RECONSTRUCTION OF NONUNIFORMLY SAMPLED BANDLIMITED SIGNALS USING TIME-VARYING DISCRETE-TIME FIR FILTERS

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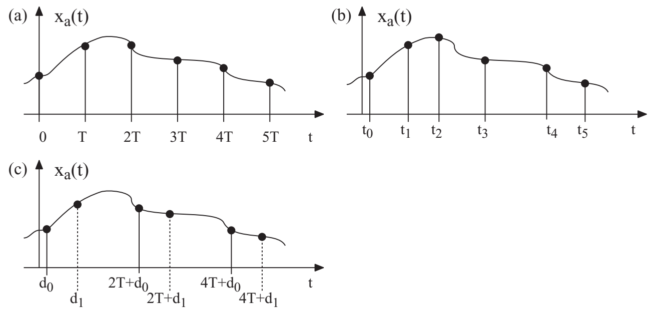
Abstract – This paper deals with reconstruction of nonuniformly sampled bandlimited continuous-time signals using time-varying discrete-time FIR filters. The points of departures are that the signal is slightly oversampled as to the average sampling frequency and that the sampling instances are known. Under these assumptions, a representation of the reconstructed sequence is derived that utilizes a time-frequency function. This representation enables a proper utilization of the oversampling and reduces the reconstruction problem to a design problem that resembles an ordinary filter design problem. Furthermore, for an important special case, corresponding to a certain type of periodic nonuniform sampling, it is shown that the reconstruction problem can be posed as a filter-bank design problem, thus with requirements on a distortion transfer function and a number of aliasing transfer functions.



igure 1. (a) Uniform sampling. (b) Nonuniform sampling. c) Periodic nonuniform sampling.

1. INTRODUCTION

Nonuniform sampling occurs in many practical applications either intentionally or unintentionally [1]. An example of the latter is found in time-interleaved analog-to-digital converters (ADCs) where static time-skew errors between the different subconverters give rise to a class of periodic nonuniform sampling [2], see Fig. 1(c).

Regardless whether the continuous-time (CT) signal, say $x_a(t)$, has been sampled uniformly [Fig. 1(a)], producing the sequence $x(n) = x_a(nT)$ or nonuniformly [Fig. 1(b)], producing the sequence $x_1(n) = x_a(t_n)$, it is often desired to reconstruct $x_a(t)$ from the generated sequence of numbers. Thus, in the nonuniform-sampling case, it is desired to retain $x_a(t)$ from the sequence $x_1(n)$. This can, in principle, be done in two different ways. The first way is to reconstruct $x_a(t)$ directly from $x_1(n)$ through CT reconstruction functions. Although it is known how to do this in principle (see e.g. [3]–[7]), problems arise when it comes to practical implementations. In particular, it is very difficult to practically implement CT functions with high precision. It is therefore desired to do the reconstruction in the digital domain, i.e., to first recover $x(n)$. One then needs only one conventional digital-to-analog converter (DAC) and a CT filter to obtain $x_a(t)$, which are much easier to implement than general complicated CT functions.

Recovering $x(n)$ from $x_1(n)$ in the digital domain can in principle be done by digital reconstruction functions obtained through sampling of a corresponding CT reconstruction function. However, the CT reconstruction functions are generally noncausal (two-sided) functions which therefore must be truncated in order to make the corresponding digital reconstruction system practically implementable. This truncation causes reconstruction errors that are not easily controlled and one should therefore seek for other techniques. One faces a similar situation when designing digital filters. It is well known that filters designed through truncation of infinite-length impulse responses (referred to as windowing techniques) exhibits large errors (referred to as Gibb's phenomenon) in the frequency domain around the cut-off frequency. Although different weights (windows) can improve the situation, it is recognized that other design methods are usually preferred [8]. When designing filters, one does normally not attempt to approximate a desired function over the whole frequency range $|\omega T| \le \pi$, ωT being the frequency variable. Instead, one allows certain region(s) referred to as the transition region(s) where no requirements are stated. In this way, one can in the remaining frequency regions fully control the filter performance. It is therefore reasonable to take the same action when reconstructing nonuniformly sampled signals. That is, instead of approximating perfect reconstruction (PR) in the whole frequency range $|\omega T| \le \pi$, in which case one is doomed to face problems, one should a priori assume a small oversampling factor in which case PR needs to be approximated only in the region $|\omega T| \le \omega_0 T < \pi$. In this way, one can fully control the reconstruction by properly designing the reconstruction system. For example, the method introduced in [9] employs causal interpolation functions and it was observed experimentally that the reconstruction deteriorates when the bandwidth approaches π, which further motivates the benefits of allowing a slight oversampling.

