ML FREQUENCY OFFSET AND CARRIER PHASE ESTIMATION IN OFDM SYSTEMS WITH NONCIRCULAR TRANSMISSIONS

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ABSTRACT

In this paper the problem of blind frequency offset and carrier phase estimation in orthogonal frequency-division multiplexing (OFDM) systems with noncircular transmissions is considered. In this case, if the number of subcarriers is sufficiently large, the OFDM signal becomes an improper complex Gaussian process. By exploiting the joint probability density function for improper (or noncircular) complex Gaussian random vectors maximum likelihood estimators for the parameters of interest are derived, and moreover, the Cramér-Rao lower bound is evaluated.

Keywords: Frequency offset and carrier phase recovery, OFDM systems, noncircular transmissions.

1. INTRODUCTION

In [1] blind maximum-likelihood (ML) estimation algorithms for frequency offset and carrier phase, separately, have been derived. Specifically, these algorithms have been obtained by maximizing the low signal-to-noise ratio (SNR) limit of the likelihood function averaged over the symbols transmitted on all the subcarriers.

However, when the number of subcarriers is sufficiently high, the OFDM signal can be modelled as a complex Gaussian process [2]. Moreover, if a noncircular signal constellation is adopted (i.e., if the mathematical expectation of the squared symbol is not equal to zero), the OFDM signal becomes an improper [3] complex random process since its pseudoautocorrelation function is different from zero [4].

In this paper it is shown that in the case of noncircular transmissions the frequency offset and the carrier phase can be jointly estimated. Specifically, by exploiting the joint probability density function (PDF) for improper (or noncircular) complex Gaussian random vectors proposed in [5], the joint ML estimator for the parameters of interest is derived. The performance of the proposed estimator, is assessed via computer simulations and compared with the Cramér-Rao lower bound.

2. SIGNAL MODEL

In the considered OFDM system the received signal is modeled by

$$r(k) = s(k)e^{j[\frac{2\pi}{N}\varepsilon k+\phi]} + n(k) \tag{1}$$

where $s(k)$ is the OFDM signal with power $\sigma_s^2 = E[|s(k)|^2]$, ε is the frequency offset normalized to the intercarrier spacing and ϕ is the carrier phase. Furthermore, $n(k)$ is proper complex white Gaussian noise with power $\sigma_n^2 = E[|n(k)|^2]$ and statistically independent of $s(k)$. The OFDM signal $s(k)$ can be written as

$$s(k) = \sigma_s \sum_{p=-\infty}^{\infty} g_p(k-pM) \tag{2}$$

with

$$g_p(q) = \begin{cases} f_p^{q+N-L}, & q = 0,1,\ldots,L-1, \\ f_p^{q-L}, & q = L, L+1,\ldots,M-1, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where N is number of subcarriers, L is prefix length and $M = N+L$ represents the effective length of the OFDM symbol. The f_p^vs are the inverse discrete Fourier transform (IDFT) of the data sequence given by

$$f_p^v = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} a_p^l e^{j\frac{2\pi}{N}lv}, \qquad v = 0,1,...,N-1, \tag{4}$$

where a_p^l denotes the symbol transmitted on the lth subcarrier during the pth OFDM symbol. It is assumed that the number of subcarriers N is sufficiently large so that the OFDM signal $s(k)$ can be modelled as a complex Gaussian process. Moreover, it is assumed that the data symbols a_p^l are statistically independent and identically distributed random variables, with zero-mean and $E[|\, a_p^l \,|^2] = 1$, belonging to a noncircular constellation with $E[(a_p^l)^2] = b \neq 0$. Noncircular constellations of particular interest are those with real data symbols. Moreover, new noncircular constellations have been also recently proposed in [6].

3. LIKELIHOOD FUNCTION

In this section the correlation and pseudocorrelation properties of the received signal are analyzed to derive the log-likelihood function for ε and ϕ.

