DESIGN OF OPTIMAL IIR FILTERS IN REDUCED-DIMENSIONALITY PARAMETER SPACES

Andrzej Tarczynski

Department of Electronic Systems, University of Westminster
115 New Cavendish Street, London, W1W 6UW, United Kingdom (Europe)
phone: +44 20 7911 5831, fax: +44 20 7911 5089, email: tarczya@wmin.ac.uk

ABSTRACT

A new method of designing WLS optimal IIR filters is proposed in this paper. The approach belongs to the emerging group of methods that search for the optimal filter parameters inside the spaces of significantly reduced dimensionalities. Efficient implementation of the proposed method is achieved by selecting suitable parameterisation of the filter's transfer function and deriving numerical formulas for calculating the WLS cost and its derivatives with respect to the optimisation parameters. The filters can thus be designed using off-the-shelf programmes tackling nonlinear optimisation problems with constraints. Owing to the reduction of the dimensionality of the search space the proposed approach is more robust and provides more accurate results than the "fully blown" traditional techniques of designing IIR filters.

1. INTRODUCTION

Let the prototype transfer function of the filter be given by $G(z) = B(z)/A(z)$, $H(\nu)$ be its required frequency response and the nonnegative function $W(\nu)$ be the weight. The functions $B(z)$ and $A(z)$ are polynomials in z^{-1} with tuneable coefficients. The objective of the Weighted Least Squares (WLS) filter design is to determine the coefficients of $B(z)$ and $A(z)$ so that the following cost is minimised.

$$J = \int_{-0.5}^{0.5} \left| G(e^{j2\pi\nu}) - H(\nu) \right|^2 W(\nu) d\nu \tag{1}$$

The roots of $A(z)$ should stay inside the unit disc. Let N_a and N_b be the degrees of $A(z)$ and $B(z)$ respectively. Traditionally, one has to search a space of $N_a + N_b + 1$ dimensions to find the optimal solution to the above problem [1]-[6]. In this paper we consider an approach that is motivated by the observation made in [6] that the search can be confined to an N_a-dimensional space. This idea might be exploited in many ways leading thus to a selection of various algorithms for designing IIR filters. For example, in [6] the WISE [4] method was modified to form an algorithm of designing filters in the reduced-dimensionality space. The attractiveness of the method introduced in [6] is that it relies on solving an optimisation problem without constraints. Its disadvantage is that the calculation of the cost and its derivatives could be time consuming. Moreover, it takes some experience to properly choose the weights for the main components of the modified cost used by that method. In this paper we formulate an algorithm that directly minimises cost (1). Unlike WISE approach, the stability of the filter is achieved by imposing constraints on the coefficients of $A(z)$ rather than by modifying cost (1).

To keep the paper self-contained we show how the problem of designing optimal filters can be solved by searching N_a- rather than $N_a + N_b + 1$-dimensional space. Suitable formulas for analytical calculation and efficient numerical approximation of the cost and its derivatives are presented. These quantities are required by most of the off-the-shell optimisation functions. Finally the proposed algorithm is tested on a numerical example.

2. REDUCTION OF SPACE DIMENSIONALITY IN DESIGN OF IIR FILTERS

The problem of designing IIR WLS optimal filters whose denominator is fixed has a closed form solution. This can be justified as follows. Let $A(z)$ be the fixed denominator of the designed filter. We define $A_f(\nu) = A(e^{j2\pi\nu})$, $B_f(\nu) = B(e^{j2\pi\nu})$ and present $B_f(\nu)$ in the following form

$$B_f(\nu) = B(e^{j2\pi\nu}) = \left[b_0, \cdots, b_{N_b} \right] \begin{bmatrix} 1 \\ \vdots \\ e^{-j2\pi\nu N_b} \end{bmatrix} = \mathbf{b}^T \mathbf{Z}_B(\nu) \tag{2}$$

