_x^2}{\sigma_v^2} \quad dB \qquad (3)$$

where σ_x^2 is the variance of the signal and
σ_v^2 is the variance of the noise.

3. YULE-WALKER SYSTEM OF EQUATIONS

The conventional Yule-Walker equations are given by [4][8]

$$\sum_{i=0}^{p_1}\sum_{j=0}^{p_2} a[i,j]r_{yy}[i-k,j-l] = -r_{yy}[-k,-l] \qquad (4)$$

for $k = 0,\cdots,p_1$ and $l = 0,\cdots,p_2$,
where $[k,l] \neq [0,0]$, $[i,j] \neq [0,0]$,
$r_{yy}[i,j] = E\{y[m,n]y[m+i,n+j]\}$,
$a[i,j]$ is the AR model coefficient,
$1 \leq m \leq M, 1 \leq n \leq N$, and $M \times N$ is the size of the given image.

(4) can be written as

$$\underline{\boldsymbol{R}}\underline{\boldsymbol{a}} = -\underline{\boldsymbol{r}}, \qquad (5)$$

where $\underline{\boldsymbol{R}}$ is a $(p_1p_2+p_1+p_2) \times (p_1p_2+p_1+p_2)$ matrix
and $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{r}}$ are both $(p_1p_2+p_1+p_2) \times 1$ vectors.

These equations give good AR model coefficient estimations when the SNR is high. However, the error increases with σ_v^2.

4. YULE-WALKER SYSTEM OF EQUATIONS IN THE THIRD-ORDER STATISTICAL DOMAIN

The equations that relate the AR model parameters to the third-order moment samples are [8][9]:

$$\sum_{i=0}^{p_1}\sum_{j=0}^{p_2} a[i,j]C_{3y}([i-k,j-l],[i-k,j-l]) =$$

$$-C_{3y}([-k,-l],[-k,-l]) \quad (6)$$

for $k = 0,\cdots,p_1$, $l = 0,\cdots,p_2$ and $[k,l] \neq [0,0]$, where

$$C_{3y}([i_1,j_1],[i_2,j_2]) = E\{y[m,n]y[m+i_1,n+j_1]y[m+i_2,n+j_2]\}.$$

These equations are insensitive to external Gaussian noise. The equations can be written in matrix form as

$$\underline{\boldsymbol{C}}\underline{\boldsymbol{a}} = -\underline{\boldsymbol{c}}, \quad (7)$$

where $\underline{\boldsymbol{C}}$ is a $(p_1p_2+p_1+p_2)\times(p_1p_2+p_1+p_2)$ matrix and $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{c}}$ are both $(p_1p_2+p_1+p_2)\times 1$ vectors.

5. THE COMBINED METHOD

In [1], a method which combines the Yule-Walker system of equations and the Yule-Walker system of equations in the third-order statistical domain is used to estimate the 2-D AR model coefficients. Mathematically, an extended system is written as:

$$\underline{\boldsymbol{D}}\begin{pmatrix}\underline{\boldsymbol{R}}\\ \underline{\boldsymbol{C}}\end{pmatrix}\underline{\boldsymbol{a}} = -\underline{\boldsymbol{D}}\begin{pmatrix}\underline{\boldsymbol{r}}\\ \underline{\boldsymbol{c}}\end{pmatrix} \quad (8)$$

where the matrix $\underline{\boldsymbol{R}}$ and vector $\underline{\boldsymbol{r}}$ are defined in (5),
the matrix $\underline{\boldsymbol{C}}$ and vector $\underline{\boldsymbol{c}}$ are defined in (7),
$\underline{\boldsymbol{D}}$ is a diagonal weighting matrix, and
$\underline{\boldsymbol{a}}$ is the vector of the unknown AR model coefficients, $[a[0,1],\cdots,a[0,p_2],\cdots,a[p_1,p_2]]^T$.

In [1], the diagonal weighting matrix, $\underline{\boldsymbol{D}}$, was taken to be an identity matrix.

6. WEIGHTING MATRIX

In this section, a novel weighting matrix is introduced. The matrix remains diagonal. However, the elements depend on the variance of the external Gaussian noise in the system, which is derived in Section 6.1. In Section 6.2, the determination of the weighting matrix is presented.

6.1 Yule-Walker System of Equations for Noisy Signals

Consider the system as shown in (2). The signal $x[m,n]$ and the noise $v[m,n]$ are assumed to be statistically independent, hence the cross correlation function samples between $x[m,n]$ and $v[m,n]$ is zero, i.e., $r_{xv}[k,l] = 0$. Consequently

$$r_{yy}[k,l] = r_{xx}[k,l] + r_{vv}[k,l]. \quad (9)$$

The Yule-Walker system of equations has in that case the following form [7]:

$$\sum_{i=0}^{p_1}\sum_{j=0}^{p_2} a[i,j]r_{yy}[i-k,j-l] = \begin{cases}\sigma_w^2+\sigma_v^2 & [k,l]=[0,0]\\ \sigma_v^2\cdot a[k,l] & [k,l]\in S'_{QP}\\ 0 & \text{elsewhere}\end{cases} \quad (10)$$

where $[i,j] \neq [0,0]$.