Although oversampling itself is undesired since it generates more samples than necessary for reconstruction, according to the Nyquist sampling theorem, it is known that a slight oversamling is required for practical implementation of conventional ADCs and DACs. It is therefore conjectured that a slight oversampling is also required in order to practically implement a device that reconstructs nonuniformly sampled signals. This is of course not a new paradigm but it seems that it is often unsatisfactorily handled when it comes to analysis, design, and implementation of practical reconstruction algorithms.

This paper deals with reconstruction through time-varying FIR filters. The point of departure is a bandlimited CT signal that is nonuniformly sampled and slightly oversampled as to the average sampling frequency, the reason for the latter assumption being as outlined above. It is further assumed that the sampling instances are known. Under these assumptions, a representation of the reconstructed sequence is derived that utilizes a time-frequency function. This representation enables a proper utilization of the oversampling and reduces the reconstruction problem to a design problem that resembles an ordinary filter design problem. Furthermore, in an important special case, corresponding to a certain type of periodic nonuniform sampling, it is shown that the design problem can be posed as a filter-bank design problem, thus with requirements on a distortion transfer function (that should approximate one) and a number of aliasing transfer functions (that should approximate zero).

A FB formulation of the problem has been done earlier in [7], but there is a major difference between the formulation in that paper and the one in this paper. In [7], it is observed that the nonuniformly sampled signal can be expressed with the aid of a regular decimated analysis filter bank with analysis filter frequency responses that are *fixed* and determined by the sampling instances. It is then shown that $x(n)$ can be retained using a synthesis filter bank with ideal non-causal multilevel filters. The issue of using practical causal synthesis filters approximating the ideal ones was not treated though. That is, it is not known how well a “practical version” of that solution will behave. In this paper, the FB formulation contains adjustable analysis filters but fixed and trivial synthesis filters (pure delays). Here, the problem is thus to properly design the analysis filters which, for many sampling patterns, can be done as good as desired with an acceptable filter order, due to the fact that a slight oversampling is used. Oversampling is not utilized in [7] which means that one most likely will face problems when designing practical causal synthesis filters approximating the ideal multilevel filters. Oversampling was utilized in the special class of synthesis FB proposed in [10], in which case PR can be approximated as close as desired.

Following this introduction, the paper first recapitulates uniform sampling in Section 2, the reason being that the reconstruction here aims at retaining $x(n)$ from $x_1(n)$, not $x_a(t)$ directly. Section 3 considers nonuniform sampling and reconstruction using time-varying FIR filters. Section 4 studies the special case of periodic nonuniform sampling and shows how the design problem can be posed as a FB design problem. Finally, Section 5 concludes the paper.