Cost (1) can thus be written as

$$J = \int_{-0.5}^{0.5} \left(\frac{\mathbf{b}^T \mathbf{Z}_B(\nu)}{A_f(\nu)} - H(\nu) \right) \left(\frac{\mathbf{Z}_B^H(\nu)\mathbf{b}}{A_f^*(\nu)} - H^*(\nu) \right) W(\nu) d\nu \tag{3}$$

Hence

$$J = \mathbf{b}^T \int_{-0.5}^{0.5} \frac{\mathbf{Z}_B(\nu)\mathbf{Z}_B^H(\nu)}{\left| A_f(\nu) \right|^2} W(\nu) d\nu \mathbf{b} -$$

$$2\int_{-0.5}^{0.5} \frac{\mathbf{Z}_B^H(\nu) H(\nu)}{A_f^*(\nu)} W(\nu) d\nu \mathbf{b} + \int_{-0.5}^{0.5} \left| H(\nu) \right|^2 W(\nu) d\nu \tag{4}$$

We put cost (4) in a more concise form

$$J = \mathbf{b}^T \mathbf{M}\mathbf{b} - 2\mathbf{n}^T \mathbf{b} + r\,. \tag{5}$$

The vector of optimal coefficients of $B(z)$ is given by

$$\hat{\mathbf{b}} = \mathbf{M}^{-1}\mathbf{n} \tag{6}$$

The minimum value of J can thus be calculated by substituting (6) in (5):

$$\hat{J} = r - \mathbf{n}^T \mathbf{M}^{-1}\mathbf{n} \tag{7}$$

Note that $\hat{J}$ depends only on the coefficients of $A(z)$, which so far were assumed to be constant. The numerator of the transfer function $B(z)$ is now bundled in as an optimally chosen polynomial and does not affect the cost any more. Therefore, in order to design an optimal filter we need to minimise (7) with respect to the coefficients of $A(z)$. This optimisation problem can be solved by searching an N_a-dimensional space.

3. DESIGN OF OPTIMAL IIR FILTERS

In this section we formulate a nonlinear optimisation problem with constraints whose solution yields the coefficients of $A(z)$. First we parameterise $A(z)$ as a product of $L = N_a/2$ second order polynomials. For simplicity we assume that $A(z)$ is of even degree. It is not difficult to expand the analysis to the cases when the degree of $A(z)$ is odd.

$$A(z) = \prod_{r=1}^{L} A_r(z) = \prod_{r=1}^{L} \left(1 + a_{1r} z^{-1} + a_{2r} z^{-2}\right) \tag{8}$$

We demand that the roots of $A(z)$ are placed in $\mathrm{o}(\rho)$ - a disc with radius $\rho < 1$ centred at the origin. Note that the roots of $D(z)$ belong to $\mathrm{o}(\rho)$ if and only if all the roots of $D(z\rho)$ are placed in the unit disc $\mathrm{o}(1)$. Therefore $A(z)$ has all its roots inside $\mathrm{o}(\rho)$ only when the following inequalities are satisfied

$$\frac{a_{2r}}{\rho^2} \le 1 \tag{9}$$

$$\frac{a_{1r}}{\rho} - \frac{a_{2r}}{\rho^2} \le 1 \tag{10}$$

$$-\frac{a_{1r}}{\rho} - \frac{a_{2r}}{\rho^2} \le 1 \tag{11}$$

for $r = 1,\cdots,L$. The design of optimal IIR filter is now reduced to solving the following optimisation problem.

Filter Design Problem
Minimise $\hat{J}$ defined by (7) with respect to $a_{11}\ a_{21}\ \cdots\ a_{1L}\ a_{2L}$ subject to the constraints (9)-(11).

Once the optimisation problem is solved we can use (8) to construct the denominator of the transfer function of the filter and (6) along with (2) to get its numerator.