Let $\boldsymbol{r} = [r(0)\cdots r(N+L-1)]^T$ the column vector containing $N+L$ samples within one OFDM symbol, it results that the samples in the cyclic prefix and their copies are pairwise correlated, i.e.,

$$E[r(k)r^*(m)] = \begin{cases} \sigma_s^2+\sigma_n^2, & m=k, & (5.1) \\ \sigma_s^2 e^{-j2\pi\varepsilon}, & k\in\tau_1, \quad m=k+N, & (5.2) \\ 0, & \text{otherwise}, & (5.3) \end{cases}$$

where the superscript * denotes complex conjugation and $\tau_1 \triangleq \{0,\ldots,L-1\}$. Moreover, due to the assumption of noncircular transmissions (i.e., $E[(a_p^l)^2] = b \neq 0$), for $N > 2L$ and $N \geq 2$, it follows that

$$E[r(k)r(m)] = \begin{cases} \gamma\sigma_s^2, & k\in\tau_1,\ m=2L-k, & (6.1) \\ \gamma\sigma_s^2, & k=m=L, & (6.2) \\ \gamma\sigma_s^2 e^{j2\pi\varepsilon}, & k\in\tau_2, \quad m=2L+N-k, & (6.3) \\ 0, & \text{otherwise}, & (6.4) \end{cases}$$

where $\tau_2 \triangleq \{L+1,\ldots,L+N-1\}$ and

$$\gamma \triangleq b\exp\left\{j\left[\frac{4\pi}{N}\varepsilon L+2\phi\right]\right\}. \tag{7}$$

Thus, in this case the complex Gaussian process modelling the OFDM signal becomes an improper [3] random process, since its pseudoautocorrelation is different from zero [4].

Let us note that, accounting for the correlation and pseudocorrelation properties of the observations $\boldsymbol{r}$ (assuming that N is even) the set of the received samples $r(k)$, $k\in\{0,1,\ldots,N+L-1\}$, can be decomposed in four subsets of random variables such that elements belonging to different subsets are statistically independent of each other. Specifically, for $k\in\tau_1$, in virtue of (5.2), (6.1) and (6.3), the samples $r(k)$, $r(2L-k)$ and $r(N+k)$ are mutually correlated noncircular complex Gaussian random variables. Moreover, from (6.3) follows that, for $k\in\{2L+1,\ldots,L+N/2-1\}$, the samples $r(k)$ and $r(2L+N-k)$ are pairwise correlated noncircular complex Gaussian random variables. Finally, from (6.2) and (6.3) follows that the samples $r(L)$ and $r(N/2+L)$, belonging to the third and fourth subset, respectively, are noncircular complex Gaussian random variables. Therefore, the log-likelihood function for ε and ϕ can be written as

$$\begin{aligned}
\Lambda(\varepsilon,\phi) &= \log f(\boldsymbol{r}\mid\varepsilon,\phi)\\
&= \log\left[\frac{f(\boldsymbol{r}\mid\varepsilon,\phi)\displaystyle\prod_{k=0}^{N+L-1}\frac{1}{\pi(\sigma_s^2+\sigma_n^2)}\exp\left[-\frac{|r(k)|^2}{\sigma_s^2+\sigma_n^2}\right]}{\displaystyle\prod_{k=0}^{N+L-1}\frac{1}{\pi(\sigma_s^2+\sigma_n^2)}\exp\left[-\frac{|r(k)|^2}{\sigma_s^2+\sigma_n^2}\right]}\right]\\
&= c_1+\log\left[\prod_{k=0}^{L-1}\frac{f(r(k),r(2L-k),r(k+N))}{\exp\left[-\frac{|r(k)|^2+|r(2L-k)|^2+|r(k+N)|^2}{\sigma_s^2+\sigma_n^2}\right]}\right.\\
&\quad\times\frac{f(r(L))}{\exp\left[-\frac{|r(L)|^2}{\sigma_s^2+\sigma_n^2}\right]}\times\frac{f(r(L+\frac{N}{2}))}{\exp\left[-\frac{|r(L+\frac{N}{2})|^2}{\sigma_s^2+\sigma_n^2}\right]}\\
&\quad\left.\times\prod_{k=2L+1}^{N/2+L-1}\frac{f(r(k),r(2L+N-k))}{\exp\left[-\frac{|r(k)|^2+|r(2L+N-k)|^2}{\sigma_s^2+\sigma_n^2}\right]}\right]
\end{aligned} \tag{8}$$

