LINEAR PRECODERS FOR OFDM WIRELESS COMMUNICATIONS WITH MMSE EQUALIZATION: FACTS AND RESULTS

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ABSTRACT

This contribution aims at analyzing the performance of **MMSE equalizers** for large linear precoded OFDM transmissions over fading wireless channels with **limited diversity** and **channel state information available only at the receiver**. Linear Precoding consists in multiplying by a $N \times K$ isometric matrix a K-dimensional vector obtained by serial to parallel conversion of a symbol sequence to be transmitted . Based on the *Free Probability Theory*, asymptotical analysis ($N \to +\infty$, $K \to \infty$ and $K/N \to \alpha \leq 1$) of the SINR is conducted to understand the different parameters involved in Linear Precoded OFDM schemes. The theoretical results are confirmed by numerical simulations when considering convolutional coding with finite memory.

1. INTRODUCTION

A multi-carrier OFDM system [1] using a Cyclic Prefix for preventing inter-block interference is known to be equivalent to multiple flat fading parallel transmission channels in the frequency domain.

When perfect channel state information is available at the transmitter, it is well known that the optimum (capacity-achieving) transmitted spectrum may be computed by the well-known "water-pouring" method: adapt the rate on each each subchannel by taking into account the signal to noise ratio on that specific subchannel

When no channel knowledge at the transmitter is available, the water pouring strategy is inappropriate and the transmitted information on one subchannel can be irremediably lost if a deep fade occurs. Different methods also known as diversity techniques have been proved efficient to cope with these channel impairments. In particular, [2] proposed a robust transmission scheme combining the advantages of CDMA with the strength of OFDM known as OFDM-CDMA, in which the information is precoded across all the carriers by a pre-coding matrix. This combination increases the overall frequency diversity of the modulator, so that unreliable carriers can still be recovered by taking advantage of the subbands enjoying a high Signal to Noise Ratio (SNR). Although originally proposed for a multiuser access scheme, this concept is extended to all single user OFDM systems and is referred in the sequel as Linear Precoded OFDM (LP OFDM) [3, 4]. [5] and [6] proposed to optimize the coefficients of the matrix by deriving an upper bound of the error of the maximum likelihood (ML) detector. More recently, Giannakis [3] generalized these results to rectangular matrices considering limited diversity channels. Even though optimal maximum-likelihood (ML) detector clearly outperforms MMSE receivers, the high computational cost of the ML detector prevents its use in practical context. This paper will therefore only focus on MMSE equalizers.

As in the context of multi-users systems, the performance of such receivers can be characterized by their associated Signal to Interference + Noise ratios (SINR), which, unfortunately have a non interpretable analytical expression. In order to overcome this difficulty, we propose to use an attractive approach already used in the context of multi-users CDMA systems, where the precoder is modeled as a certain type of random matrix. In [7], the performance of the MMSE receiver considering ergodic channels with i.i.d flat frequency fading and particular rectangular random precoders has been studied. It was in particular shown that the SINR converges almost surely toward a deterministic constant not depending on the particular realization of the precoding matrix. This contribution extends the analysis to **limited diversity fading channels** for which we derive the SINR based on asymptotical analysis ($K \to \infty$, $N \to \infty$ and $K/N \to \alpha \leq 1$) and give tools for understanding the parameters involved in LP-OFDM schemes (especially the nature and size of the LP matrix).

The system and channel model are described in section 2, followed by an asymptotical analysis of the SINR of the MMSE receiver in section 3. Finally, section 4 is devoted to performance results according to certain criteria. Some conclusions are drawn in Section 5.

2. LP-OFDM TRANSCEIVER MODEL

In the following, upper (lower boldface) symbols will be used for matrices (column vectors) whereas lower symbols will represent scalar values, $(.)^T$ will denote transpose operator, $(.)^\star$ conjugation and $(.)^H = \left((.)^T\right)^\star$ hermitian transpose.