2. UNIFORM SAMPLING

In uniform sampling, the sequence $x(n)$ is obtained by sampling the CT input signal $x_a(t)$ uniformly at the time instances nT, for all integers n, i.e.,

$$x(n) = x_a(nT), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{1}$$

where T is the sampling period and $f_{\text{sample}} = 1/T$ is the sampling frequency. The Fourier transforms of $x(n)$ and $x_a(t)$ are related according to Poisson’s summation formula as

$$X(e^{j\omega T}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\omega - j\frac{2\pi r}{T}\right) \tag{2}$$

Since the spectrum of $x(n)$ is periodic with a period of 2π (2π-periodic) with respect to ωT, it suffices to consider $X(e^{j\omega T})$ in the interval $-\pi \le \omega T \le \pi$. Throughout this paper, it is assumed that $x_a(t)$ is bandlimited according to

$$X_a(j\omega) = 0, \quad 0 < \omega_0 \le |\omega|, \quad \omega_0 \le \pi/T \tag{3}$$

(see also Fig. 2(a)). That is, the Nyquist criterion for sampling with a sampling frequency of $1/T$ without aliasing is fulfilled. In this case,

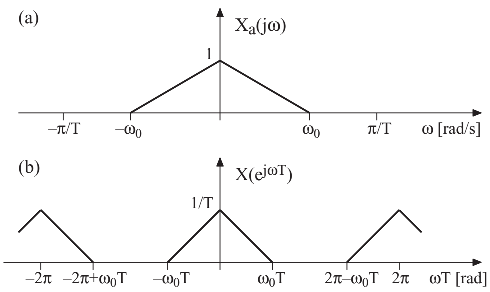


Figure 2. Spectra of a bandlimited signal $x_a(t)$ and the sequence $x(n) = x_a(nT)$. (Uniform sampling).

$$X(e^{j\omega T}) = \frac{1}{T}X_a(j\omega), \quad -\pi \le \omega T \le \pi \tag{4}$$

(see also Fig. 2(b)). Equation (4) implies that $x_a(t)$ can be recovered from $x(n)$. In practice, this is done using a DAC followed by an CT reconstruction filter. We also note that $x_a(t)$ is oversampled unless $\omega_0 = \pi/T$.

3. NONUNIFORM SAMPLING

Throughout this paper, it is assumed that the nonuniform sampling of the CT signal $x_a(t)$ is done in such a way that the so obtained sequence, say $x_1(n)$, is given by

$$x_1(n) = x_a(t_n) \tag{5}$$

where

$$t_n = nT + \varepsilon_n T \tag{6}$$

with $\varepsilon_n T$ representing the distance between the “nonuniform sampling instance” t_n and the “uniform sampling instance” nT. The average sampling frequency is thus still $1/T$. It is also assumed that the sampling instances are distinct, i.e., $t_n \ne t_m$, $n \ne m$, and that $t_n < t_m$, $n < m$.

Given $x_1(n)$, a new sequence, $y(n)$, is formed through some reconstruction formula. It is desired to achieve $y(n) = x(n)$ because, then, $x_a(t)$ can due to (4) be recovered using conventional reconstruction methods for uniformly sampled signals. The equality $y(n) = x(n)$ corresponds in the frequency domain to $Y(e^{j\omega T}) = X(e^{j\omega T})$. If these equations hold, the reconstruction system is said to be a *perfect reconstruction system*.

In this paper, the reconstruction is performed using a time-varying FIR filter characterized by the impulse responses $h_n(k)$. It is assumed here that the order of the FIR filter is $2N$ and thus even. It is convenient and possible to let the FIR filter be noncausal which implies that $h_n(k)$ in this even-order case is non-zero for $k = -N, -N+1, \ldots, N$. In a practical implementation, the corresponding causal filter is obtained by simply introducing a delay of N samples. In the odd-order case, one has instead $k = -N, -N+1, \ldots, N-1$ or $k = -N+1, -N+1, \ldots, N$, but that will not change the principles dealt with in this paper. Henceforth, only the even-order case is therefore considered for the sake of simplicity.