where the constant c_1 is independent of (ε,ϕ). Note that the one-, two- and three-dimensional PDFs $f(\cdot)$ involved in (8) are PDFs of noncircular complex Gaussian random variables. Their expression is derived in Appendix A taking into account the correlation and pseudocorrelation properties of the observations $\boldsymbol{r}$ reported in (5) and (6) and by exploiting the generalized multivariate distribution for improper complex Gaussian random vectors proposed in [5]. Specifically, by disregarding additive constants independent of the desired parameters, the log-likelihood function takes the form

$$\Lambda(\varepsilon,\phi) = \frac{c_2}{\sigma_s^2+\sigma_n^2}\Re\left[Ae^{-j2\pi\varepsilon}+\gamma^*\left(C+De^{-j2\pi\varepsilon}\right)\right], \tag{9}$$

where $\Re[\cdot]$ denotes real part, γ is defined in (7),

$$A \triangleq (1-\rho|\gamma|^2)\sum_{k=0}^{L-1}r^*(k)r(k+N), \tag{10}$$

$$C \triangleq (1-\rho)\sum_{k=0}^{L-1}r(k)r(2L-k)+\frac{\rho}{c_2(1-\rho^2|\gamma|^2)}r^2(L), \tag{11}$$

$$\begin{aligned}
D \triangleq\ &(1-\rho)\sum_{k=0}^{L-1}r(k+N)r(2L-k)\\
&+\frac{\rho}{c_2(1-\rho^2|\gamma|^2)}\sum_{k=2L+1}^{N-1}r(k)r(N+2L-k),
\end{aligned} \tag{12}$$

$$c_2 \triangleq \frac{2\rho}{(1-2\rho^2|\gamma|^2-\rho^2+2\rho^3|\gamma|^2)} \tag{13}$$

and

$$\rho \triangleq \frac{\sigma_s^2/\sigma_n^2}{1+\sigma_s^2/\sigma_n^2} = \frac{SNR}{1+SNR}. \tag{14}$$

In low SNR conditions and for circular transmissions (i.e., for $\gamma=0$), the log-likelihood function (9) takes the form

$$\Lambda_C(\varepsilon) = \frac{2\sigma_s^2}{\sigma_n^4}\Re\left\{\sum_{k=0}^{L-1}r^*(k)r(k+N)e^{-j2\pi\varepsilon}\right\}. \tag{15}$$

This low SNR approximation of the log-likelihood function, previously derived in [1] without the Gaussian assumption for the OFDM signal, is independent of the phase ϕ and, then, only the frequency shift ε can be estimated. However, also in low SNR conditions by considering non circular transmissions both the parameters of interest can be estimated.

4. PROPOSED ESTIMATORS

In this section, the proposed estimators are derived and the Cramér-Rao lower bound in the estimation of ε and ϕ is evaluated.

The maximum likelihood estimator is obtained by searching the value of the vector (ε,ϕ) that maximizes the log-likelihood function. To proceed we keep ε fixed and let ϕ vary. In these conditions $\Lambda(\varepsilon,\phi)$ achieves a maximum for

$$\hat{\phi}_{MLNC}(\varepsilon) = -\frac{2\pi}{N}L\varepsilon+\frac{1}{2}\angle\left[C+De^{-j2\pi\varepsilon}\right]+n\pi, \tag{16}$$

where n is an integer and $\angle[\cdot]$ denotes the argument of a complex number. We assume that the condition $|\phi|\le\frac{\pi}{2}$ is satisfied; thus we set $n=0$. Accounting for (9) and (16), the ML frequency shift estimate is obtained by

$$\hat{\varepsilon}_{MLNC} = \arg\max_{\tilde{\varepsilon}}\left[\Re\left\{Ae^{-j2\pi\tilde{\varepsilon}}\right\}+\left|C+De^{-j2\pi\tilde{\varepsilon}}\right|\right] \tag{17}$$

where $\tilde{\varepsilon}$ denotes a trial value of ε.

