A NOVEL AND FAST 1-BIT FFT SCHEME WITH TWO DITHER-QUANTIZED CHANNELS

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ABSTRACT

The paramount importance enjoyed by the FFT algorithm and its variants is amply demonstrated by the plethora of applications it currently enjoys in a myriad of practical areas. As this algorithm is invariably digitally implemented, its computational accuracy relies on its two inputs having a sufficiently fine quantization. This precludes the use of a coarse quantization scheme for the 2 FFT inputs and the exploitation of all the concomitant and attractive practical advantages that this scheme would bring to the FFT application at hand. This paper proposes a new theory that resolves this conflict between exploiting these practical advantages and retaining an acceptable computational accuracy of the FFT. This theory is tested with the smallest possible quantization resolution (1-bit) at which all potential practical advantages are maximized. The simulation work, which includes both clean and noisy signals, corroborates the proposed theory quite well even in severely noisy environments.

1. INTRODUCTION

The wide applicability of the Fourier transform (FT) in a multitude of engineering fields such as signal and image processing, control, communications, filtering, geophysics, seismics, optics, acoustics, radar and sonar signal processing, clearly reflects the paramount importance of this tool in these and other areas.
The explosion in the application of the FT was ushered in by the introduction of the now-famous Fast Fourier transform (FFT) algorithm which revolutionized the way in which the classical discrete FT (DFT) was digitally computed [1]. Ever since its introduction, the original radix-2 underwent a series of modifications and adaptation to data lengths other than powers of 2. An interesting contemporary view and a review of the state of the art of the FFT can be found in [2] and [3], respectively. For an accurate computation of the FT, both the original (radix-2) FFT algorithm and all its variants rely on both their inputs and basis functions (sines and cosines) being highly quantized ($\geq$ 8 bits). A straightforward reduction of the quantization resolution of the input and/or the basis functions is bound to lead to an irrecoverable loss of computational accuracy. The main motivation behind the desire to reduce the quantizaiton resolution of the signals involved stems from the practical advantages that will accrue from such a reduction process. These advantages become maximal when the coarsest (i.e. 1-bit) quantization scheme is used, and include a simpler structure, a lower implementational cost and a higher computational speed. These advantages will in turn lead to a cost-effective fully parallel FFT estimation scheme, if needed.
The highly attractive 1-bit quantization scheme has already been successfully tested on a 1-channel (input) – quantized FFT estimator, termed the modified relay FFT (or MR-FFT), using both noise-free and noisy input [4,5]. In the MR-FFT scheme, only the input is 1-bit quantized. The success of the MR-FFT estimator relied on a frequency-domain extension of the exact moment recovery (EMR) theory [6] that is based on the non-substractively dithered quantization (NSDQ) scheme. This theory has also enjoyed other successful applications [7,8].
The objective of this paper is two-fold: (a) to extend the successful application of the 1-bit NSDQ scheme, used in the MR-FFT estimator and reported in [3,4], from one quantized-channel estimator (MR-FFT) to a 2 quantized-channel estimator, i.e. one, where both the input and the basis functions are 1-bit NSDQ-quantized and (b) to test the estimation robustness to different noisy environments. This new 2-channel 1-bit NSDQ-quantized FFT estimator will henceforth be referred to as the modified polarity coincidence FFT estimator (or MPC-FFT). A block diagram description of the MPC-FFT estimator is given in Fig. 1, where the measurement noise (*N(n)*), added to the FFT input *(x(n)),* is assumed to be uncorrelated with both *x(n)* and the 2 basis signals *s(n)* and *c(n)*.

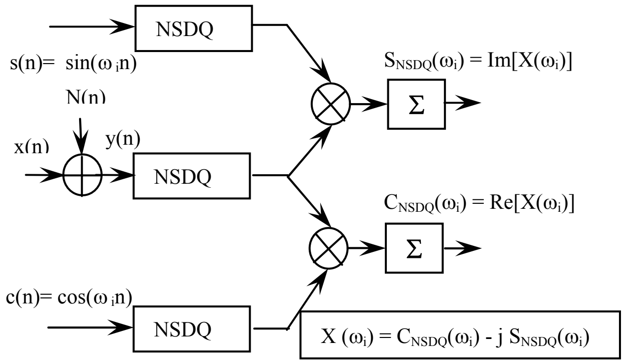

Fig. 1: MPC-FFT estimation scheme

2. RELEVANT RESULTS ON THE NEW 2-D EXACT RECOVERY THEORY OF THE DFT

In this section, we present some key results which are based on [6] and which include a new theorem on the moment-sense equivalence between the NSDQ quantization-based DFT and a frequency domain polynomial mapping.