4. EFFICIENT CALCULATION OF THE COST AND ITS GRADIENT

In order to numerically solve the optimisation problem it is important to find efficient ways of calculating cost (7) and its derivatives. This task is not trivial. As (4) shows the elements of matrix $\mathbf{M}$ and vector $\mathbf{n}$ require integrations. We did not derive the gradient of cost (7) yet but one may expect that the derivatives of (7) with respect to $a_{11}\ a_{21}\ \cdots\ a_{1L}\ a_{2L}$ are even more complicated than (4). The observations below will help us to formulate efficient algorithms for calculating (7) and its derivatives.
If $X(\nu)$ is a Hermitian-symmetric periodic function with period 1 then the samples of signal $x(n)$, whose spectrum is $X(\nu)$, can be calculated as follows

$$x(n) = \int_{-0.5}^{0.5} X(\nu) e^{j2\pi\nu n} d\nu = \int_{0}^{1} X(\nu) e^{j2\pi\nu n} d\nu\,. \tag{12}$$

By approximating the second integral in (12) with summation we get

$$x(n) \approx \frac{1}{N_f} \sum_{m=0}^{N_f - 1} X\!\left(\frac{m}{N_f}\right) e^{j2\pi\frac{mn}{N_f}} = \tilde{x}(n)\,. \tag{13}$$

Note that (13) is the inverse DFT of $X(\nu)$. If the number of frequency points N_f is a power of 2, (13) can be efficiently calculated using inverse FFT. In some cases we may need to calculate $x(n)$ for negative values of n. Since inverse FFT does not yield such results we resolve the problem by noting that for real-valued $X(\nu)$ we have $\tilde{x}(-n) = \tilde{x}(n)$. When, $X(\nu)$ is complex-valued then a more general formula can be used $x(-n) \approx \tilde{x}(-n) = \tilde{x}(N_f - n)$. In subsequent analyses we also exploit the fact that:

$$\int_{-0.5}^{0.5} X^*(\nu) e^{j2\pi\nu n} d\nu = x(-n) \tag{14}$$

4.1 Numerical Calculation of Cost $\hat{J}$

Let $S_1(\nu) = \dfrac{W(\nu)}{\left|A_f(\nu)\right|^2}$ and $S_2(\nu) = \dfrac{G(\nu)W(\nu)}{A_f^*(\nu)}$. We define

$$s_1(n) = \int_{-0.5}^{0.5} S_1(\nu) e^{j2\pi\nu n} d\nu \tag{15}$$

and

$$s_2(n) = \int_{-0.5}^{0.5} S_2(\nu) e^{j2\pi\nu n} d\nu \tag{16}$$

Since $S_1(\nu)$ is a real-valued function, it follows from (4), (5) and (15) that $\mathbf{M}$ is the following Toeplitz matrix

$$\mathbf{M}=\begin{bmatrix} s_1(0) & s_1(1) & \cdots & s_1(N_b) \\ s_1(1) & s_1(0) & \cdots & s_1(N_b-1) \\ \vdots & \vdots & \ddots & \vdots \\ s_1(N_b) & s_1(N_b-1) & \cdots & s_1(0) \end{bmatrix}. \quad (17)$$

Similarly, vector $\mathbf{n}$ is given by

$$\mathbf{n}=\left[s_2(0),\ s_2(1),\ \cdots,\ s_2(N_b)\right] \quad (18)$$

To summarise, in order to create matrix $\mathbf{M}$ and vector $\mathbf{n}$ we need, first, to use IFFT to get $\mathbf{m}=\left[s_1(0),\ \cdots,\ s_1(N_b)\right]$ and $\mathbf{n}=\left[s_2(0),\ \cdots,\ s_2(N_b)\right]$. Then $\mathbf{M}$ can be formed using (17). Note that r in (17) does not change with the filter parameters. Therefore, in most cases there is no need for an accurate estimate of its value. In fact from the point view of minimising (7) r can be set to any number, e.g. $r=0$.