In the case of circular constellations the low SNR approximation of the log-likelihood function (15) is maximized for

$$\hat{\varepsilon}_{MLC} = -\frac{1}{2\pi}\angle\sum_{k=0}^{L-1}r(k)r^*(k+N). \tag{18}$$

This estimator was previously derived in [2] and [1].

From (16) follows that the carrier phase estimator $\hat{\phi}_{MLNC}$ gives unambiguous estimates if $|\phi|\le\frac{\pi}{2}$. Moreover, since the function to be maximized with respect to ε in the right hand side (RHS) of (17) is a periodic function of unit period, and, in addition, accounting for (18), it follows that the frequency offset estimators $\hat{\varepsilon}_{MLNC}$ and $\hat{\varepsilon}_{MLC}$ give ambiguous estimates unless $|\varepsilon|\le 0.5$.

Let us now evaluate the Cramér-Rao lower bound (CRB) in the estimation of ε and ϕ. Accounting for (9) the Fisher information matrix is given by

$$\boldsymbol{F} = 4\rho c_2 \times\begin{bmatrix}\pi^2\left[k_1+k_4^2k_2+(k_4+1)^2k_3\right] & \pi\left[k_4k_2+(k_4+1)k_3\right]\\ \pi\left[k_4k_2+(k_4+1)k_3\right] & k_2+k_3\end{bmatrix}$$

where

$$k_1 \triangleq (1-\rho|\gamma|^2)L,$$

$$k_2 \triangleq (1-\rho)|\gamma|^2L+|\gamma|^2\frac{\rho}{c_2(1-\rho^2|\gamma|^2)},$$

$$k_3 \triangleq (1-\rho)|\gamma|^2L+|\gamma|^2\frac{\rho}{c_2(1-\rho^2|\gamma|^2)}(N-2L-1)$$

and

$$k_4 \triangleq \frac{2L}{N}.$$

The Cramér-Rao lower bound of the frequency offset and the carrier phase estimate is given by the corresponding diagonal element of inverse of Fisher information matrix $\boldsymbol{F}^{-1}$, that is

$$CRB_\varepsilon = \quad [\boldsymbol{F}^{-1}]_{11} = \frac{k_2+k_3}{4\pi^2\rho k_5 c_2}\,, \tag{20}$$

$$CRB_\phi = \quad [\boldsymbol{F}^{-1}]_{22} = \frac{k_1+k_4^2k_2+(k_4+1)^2k_3}{4\rho k_5 c_2}\,, \tag{21}$$

where $k_5 \triangleq k_1k_2+k_1k_3+k_3k_2$. The RHS of (20) and (21), depend, through an unmanageable relation, on the number of subcarriers N, the cyclic prefix length L, SNR and the signal constellation (through the parameter γ). Therefore, in the following several approximations of (20) and (21) are derived.

In the case of noncircular constellations with $|\gamma| = 1$ (e.g., real data symbols) for high values of SNR and for $N \gg L$, CRB_ε and CRB_ϕ take the form

$$CRB_\varepsilon \approx \frac{3}{2\pi^2 SNR(8L+3)} \tag{22}$$

$$CRB_\phi \approx \frac{3}{2SNR(8L+3)}. \tag{23}$$

Moreover, in the case of noncircular constellations with $|\gamma| \neq 1$ and for high values of SNR, CRB_ε and CRB_ϕ can be approximated by

$$CRB_\varepsilon \approx \frac{1}{4\pi^2 LSNR} \tag{24}$$

$$CRB_\phi \approx \frac{(1-|\gamma|^2)}{4N|\gamma|^2}\,. \tag{25}$$

Finally, in the case of circular transmissions ($\gamma = 0$) only the frequency shift can be estimated and, for high values of SNR, the lower bound is

$$CRB_\varepsilon \approx \frac{1}{4\pi^2 LSNR}\,. \tag{26}$$

This expression is coincident with the high SNR approximation of the variance, derived in [1], of the frequency shift estimator (18).