Under the above assumptions, $y(n)$ is now formed according to

$$y(n) = \sum_{k=-N}^{N} x_1(n-k)h_n(k) \tag{7}$$

It is desired to select $h_n(k)$ so that $y(n)$ approximates $x(n)$

as close as possible (in some sense). To see how to choose $h_n(k)$, $x_1(n)$ is first written in terms of the inverse Fourier transform of $x_a(t)$ by which we obtain, due to (3), (5) and (6),

$$x_1(n) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega T\varepsilon_n}X(j\omega)e^{j\omega Tn}d\omega \tag{8}$$

Inserting (8) into (7), and interchanging the summation and integration, one obtains

$$y(n) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} H_n(e^{j\omega T})X(j\omega)e^{j\omega Tn}d\omega \tag{9}$$

where

$$H_n(e^{j\omega T}) = \sum_{k=-N}^{N} h_n(k)e^{-j\omega T(k-\varepsilon_{n-k})} \tag{10}$$

When (4) holds, (9) can equivalently can be written as

$$y(n) = \frac{1}{2\pi}\int_{-\omega_0 T}^{\omega_0 T} H_n(e^{j\omega T})X(e^{j\omega T})e^{j\omega Tn}d(\omega T) \tag{11}$$

Equation (11) represents $y(n)$ with the aid of the functions $H_n(e^{j\omega T})$ which can be viewed either as an infinite set of frequency functions or one time-frequency function. Further, $x(n)$ can be expressed in terms of its inverse Fourier transform according to

$$x(n) = \frac{1}{2\pi}\int_{-\omega_0 T}^{\omega_0 T} X(e^{j\omega T})e^{j\omega Tn}d(\omega T) \tag{12}$$

Comparing (11) with (12), it is seen that *perfect reconstruction* is obtained if

$$H_n(e^{j\omega T}) = 1, \quad \omega T \in [-\omega_0 T, \omega_0 T] \tag{13}$$

for all n.

Defining the error $e(n)$ as $e(n) = y(n)–x(n)$ one obtains from (11) and (12) that

$$e(n) = \frac{1}{2\pi}\int_{-\omega_0 T}^{\omega_0 T} (H_n(e^{j\omega T})-1)X(e^{j\omega T})e^{j\omega Tn}d(\omega T) \tag{14}$$

Apparently, $e(n) = 0$ in the PR case since, then, $H_n(e^{j\omega T}) = 1$. In practice, $H_n(e^{j\omega T})$ can only approximate one in the frequency range of interest. The goal is then to determine the coefficients $h_n(k)$ so that the error $e(n)$ is minimized according to some criterion. A problem is that $e(n)$ does not only depend on $h_n(k)$ but also on $X(e^{j\omega T})$ which means that one generally must have knowledge about the input signal spectrum in order to determine $h_n(k)$ in the best possible way. If one does not have complete knowledge about $X(e^{j\omega T})$, which often is the case in practice, one has to accept a suboptimum solution instead.

The simplest way to obtain a suboptimum solution is to determine each $h_n(k)$ separately so that each $H_n(e^{j\omega T})$ in (10) approximates one in the frequency region $\omega T \in [-\omega_0 T, \omega_0 T]$, $\omega_0 T < \pi$, as good as possible according to some criteria. The rationale behind this is that, regardless of $X(e^{j\omega T})$, one can generally say that the closer $H_n(e^{j\omega T})$ is to one the closer $e(n)$ is to zero, with "close" being interpreted in a wide sense. By utilizing the representation of $y(n)$ in (11), this design problem resembles an ordinary filter design problem. One difference is however that the functions $H_n(e^{j\omega T})$ are unconventional in the sense that they make use of non-integer delays [see (10)]. (The non-integer delays are merely a consequence of the nonuniform sampling and the problem formulation; thus, they are not actually implemented which of course would cause problems.) Another difference is, of course, that a new design has to be done for each n. However, in an important special case, where the sampling is periodically nonuniform, it suffices to design only a few filters. In this case, the design problem can be conveniently posed as a FB design problem which eases the design and analysis. This is the topic of the next section.