4.2 Numerical Calculation of the Gradient of Cost $\hat{J}$

The gradient of (7) is a vector consisting of $\frac{\partial J}{\partial a_{kr}}$, where $k=1,2$ and $r=1,\cdots,L$. Note that $\frac{\partial J}{\partial a_{kr}}=\frac{\partial}{\partial a_{kr}}\left(r-\mathbf{n}^T\mathbf{M}^{-1}\mathbf{n}\right)=-2\frac{\partial \mathbf{n}^T}{\partial a_{kr}}\mathbf{M}^{-1}\mathbf{n}-\mathbf{n}^T\frac{\partial \mathbf{M}^{-1}}{\partial a_{kr}}\mathbf{n}$. It can be proven that $\frac{\partial \mathbf{M}^{-1}}{\partial a_{kr}}=-\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial a_{kr}}\mathbf{M}^{-1}$. Therefore

$$\frac{\partial J}{\partial a_{kr}}=\hat{\mathbf{b}}^T\frac{\partial \mathbf{M}}{\partial a_{kr}}\hat{\mathbf{b}}-2\frac{\partial \mathbf{n}^T}{\partial a_{kr}}\hat{\mathbf{b}} \quad (19)$$

It follows from (17)-(19) that the main numerical problem related to the calculation of the gradient of the cost is to approximate the derivatives of $s_1(n)$ and $s_2(n)$. Note that

$$\frac{\partial s_1(n)}{\partial a_{kr}}=\int_{-0.5}^{0.5}W(\nu)e^{j2\pi\nu n}\frac{\partial}{\partial a_{kr}}\frac{1}{A_f(\nu)A_f^*(\nu)}d\nu \quad (20)$$

and

$$\frac{\partial}{\partial a_{kr}}\frac{1}{A_f(\nu)A_f^*(\nu)}=\frac{-1}{\left|A_f(\nu)\right|^2}\left(\frac{e^{-j2\pi\nu k}}{A_r(e^{j2\pi\nu})}+\frac{e^{j2\pi\nu k}}{A_r^*(e^{j2\pi\nu})}\right) \quad (21)$$

We introduce polynomials $\overline{A}_r(z)$ defined by

$$\overline{A}_r(z)=\frac{A(z)}{A_r(z)}=1+\overline{a}_{1r}z^{-1}+\cdots+\overline{a}_{N_a-2,r}z^{-(N_a-2)} \quad (22)$$

Now we represent (20) as

$$\frac{\partial s_1(n)}{\partial a_{kr}}=$$
$$\sum_{l=0}^{N_a-2}\overline{a}_{lr}\int_{-0.5}^{0.5}\frac{-W(\nu)}{\left|A_f(\nu)\right|^2}\left(\frac{e^{j2\pi\nu(n-k-l)}}{A_f(\nu)}+\frac{e^{j2\pi\nu(n+k+l)}}{A_f^*(\nu)}\right)d\nu \quad (23)$$

Let $S_3(\nu)=\frac{-W(\nu)}{\left|A_f(\nu)\right|^2A_f^*(\nu)}$ and

$$s_3(n)=\int_{-0.5}^{0.5}S_3(\nu)e^{j2\pi\nu n}d\nu \quad (24)$$

We use (14) and (24) to rewrite (23) in the following form

$$\frac{\partial s_1(n)}{\partial a_{kr}}=\sum_{l=0}^{N_a-2}\overline{a}_{lr}\left[s_3(k+l-n)+s_3(k+l+n)\right] \quad (25)$$