Note that in the case of noncircular constellations with $|\gamma| \neq 1$ the CRB_ε is coincident with the Cramér-Rao lower bound of the frequency offset in OFDM systems with circular transmissions (see (24) and (26)) and both coincide, for high values of SNR, with the variance of $\hat{\varepsilon}_{MLC}$ in (18). Thus, no gain can be expected, for large values of SNR, in the frequency offset estimation by exploiting noncircularity when $|\gamma| \neq 1$. However, a gain can be achieved in the case where $|\gamma| = 1$ (see (22) and (26)).

5. NUMERICAL RESULTS

The performance of MLNC and MLC estimators is assessed via computer simulation and compared with the Cramér-Rao lower bound. Specifically, Fig.1 presents the bias and the variance for frequency offset and carrier phase estimates, evaluated by averaging over 20000 runs, as a function of SNR in an AWGN channel. An OFDM system with $N = 256$ BPSK subcarriers and a cyclic prefix length fixed at $L = 1$ (dashdot lines), $L = 2$ (dashed lines) and $L = 12$ (solid lines) is considered. MLNC and MLC estimators are indicated by '$*$' and '$\circ$' markers, respectively. The results show that performance improvement of the MLNC frequency offset estimator with respect to the MLC estimator, increases as the prefix length decreases. Note that the MLNC estimator achieves, for sufficiently high values of SNR and for high values of L, the Cramér-Rao lower bound (line without markers). Moreover, only the performance of MLNC phase estimator is reported, since the method proposed in [1] does not allow to jointly estimate the frequency offset and the carrier phase. The performance of the MLNC estimator improves as the size of the cyclic prefix increases.

Appendix A

In this appendix we derive the expression of the PDFs of noncircular complex Gaussian random variables involved in (8).

Let us consider the random variables $x_1 \triangleq r(k)$, $x_2 \triangleq r(2L-k)$ and $x_3 \triangleq r(k+N)$ where $k \in \tau_1$. The noncircular complex Gaussian random vector $\boldsymbol{v}_1 \triangleq [x_1\,, x_1^*\,, x_2\,, x_2^*\,, x_3\,, x_3^*]^T$ is characterized by the joint PDF [5]

$$f(\boldsymbol{v}_1) = \quad \frac{1}{\pi^3\sqrt{\det(\boldsymbol{V}_1)}} \exp\left[-\frac{\boldsymbol{v}_1^H\boldsymbol{V}_1^{-1}\boldsymbol{v}_1}{2}\right]$$

where the superscript ^{H} denotes complex conjugate transposition and the covariance matrix $\boldsymbol{V}_1$, accounting for (5) and (6), is given by

$$\boldsymbol{V}_1 \triangleq E[\boldsymbol{v}_1\boldsymbol{v}_1^H] = (\sigma_s^2+\sigma_n^2)\begin{bmatrix} \boldsymbol{I} & \boldsymbol{R}_1 & \boldsymbol{R}_2 \\ \boldsymbol{R}_1 & \boldsymbol{I} & \boldsymbol{R}_3 \\ \boldsymbol{R}_2^* & \boldsymbol{R}_3 & \boldsymbol{I} \end{bmatrix}$$

where $\boldsymbol{I}$ is the 2×2 identity matrix,

$$\boldsymbol{R}_1 \quad \triangleq \quad \begin{bmatrix} 0 & \rho\gamma \\ \rho\gamma^* & 0 \end{bmatrix},$$

$$\boldsymbol{R}_2 \quad \triangleq \quad \begin{bmatrix} \rho e^{-j2\pi\varepsilon} & 0 \\ 0 & \rho e^{j2\pi\varepsilon} \end{bmatrix}$$

and, moreover,

$$\boldsymbol{R}_3 \quad \triangleq \quad \begin{bmatrix} 0 & \gamma\rho e^{j2\pi\varepsilon} \\ \gamma^*\rho e^{-j2\pi\varepsilon} & 0 \end{bmatrix}.$$