2.1 Definition of the NSDQ quantization scheme

Given an input x and a (user-defined) dither signal D that is statistically independent of x, then a non-subtractively dithered quantization (NSDQ) of x is equivalent to the classical quantization (Q) of the dithered signal $y = x + D$, i.e.

$$x \rightarrow x_{NSDQ} = NSDQ(x) = Q_a(y) = y_Q \quad (1)$$

Here, Q_a represents the entire class of uniform classical quantizers parametrized by the step (q) and the shift factor given by $a \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, i.e.:

$$y_Q = \left(a + n + \frac{1}{2}\right) q \qquad if \quad y \in \left[(a+n)q, (a+n+1)q\right] \quad (2)$$

2.2 Definition of the p^th order class of linearizing dither signals $\mathcal{D}_p$

Given an ergodic and stationary dither signal D and its characteristic function $W_D(u)$, then:

$$D \in \mathrm{D_P} \Leftrightarrow W_D^{(r)}\left(\frac{2n\pi}{q}\right) = 0,\ \forall\, r \in [0, p-1]\ and\ n \neq 0 \quad (3)$$

According to the closure property of $\mathcal{D}_p$ [6], we can say that if $D \in \mathcal{D}_p$ and for any signal x that is statistically independent of D, then the dithered signal $y = (x + D) \in \mathcal{D}_p$.

2.3 Extension to, and Characterization of, the 2-D NSDQ quantization scheme

A 2-D vector NSDQ-quantizer is defined here as being made of 2 separate uniform (step q) scalar NSDQ quantizers, each of which is characterized by its own triplets of signals $\left(x_l, x_{NSDQ_l}, D_l\right)$ for l=1,2 where x_l, x_{NSDQ_l}, *and* D_l represent the input, output and dither signals of the l^{th} scalar quantizer, respectively, and the 2 zero-mean dither signals D_1 and D_2 are assumed to be statistically independent of each other and of x_1 and x_2.

Furthermore, each scalar NSDQ quantizer is statistically characterized by its own p^th order moment-sense input/output function (MSIOF) given by:

$$h_{p_i}(x_i) = \sum_{k_i=0}^{p_i} c_{k_i}\, x_i^{k_i} \quad (4)$$

$$c_{k_i} = \sum_{t_i=0}^{p_i-k_i} \frac{p_i!}{(p_i - k_i - t_i + 1)!\; k_i!\, t_i!}\left(\frac{q}{2}\right)^{p_i-k_i-t_i} \cdot E\left[R^{t_i}\right]\left[p_i \oplus k_i \oplus t_i \oplus 1\right]$$

where $\oplus$ denotes modulo-2 operation and i=1,2 .

Note here that, because D_1 is statistically independent of D_2, the 2-D MSIOF becomes separable, i.e: $h_{p_1,p_2}(x_1, x_2) = h_{p_1}(x_1).h_{p_2}(x_2)$

2.4 A Key Theorem and its application to the Exact-recovery of the MPC-FFT

Theorem 1:

Given a 2-D vector NSDQ quantizer, characterized by its 2 signal triplets $\left(x_l, x_{NSDQ_l}, D_l\right)$, l=1,2, and its 2-D (p_1, p_2)-th order MSIOF, $h_{p_1,p_2}(x_1, x_2)$, and given that for $i \in [1, N]$, $X_{1k}(\omega_i)$ and $H_{p_1,p_2}(\omega_1, \omega_2)$ are the DFTs of the signal x_1^k and the MSIOF $h_{p_1,p_2}(x_1, x_2)$, respectively, and that $X_{NSDQ_{p_1}}(\omega_i) \underline{\Delta} \sum_{n=0}^{N-1} x_{NSDQ_1}^{p_1}(n) \cdot k_{NSDQ}(n)$, where $k_{NSDQ}(n)$ is the NSDQ-quantized Fourier kernel $k(n) = e^{-j\omega_i n}$, to be used as the second input $x_2(n)$ of the vector NSDQ-quantizer, then, for any quantization resolution (q) and for any finite order p_1, the fully NSDQ-quantized DFT, $X_{NSDQ}(\omega_i)$, is statistically (moment-sense) equivalent to a frequency-domain p_1-th order polynomial mapping $H_{p_1}(\omega_i)$ involving only lower-order (i.e. k ≤ p_1) DFTs of the NSDQ-quantizer input $x_1(n)$, i.e

$$E\left[X_{NSDQ_{p_1}}(\omega_i)\right] = E\left[H_{p_1}(\omega_i)\right] \quad \forall\, i \in [1, N]\ and\ \forall q$$

where

$$E\left[H_{p_1}(\omega_i)\right] \underline{\Delta} \sum_{n=0}^{N-1} h_{p_1}(x_1(n)) \cdot k(n) = DFT\left[h_{p_1}(x_1(n))\right] = \sum_{k=0}^{p_1} c_k\, E\left[X_{1k}(\omega_i)\right] \quad (5)$$

with c_k as given above in (4).