In the matrix-vector form the above becomes (11) on the next page.

Note that $\sigma_v^2\begin{pmatrix}\boldsymbol{a}_0\\ \boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{p_1}\end{pmatrix} = \sigma_v^2\underline{\boldsymbol{I}}\begin{pmatrix}\boldsymbol{a}_0\\ \boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{p_1}\end{pmatrix}$

$$= \begin{pmatrix}\sigma_v^2\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0}\\ \boldsymbol{0} & \sigma_v^2\boldsymbol{I} & \cdots & \boldsymbol{0}\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \sigma_v^2\boldsymbol{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{a}_0\\ \boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{p_1}\end{pmatrix} \quad (12)$$

and the system (11) may be written as (13) on the next page, where $\boldsymbol{a}_i = [a[i,0],a[i,1],\cdots,a[i,p_2]]^T$ is a vector of size $(p_2+1)\times 1$,

$\boldsymbol{h}_1 = [1,0,\cdots,0]^T$ is a vector of size $(p_2+1)\times 1$,

$\boldsymbol{0} = [0,0,\cdots,0]^T$ is a vector of size $(p_2+1)\times 1$, and

$$\boldsymbol{R}_{yy}[i] = \begin{pmatrix} r_{yy}[i,0] & r_{yy}[i,-1] & \cdots & r_{yy}[i,-p_2]\\ r_{yy}[i,1] & r_{yy}[i,0] & \cdots & r_{yy}[i,-(p_2-1)]\\ \vdots & \vdots & \ddots & \vdots\\ r_{yy}[i,p_2] & r_{yy}[i,p_2-1] & \cdots & r_{yy}[i,0]\end{pmatrix}$$

is a matrix of size $(p_2+1)\times(p_2+1)$.

After expanding the equations (11), we can remove the first row of the equations since the variance of the driving noise $w[m,n]$ is unknown. The coefficient $a[0,0]$ is assumed to be 1, so we can move the first column of the matrix on the left-hand side to the right-hand side of the equation. After the rearranging step, the equations can be written in matrix form as

$$\underline{\boldsymbol{R}}_{yy}\underline{\boldsymbol{a}} + \underline{\boldsymbol{r}}_{yy} = \sigma_v^2\underline{\boldsymbol{I}}\underline{\boldsymbol{a}} \quad (14)$$

where the AR model coefficients estimation $\underline{\boldsymbol{a}}$ is obtained from (7) using

$$\underline{\boldsymbol{a}} = -\underline{\boldsymbol{C}}^{-1}\underline{\boldsymbol{c}}. \quad (15)$$

Let $\underline{\boldsymbol{r}}_1 = \underline{\boldsymbol{R}}_{yy}\underline{\boldsymbol{a}} + \underline{\boldsymbol{r}}_{yy}$, where $\underline{\boldsymbol{a}}$ is obtained from (15). The variance of the noise $v[m,n]$ can be calculate using

$$\sigma_v^2 = (\underline{\boldsymbol{a}}^T\underline{\boldsymbol{a}})^{-1}\underline{\boldsymbol{a}}^T\underline{\boldsymbol{r}}_1 \quad (16)$$

6.2 Determination of the Weighting Matrix

The weighting diagonal matrix, $\underline{\boldsymbol{D}}$, is determined as (17) on the next page. where $\lceil x\rceil$ denotes rounding toward infinity.

7. SIMULATION RESULTS

Synthetic images generated from the following 2×2 stable and separable AR model are used for simulation purposes. The $2-D$ stable AR model coefficients are obtained from $\underline{a} = \underline{a}_1^T \times \underline{b}_1$, where $\underline{a}_1$ and $\underline{b}_1$ are both stable $1-D$ AR model coefficients.

$x[m,n] = -0.16x[m-2,n-2] - 0.2x[m-2,n-1] - 0.4x[m-2,n] - 0.2x[m-1,n-2] - 0.25x[m-1,n-1] - 0.5x[m-1,n] - 0.4x[m,n-2] - 0.5x[m,n-1] + w[m,n]$

The driving noise, $w[m,n]$, is zero-mean exponentially-distributed. Additional Gaussian noise, $v[m,n]$, with zero-mean and unity variance is added onto $x[m,n]$ to yield