The derivative of matrix $\mathbf{M}$ in (19) is therefore a Toeplitz matrix generated by the vector $\frac{\partial \mathbf{m}}{\partial a_{kr}}=\left[\frac{\partial s_1(0)}{\partial a_{kr}},\ \frac{\partial s_1(1)}{\partial a_{kr}},\ \cdots,\ \frac{\partial s_1(N_b)}{\partial a_{kr}}\right]^T$. This vector can be obtained from

$$\frac{\partial \mathbf{m}}{\partial a_{kr}}=\left\{s_3\begin{pmatrix} k & k+1 & \cdots & k+N_a-2 \\ k-1 & k & \cdots & k+N_a-3 \\ \vdots & \vdots & \ddots & \vdots \\ k-N_b & k-N_b+1 & \cdots & k-N_b+N_a-2 \end{pmatrix}+\right.$$
$$\left.s_3\begin{pmatrix} k & k+1 & \cdots & k+N_a-2 \\ k+1 & k+2 & \cdots & k+N_a-1 \\ \vdots & \vdots & \iddots & \vdots \\ k+N_b & k+N_b+1 & \cdots & k+N_b+N_a-2 \end{pmatrix}\right\}\begin{bmatrix} \overline{a}_{0r} \\ \overline{a}_{1r} \\ \vdots \\ \overline{a}_{N_a-2,r} \end{bmatrix} \quad (26)$$

Note that we have used a simplified notation in (26). Instead of writing matrices of samples of signal $s_3(k)$ we used matrices of time instants for which the samples of the signal are needed and put the name of the signal in the front of each matrix.

The derivative of $s_2(n)$ with respect to a_{kr}, needed to form the second term of (19), can be generated as follows

$$\frac{\partial s_2(n)}{\partial a_{kr}}=\int_{-0.5}^{0.5}G(\nu)W(\nu)e^{j2\pi\nu n}\frac{\partial}{\partial a_{kr}}\frac{1}{A_f^*(\nu)}d\nu \quad (27)$$

This could be further processed using

$$\frac{\partial}{\partial a_{kr}}\frac{1}{A_f^*(\nu)}=\frac{-e^{j2\pi\nu k}}{A_f^*(\nu)A_r^*(e^{j2\pi\nu})}=\frac{-\overline{A}_r^*(e^{j2\pi\nu})e^{j2\pi\nu k}}{A_f^{*2}(\nu)} \quad (28)$$

Let $S_4(\nu) = \dfrac{-G(\nu)W(\nu)}{A_f^{*2}(\nu)}$ and

$$s_4(n) = \int_{-0.5}^{0.5} S_4(\nu) e^{j2\pi\nu n} d\nu \tag{29}$$

then

$$\frac{\partial s_2(n)}{\partial a_{kr}} = \sum_{l=0}^{N_a-2} \bar{a}_{lr} s_4(n+k+l) \tag{30}$$

Therefore

$$\frac{\partial \mathbf{n}}{\partial a_{kr}} =$$

$$s_4 \begin{pmatrix} k & k+1 & \cdots & k+N_a-2 \\ k+1 & k+2 & \cdots & k+N_a-1 \\ \vdots & \vdots & \iddots & \vdots \\ k+N_b & k+N_b+1 & \cdots & k+N_b+N_a-2 \end{pmatrix} \times \begin{bmatrix} \bar{a}_{0r} \\ \bar{a}_{1r} \\ \vdots \\ \bar{a}_{N_a-2,r} \end{bmatrix} \tag{31}$$

5. NUMERICAL EXAMPLE

The proposed approach to designing IIR filters has been tested numerically. The related optimisation problem has been solved using MATLAB optimisation toolbox function *fmincon* that tackles nonlinear optimisation problems with constraints. In order to use that function a programme calculating cost (7) and its gradient with respect to a_{kr} has been written. The numerical algorithms presented in the previous section have been implemented.

The filter prototype, target frequency response and the weight function used in this test are the same as those defined in example 1 in [3]:

$$H(\nu) = \begin{cases} 0 & \text{when } |\nu| < 0.25 \\ e^{j24\pi\nu} & \text{when } |\nu| > 0.25 \end{cases} \tag{32}$$

$$W(\nu) = \begin{cases} 0 & \text{when } \nu \in [0.2375, \quad 0.2625] \\ 1 & \text{when } \nu \in [0, \quad 0.2375] \cup [0.2625, \quad 0.5] \end{cases} \tag{33}$$

The structure of the filter is defined by $N_a = N_b = 14$. We request that all the poles of the filter are placed in $\mathrm{o}(\rho)$ where $\rho = 0.95$. Note that the last specification has not been taken from [3] since the authors of [3] used a different mechanism of maintaining stability of the filter.