Thus, it follows that

$$\begin{aligned} f(\boldsymbol{v}_1) = \;& \frac{1}{\pi^3\sqrt{\det(\boldsymbol{V}_1)}} \\ & \times \exp\Big\{-\frac{c_2}{2\rho(\sigma_s^2+\sigma_n^2)}\Big[(|x_1|^2+|x_3|^2)(\rho^2|\gamma|^2-1) \\ & +|x_2|^2(\rho^2-1)-2\rho(\rho-1)\Re\left(x_1x_2\gamma^*+x_2x_3\gamma^*e^{-j2\pi\varepsilon}\right) \\ & -2\rho(\rho|\gamma|^2-1)\Re\left(x_1^*x_3e^{-j2\pi\varepsilon}\right)\Big]\Big\}, \end{aligned}$$

where the determinant $\det(\boldsymbol{V}_1)$ is independent of ε and ϕ.

Let us now consider the random variables $x_4 \triangleq r(k)$ and $x_5 \triangleq r(N+2L-k)$ where $k \in \{2L+1,\ldots,N/2+L-1\}$. The noncircular complex Gaussian random vector $\boldsymbol{v}_2 \triangleq [x_4\,, x_4^*\,, x_5\,, x_5^*]^T$ has covariance matrix

$$\boldsymbol{V}_2 = (\sigma_s^2+\sigma_n^2)\begin{bmatrix} \boldsymbol{I} & \boldsymbol{R}_3 \\ \boldsymbol{R}_3 & \boldsymbol{I} \end{bmatrix}$$