2.5 Application of Theorem 1 to the accurate estimation of the DFT of a noisy signal from its 1-bit NSDQ-quantized counterpart

Let the signal of interest be a noisy one, i.e $y(n) = x(n) + N(n)$ where $x(n)$ is the noise-free input and $N(n)$ is the measurement noise assumed to be statistically independent of $x(n)$ and of the two dither signals D_1 and D_2. If we let $p_1 = 1$ and $x_1(n) = y(n)$, it can be readily shown from (5) that: $H_1(\omega_i) = Y(\omega_i)$ and hence $E[Y_{NSDQ}(\omega_i)] = E[Y(\omega_i)] \quad \forall\, i \in [1, N]$. Moreover since $N(n)$ is zero-mean, it follows that $E[N(\omega_i)] = 0 \quad \forall\, i \in [1, N]$. Note also that $E[X(\omega_i)] = X(\omega_i)$ if $x(n)$ is a deterministic signal. In the simulation work, since $k(n)$ is complex, two 2-D NSDQ quantizers, with one common channel, have to be used: one called the real part quantizer driven by $c(n) = \mathrm{Re}[k(n)] = \cos\omega_i n$ and $y(n)$, and the other, termed the imaginary part quantizer, is driven by $s(n) = \mathrm{Im}[k(n)] = \sin\omega_i n$ and $y(n)$. In a sense, one quantizer was used to estimate the cosine transform of $x(n)$ and the other its sine transform. Combining the estimates of both transforms then leads to the estimate of the DFT of the noise-free signal $x(n)$.

3. SIMULATION RESULTS

To test the above-mentioned theoretical developments, two cases were considered. In the first case, the MPC-FFT estimator was used to estimate the FFT of a clean and a noisy single sinusoidal signal. In the second case, the MPC-FFT estimator was utilized to estimate a clean and a noisy multisine signal. Note here that a multisine signal is representative of the general class of complex signals. A description of the simulation work now follows.

A sinusoidal signal of amplitude A=10 and frequency f=1000 Hz was used as the input signal $x(n)$. A total of 6400 points were used for the estimation of the FFT magnitude spectrum. Fig. 2 shows the amplitude spectra of the original (i.e. non-quantized) signal and a non-dithered 1-bit quantized signal. It can be seen that the two spectra are quite different, as non-negligible signal peaks at frequencies other than the test frequency are present in the non-dithered quantized spectrum. Also at the test frequency, the relative error in the FFT magnitude spectrum is around 35%. In the second simulation, a zero-mean signal of peak-to-peak amplitude equal to that of the input signal, is added to the input and their sum is 1-bit quantized. Both the sine and the cosine basis functions are also 1-bit NSDQ-quantized. Then the FFT magnitude spectrum is evaluated using the proposed scheme. The results in Fig. 3 clearly demonstrate that the MPC-FFT estimator has not only fully recovered the FFT magnitude spectrum, with an error in the range of 5%, but has also replaced the structured harmonics-related error by a random-like error pattern. The magnitude of this random error is almost 5% of the peak magnitude spectrum. To test the robustness of the proposed scheme, we have also estimated the FFT-spectrum of a noisy signal, with a SNR of 15dB. The results, shown in Fig. 4, clearly show that a relative estimation error of less than 10% was achieved for the FFT magnitude spectrum at the test frequency.