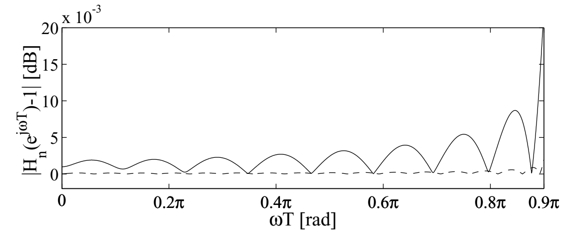


Figure 3. Illustration (for one value of n) of reduced reconstruction error when the filter order $2N$ increases. Solid line: $2N = 16$. Dashed line: $2N = 32$.

Figure 3 illustrates that the reconstruction error reduces as the filter order increases. The sampling instances are here as in (6), with $\varepsilon_n T$ being randomly chosen numbers in the interval $(–0.5T, 0.5T)$[1], whereas $\omega_0 T = 0.9\pi$. The filters have been designed using a least-squares approach where each $h_n(k)$ is analytically computed through matrix inversion. Due to the limited space, design details are omitted in this paper but will instead be treated in another paper.

4. PERIODIC NONUNIFORM SAMPLING

An important special case of nonuniform sampling occurs when the distances between the "nonuniform sampling instance" $\varepsilon_n T$ exhibits periodicity according to

$$\varepsilon_n T = \varepsilon_{n+M}T \tag{15}$$

for all n, with M being the period. In this case, the sampling is said to be *periodic nonunifom sampling* with period M. An example is given in Fig. 1(c) with period $M = 2$, $d_0 = \varepsilon_0 T = \varepsilon_{2m}T$ and $d_1 = \varepsilon_1 T = \varepsilon_{2m+1}T$, for all integers m.

When $\varepsilon_n T$ satisfies (15), it is obvious that $h_n(k)$ and $H_n(e^{j\omega T})$ exhibits the same periodicity, i.e.,

$$h_n(k) = h_{n+M}(k) \tag{16}$$

and

$$H_n(e^{j\omega T}) = H_{n+M}(e^{j\omega T}) \tag{17}$$

for all n. This makes it possible to pose the design problem as a FB design problem. To see this, consider first the case where the output $y(n)$ is obtained by applying $x(n)$ to a time-

1. In the literature, one sometimes finds the assertion that $|\varepsilon_n T|<0.25T$ must be fulfilled to enable reconstruction but this is generally not required. In particular, when a slight oversampling is allowed, one circumvents many of the problems associated with reconstruction of critically sampled signals.

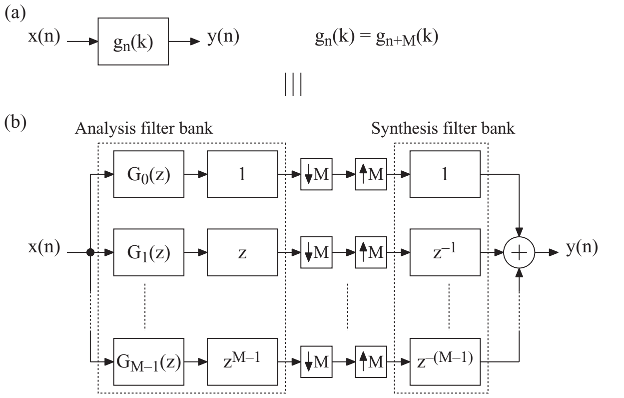


Figure 4. (a) Time-varying filter. (b) Equivalent FB representation.