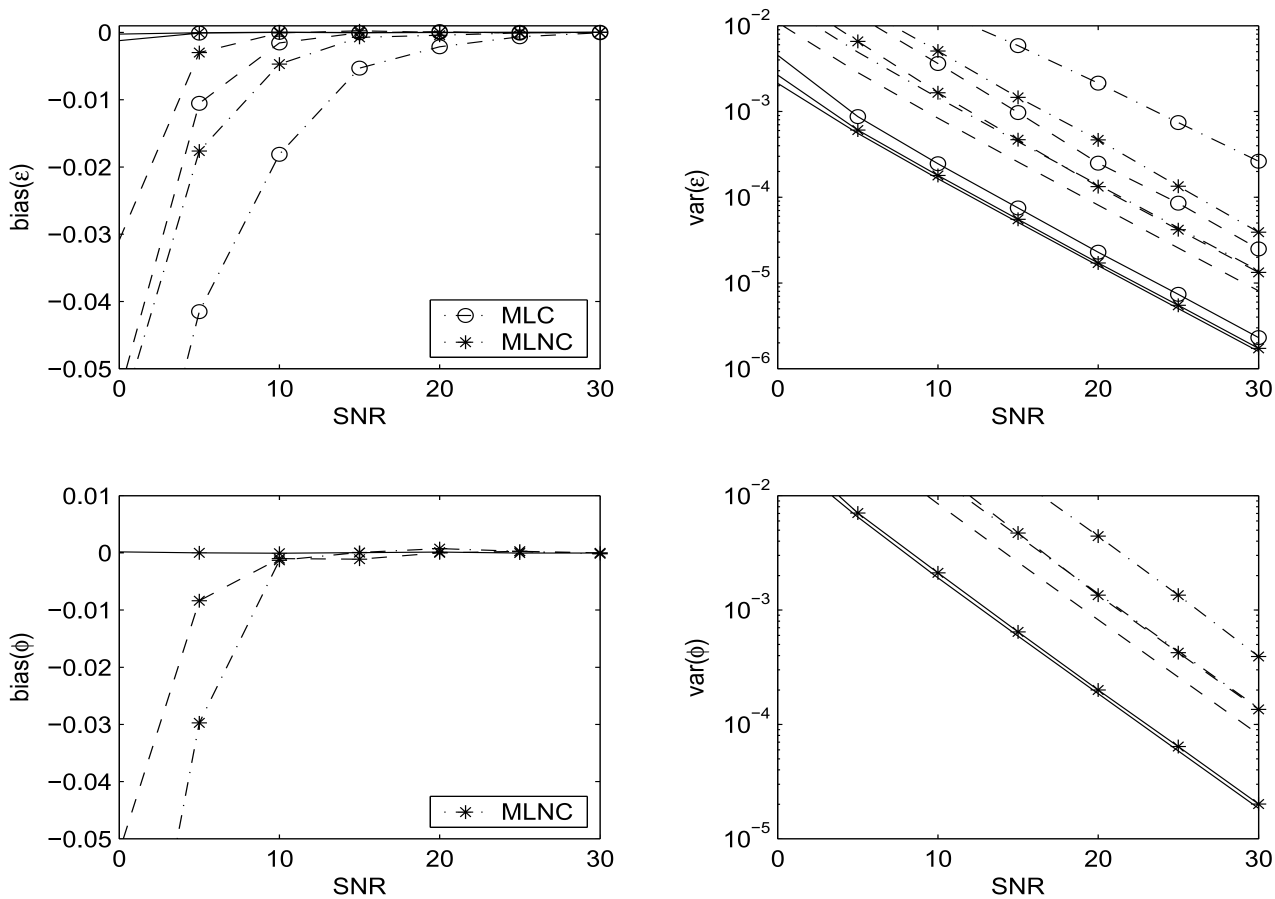

Fig.1. Bias (left) and variance (right) of MLNC and MLC estimators as a function of SNR for $N = 256$ and for a cyclic prefix length fixed at $L = 1$ (dashdot curves), $L = 2$ (dashed curves) and $L = 12$ (solid curves). MLNC and MLC estimators are indicated by '$*$' and '$\circ$' markers, respectively. The Cramér-Rao lower bounds are indicated with lines without markers.

and is characterized by the joint PDF

$$f(\boldsymbol{v}_2) = \frac{1}{\pi^2(\sigma_s^2+\sigma_n^2)^2(1-\rho^2|\gamma|^2)} \times \exp\left[-\frac{|x_4|^2+|x_5|^2-2\rho\Re\left(x_4x_5\gamma^*e^{-j2\pi\varepsilon}\right)}{(\sigma_s^2+\sigma_n^2)(1-\rho^2|\gamma|^2)}\right].$$

Finally, the noncircular complex Gaussian random variables $x_6 \triangleq r(L)$ and $x_7 \triangleq r(N/2+L)$ are characterized by the PDFs

$$f(x_6,x_6^*) = \frac{1}{\pi(\sigma_s^2+\sigma_n^2)\sqrt{1-\rho^2|\gamma|^2}} \times \exp\left[-\frac{|x_6|^2-\rho\Re\left(x_6^2\gamma^*\right)}{(\sigma_s^2+\sigma_n^2)(1-\rho^2|\gamma|^2)}\right]$$

and

$$f(x_7,x_7^*) = \frac{1}{\pi(\sigma_s^2+\sigma_n^2)\sqrt{1-\rho^2|\gamma|^2}} \times \exp\left[-\frac{|x_7|^2-\rho\Re\left(x_7^2\gamma^*e^{-j2\pi\varepsilon}\right)}{(\sigma_s^2+\sigma_n^2)(1-\rho^2|\gamma|^2)}\right],$$

respectively.

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