$$y[m,n] = x[m,n] + v[m,n].$$

$$\begin{pmatrix} \boldsymbol{R}_{yy}[0] & \boldsymbol{R}_{yy}[-1] & \cdots & \boldsymbol{R}_{yy}[-p_1] \\ \boldsymbol{R}_{yy}[1] & \boldsymbol{R}_{yy}[0] & \cdots & \boldsymbol{R}_{yy}[-(p_1-1)] \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{R}_{yy}[p_1] & \boldsymbol{R}_{yy}[p_1-1] & \cdots & \boldsymbol{R}_{yy}[0] \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{p_1} \end{pmatrix} = \sigma_w^2 \begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix} + \sigma_v^2 \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{p_1} \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} \boldsymbol{R}_{yy}[0]-\sigma_v^2\boldsymbol{I} & \boldsymbol{R}_{yy}[-1] & \cdots & \boldsymbol{R}_{yy}[-p_1] \\ \boldsymbol{R}_{yy}[1] & \boldsymbol{R}_{yy}[0]-\sigma_v^2\boldsymbol{I} & \cdots & \boldsymbol{R}_{yy}[-(p_1-1)] \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{R}_{yy}[p_1] & \boldsymbol{R}_{yy}[p_1-1] & \cdots & \boldsymbol{R}_{yy}[0]-\sigma_v^2\boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{p_1} \end{pmatrix} = \sigma_w^2 \begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix} \tag{13}$$

$$D[i,i] = \begin{cases} 1 & \text{for} \quad 1 \le i \le (p_1+1)(p_2+1)-1 \\ \lceil 50\sigma_v^2 \rceil & \text{for} \quad (p_1+1)(p_2+1) \le i \le 2(p_1+1)(p_2+1)-2 \end{cases} \tag{17}$$

Parameter	Real Value	Mean Estimated Value (SNR= 5 dB)	Variance (10^{-4})
$a[0,1]$	0.5	0.3481	0.2371
$a[0,2]$	0.4	0.2632	0.2396
$a[1,0]$	0.5	0.3478	0.1949
$a[1,1]$	0.25	0.1013	0.2787
$a[1,2]$	0.2	0.07000	0.2859
$a[2,0]$	0.4	0.2620	0.2499
$a[2,1]$	0.2	0.06993	0.2442
$a[2,2]$	0.16	0.04686	0.2966
Relative Error		0.4331	

Table 1: The results of the combined method using an identity weighting matrix for a $2-D$ symmetrical AR model with SNR equal to 5 dB.

Parameter	Real Value	Mean Estimated Value (SNR= 30 dB)	Variance (10^{-4})
$a[0,1]$	0.5	0.4992	0.2370
$a[0,2]$	0.4	0.3987	0.2753
$a[1,0]$	0.5	0.4987	0.2606
$a[1,1]$	0.25	0.2485	0.4745
$a[1,2]$	0.2	0.1986	0.4453
$a[2,0]$	0.4	0.3991	0.3323
$a[2,1]$	0.2	0.1991	0.4436
$a[2,2]$	0.16	0.1588	0.4987
Relative Error		0.01662	

Table 2: The results of the combined method using an identity weighting matrix for a $2-D$ symmetrical AR model with SNR equal to 30 dB.

Parameter	Real Value	Mean Estimated Value (SNR= 5 dB)	Variance (10^{-3})
$a[0,1]$	0.5	0.4922	0.1239
$a[0,2]$	0.4	0.3924	0.2682
$a[1,0]$	0.5	0.4902	0.1215
$a[1,1]$	0.25	0.2460	0.1854
$a[1,2]$	0.2	0.1952	0.4411
$a[2,0]$	0.4	0.3900	0.3247
$a[2,1]$	0.2	0.1958	0.4894
$a[2,2]$	0.16	0.1568	0.9877
Relative Error		0.05727	

Table 3: The results of the combined method using the new proposed weighting matrix for a $2-D$ symmetrical AR model with SNR equal to 5 dB.

Parameter	Real Value	Mean Estimated Value (SNR= 5 dB)	Variance (10^{-3})
$a[0,1]$	0.5	0.4994	0.2417
$a[0,2]$	0.4	0.3990	0.2541
$a[1,0]$	0.5	0.4994	0.1833
$a[1,1]$	0.25	0.2487	0.3735
$a[1,2]$	0.2	0.1983	0.3938
$a[2,0]$	0.4	0.3993	0.3132
$a[2,1]$	0.2	0.1993	0.3896
$a[2,2]$	0.16	0.1588	0.4144
Relative Error		0.01578	

Table 4: The results of the combined method using the new proposed weighting matrix for a $2-D$ symmetrical AR model with SNR equal to 30 dB.

The variance of $v[m,n]$ is adjusted so that SNR is equal to 5 dB for heavily noisy case and 30 dB for almost noise-free case.

The results obtained from the conventional method can be found in Table 1 and Table 2 for the SNR equal to 5 dB and 30 dB respectively. The results arisen from the proposed method may be found in Table 3 and 4 for SNR equal to 5 dB and 30 dB respectively. The average value of the lower half of the diagonal matrix is 8 when the SNR is 5 dB and 1 for SNR equal to 30 dB.

8. SUMMARY AND CONCLUSION

In this paper, we reviewed two of the most widely used methods for two-dimensional autoregressive (AR) modelling: the Yule-Walker system of equations and the Yule-Walker system of equations in the third-order statistical domain, as well as a method which uses the combination of the above systems. A new weighting scheme is purposed to control the contribution of each system by calculating the external Gaussian noise variance. The simulation results show that the new weighting scheme enables the method to estimate AR model coefficients in both low and high SNR systems.

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