varying filter described by the impulse responses $g_n(k)$, thus

$$y(n) = \sum_{k=-N}^{N} x(n-k)g_n(k) \tag{18}$$

Assume now that $g_n(k)$ satisfies $g_n(k) = g_{n+M}(k)$. By making use of the properties of downsamplers and upsamplers [11], one can in this case readily establish that $y(n)$ in (18), and thus Fig. 4(a), is identical to the output $y(n)$ in the maximally decimated FB shown in Fig. 4(b). That is, the output is obtained as the output of a maximally decimated FB with the analysis filters $z^nG_n(z)$, with $G_n(z)$ being the transfer function of $g_n(k)$, and with the trivial synthesis filters $F_n(z) = z^{-n}$. In the frequency domain, the relation between the input and output can therefore be expressed as [11]

$$Y(e^{j\omega T}) = \sum_{m=0}^{M-1} V_m(e^{j\omega T})X(e^{j(\omega T - 2\pi m/M)}) \tag{19}$$

where

$$V_m(e^{j\omega T}) = \frac{1}{M}\sum_{n=0}^{M-1} e^{-j2\pi mn/M}G_n(e^{j(\omega T - 2\pi m/M)}) \tag{20}$$

with $G_n(e^{j\omega T})$ being the Fourier transform of $g_n(k)$, thus satisfying

$$G_n(e^{j\omega T}) = G_{n+M}(e^{j\omega T}) \tag{21}$$

for all n. The term $V_0(e^{j\omega T})$ is the distortion function whereas the remaining $V_m(e^{j\omega T})$, m = 1, 2, ..., M–1, are aliasing functions. Perfect reconstruction is obtained when $V_0(e^{j\omega T}) = 1$ and $V_m(e^{j\omega T}) = 0$, m = 1, 2, ..., M–1.

Further, utilizing the inverse Fourier transform, the output $y(n)$ in Fig. 4(b) can be expressed as

$$y(n) = \frac{1}{2\pi}\int_{-\omega_0 T}^{\omega_0 T} G_n(e^{j\omega T})X(e^{j\omega T})e^{j\omega Tn}d(\omega T) \tag{22}$$

Comparing (22) with (11), it is seen that the outputs in the two cases are identical provided that

$$G_n(e^{j\omega T}) = H_n(e^{j\omega T}) \tag{23}$$

Hence, when the sampling is periodically nonuniform, the reconstruction using time-varying filters can be conveniently represented by the FB in Fig. 4(b) with analysis filters as given by (23) and (10). The design problem can in this case be posed as a FB design problem where the goal is to determine the M impulse responses $h_n(k)$, n = 0, 1, ..., M–1, so that $V_0(e^{j\omega T})$ and $V_m(e^{j\omega T}) = 0$, m = 1, 2, ..., M–1 approximate one and zero, respectively, as good as desired according to some criteria. Here, the problem is thus to properly design the analysis filters since the synthesis filters are fixed to pure delays. As already discussed in the introduction, this is different from the FB formulation in [7] where the analysis filters are fixed and the synthesis filters are to be designed. Another difference is, as mentioned earlier, that the analysis filters are here unconventional in the sense that they make use of non-integer delays [see (10)].

Finally, it is stressed that the FB in Fig. 4(b) is used here only with the purpose of easing the analysis and design of the reconstructing system. That is, $y(n)$ is not obtained by implementing the FB which is obvious because that assumes that we already have available the "uniform samples" $x(n)$ which are precisely the samples we want to recover from the "nonuniform samples" $x_1(n)$. The output $y(n)$ is of course still obtained from (7) whereas the FB in Fig. 4(b) is a convenient way of representing the reconstruction in the case of periodic nonuniform sampling.

5. CONCLUDING REMARKS

The representation of the reconstructed sequence derived in this paper enables a proper utilization of the oversampling assumed and reduces the reconstruction problem to a design problem that resembles an ordinary filter design problem. This representation is thus a useful starting point for the analysis and development of design techniques of different classes of input signals and nonuniform sampling patterns. Due to the limited space, design issues were not included in this paper but will be considered in future papers. However, for one type of sampling pattern, it was illustrated by means of an example how the reconstruction error reduces as the filter order increases, using a least-square design technique, details of which will be published elsewhere.

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