Cost (1) calculated for the filter designed in [3] is $J = 0.00050$. The filter designed with the method proposed here compares very favourably with [3] as the cost is about three times smaller $J = 0.00016$. Figure 1 shows the plots of the magnitude of the weighted frequency response errors of both filters. The method proposed in this paper clearly provides smaller errors for most of the frequencies.

6. CONCLUSIONS

A new method of designing IIR filters has been proposed in this paper. One of the main advantages of the method is that it allows the user to significantly reduce the dimensionality of the space within which the optimal parameters of the filter are looked for. This feature is particularly useful when designing "traditional" types of filters like: lowpass, highpass, bandstop etc. Experience shows [4] that in such cases the number of poles needed to achieve good filter performance can be significantly smaller than the number of zeros ($N_a << N_b$). Such filters benefit most from the proposed approach. The method has been tested numerically and provided good quality results comparing favourably with those reported in the research literature. The tests have shown that the proposed design process is very robust and yields good approximations of the local minima.

REFERENCES

[1] A. Deczky, "Synthesis of recursive filters using the minimum p-error criterion", *IEEE Trans. Audio and Electroac.*, vol. AU-20, pp. 257-263, 1972.

[2] Y. C. Lim, J. H. Lee, C. K. Chen and R. H. Yang, "A weighted least squares algorithm for quasi-equiripple FIR and IIR digital filter design", *IEEE Trans. SP*, vol. 40, pp. 551-558, 1992.

[3] W. S. Lu, S. C. Pei and C. C. Tseng, "A weighted least-squares method for the design of stable 1-D and 2-D IIR digital filters", *IEEE Trans. SP*, vol. 46, pp. 1-10, 1998.

[4] A. Tarczynski, G.D. Cain, E. Hermanowicz and M.Rojewski, "A WISE Method for Designing IIR Filters", *IEEE Trans. SP*, vol. 49, pp. 1421-1432, 2001.

[5] W. S. Lu and T. Hinamoto, "Optimal Design of IIR Digital Filters With Robust Stability Using Conic-Quadratic-Programming Updates", *IEEE Trans. on SP*, vol. 51, pp. 1581-1592, 2003.

[6] A. Tarczynski "Problem Dimensionality Reduction in Design of Optimal Filters", in *Proc. IEEE Int. Symp. Circuits and Systems ISCAS'2002*, vol. 5, pp. 717-720, 2002.

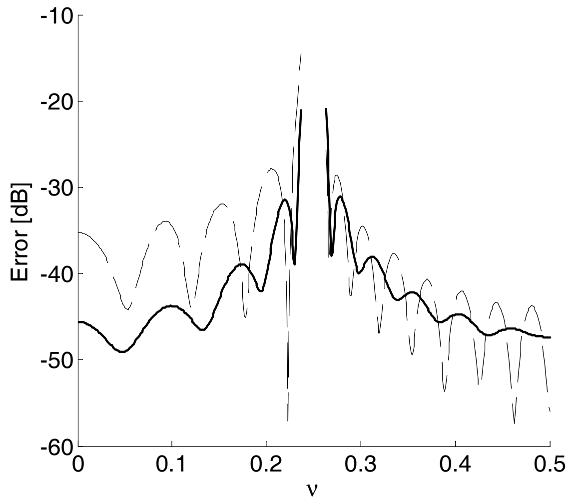

Figure 1: Weighted errors of filter frequency responses for the method proposed in [3] (thin broken line) and for the method proposed here (thick continuous line).