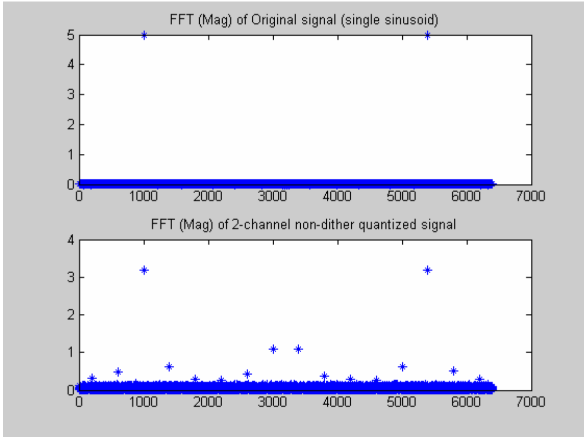

Fig. 2: Magnitude response of clean, undithered and unquantized single sine

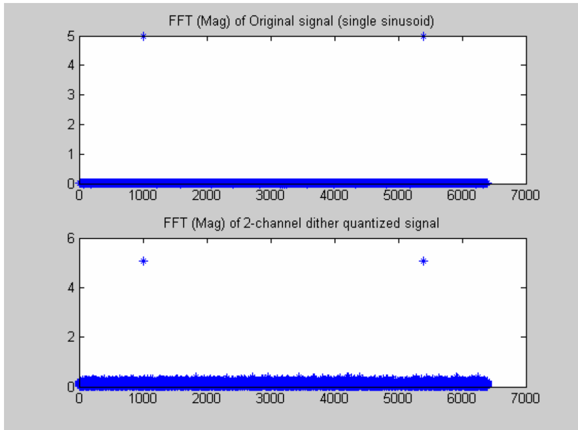

Fig. 3: A clean 1-bit NSDQ-quantized single sine

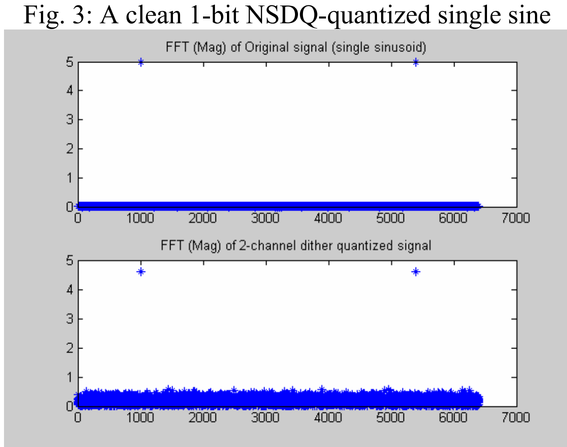

Fig. 4: A noisy (15dB) 1-bit NSDQ-quantized single sine

A multi-sine input, made of 3-sinusoidal signals of amplitudes 10,9 and 8 at frequencies 200, 300 and 500 Hz, respectively, is now considered. The simulation results (Fig. 5) show a small error in estimating the FFT magnitude spectrum when a noise-free input is used. However, when a noisy input is used with a SNR of 15dB (Fig. 6), the estimation error increases to about 10%. Moreover, the peak amplitude of the random-like pattern at the non-test frequencies also increases to about 20% of the peak value in the FFT magnitude spectrum. Processing a larger number of samples will lead to some reduction in this relative error and in the level of the noise floor too.

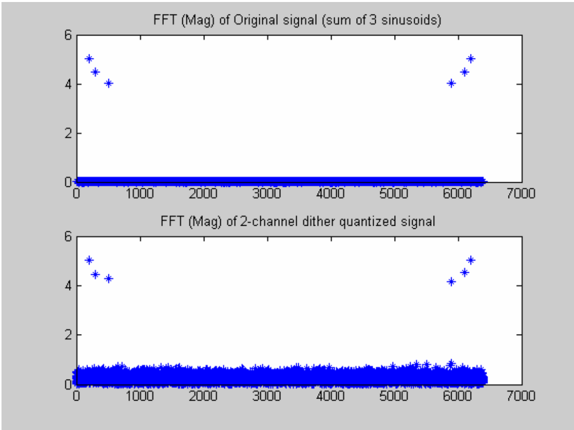

Fig. 5: FFT amplitude spectrum of a clean multisine input

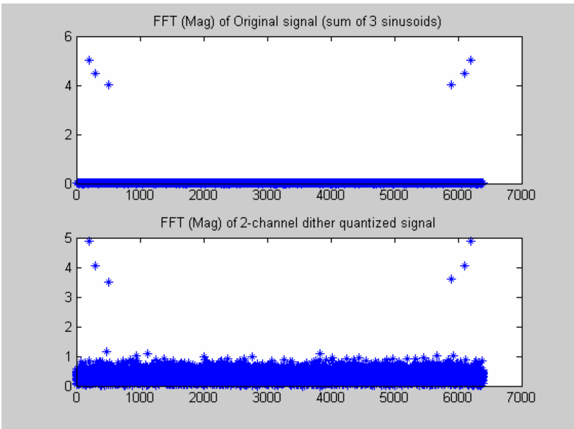

Fig. 6: FFT amplitude spectrum of a noisy multisine input (SNR =15 dB)

4. CONCLUSIONS

In this paper, a new 2-channel NSDQ-quantized estimator of the FFT magnitude spectrum is introduced. Its operation hinges upon a new theorem derived as an extension of the earlier-derived exact moment recovery theory which was itself successfully applied to other areas. The proposed fast dither-quantized FFT estimator was rigorously tested using the coarsest possible quantization (1-bit) scheme and in both clean and noisy environments. The simulation results are in very good agreement with the proposed theory. This fact, coupled with the simple 1-bit structure of the proposed estimator, makes a VLSI implementation of this estimator both a desirable and a tangible goal.

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