Overall system model: since a N carrier OFDM system [1] using a cyclic prefix is equivalent in the frequency domain to N flat fading parallel transmission channels, the baseband discrete-time block equivalent model of a LP-OFDM system can be depicted in figure 1. The $N \times 1$ received vector at time n, $\mathbf{r}(n) = [r_1(n), \cdots, r_N(n)]^T$ can be expressed as a function of the emitted symbol vector n, $\mathbf{s}(n) = [s_1(n), \cdots, s_K(n)]^T$ and of the additive noise $\mathbf{w}(n) = [w_1(n), \cdots, w_N(n)]^T$ vector using a block representation: $\mathbf{r}(n) = \mathbf{H}_N(n)\mathbf{W}_K\mathbf{s}(n) + \mathbf{w}(n)$. $\mathbf{W}_K$ is a $N \times K$ LP matrix whereas $\mathbf{H}_N(n) = \mathrm{diag}[h_1(n), \cdots, h_N(n)]$ is a diagonal matrix of the frequency domain channel attenuations at block n. We will also assume channel knowledge and perfect channel synchronization at the receiver. Assuming that the system bandwidth is much larger than the coherence bandwidth of the channel, the symbols are therefore transmitted over a static L-path channel whose (time varying) impulse response is given at time

τ by $h(\tau,t) = \frac{1}{L}\sum_{l=0}^{L-1}\alpha_l(\tau)\delta(t-d_l(\tau))$ and the $(d_l(\tau))_{l=1,\dots,L}$ are the corresponding time delays. Here, the complex gains $(\alpha_l(\tau))_{l=1,\dots,L}$ are centered Gaussian random signals with unit variance. Both the gains and time delays of the paths are assumed static during the transmission of one K-dimensional codeword (but can change from codeword to codeword). In this case, the time dependence has no impact and will be omitted from now on. The equivalent baseband channel transfer function is therefore : $h(f) = \frac{1}{L}\sum_{l=0}^{L-1}\alpha(lT)e^{-i2\pi lfT}$ where T is the sampling rate and $-\frac{1}{2T} \leq f \leq \frac{1}{2T}$. The channel attenuation on carrier k is thus given by: $h_k = \frac{1}{L}\sum_{l=0}^{L-1}\alpha_l e^{-i2\pi l\frac{k-1}{N}}$ $(1 \leq k \leq N)$.

3. ASYMPTOTICAL ANALYSIS

In this section, we study the asymptotic behavior of the SINR when the linear precoder matrix $\mathbf{W}_K$ coincides with a realization of a certain kind of isometric random matrices. The output of the Wiener filter is the vector $\mathbf{y}$ given by $\mathbf{y} = \mathbf{Gr}$, where the matrix $\mathbf{G}$ is defined as

$$\begin{aligned}\mathbf{G} &= \text{argmin}_W \mathbb{E}\|\mathbf{W}^H\mathbf{r}-\mathbf{s}\|^2 \\ &= \mathbf{W}_K^H\mathbf{H}_N^H\left(\mathbf{H}_N\mathbf{W}_K\mathbf{W}_K^H\mathbf{H}_N^H+\sigma^2\mathbf{I}_N\right)^{-1}\end{aligned}$$

The SINR $\beta_{\mathbf{w}_i}$ of the ith symbol is easily shown to express as $\beta_{\mathbf{w}_i} = \frac{\eta_{\mathbf{w}_i}}{1-\eta_{\mathbf{w}_i}}$ where

$$\eta_{\mathbf{w}_i} = \mathbf{w}_i^H\mathbf{H}_N^H\left(\mathbf{H}_N\mathbf{W}_K\mathbf{W}_K^H\mathbf{H}_N^H+\sigma^2\mathbf{I}_N\right)^{-1}\mathbf{H}_N\mathbf{w}_i \qquad (1)$$

$\mathbf{w}_i$ is the i_{th} column of matrix $\mathbf{W}_K$.

For a fixed K and N, it is extremely difficult to get insight on the performance of the MMSE receiver from the expression (1). In order to provide useful expressions, several papers [8, 9] have recently analyzed, in the CDMA context, the behavior of the asymptotic SINR ($N \to \infty$, $K \to \infty$ and $\frac{K}{N} \to \alpha$) when the entries of $\mathbf{W}_K$ are independent and identically distributed random variables. However, in our case, since synchronization is ensured at the emitter site, only isometric precoders can be considered. **For calculus purpose**, the Linear Precoder is assumed to be an isometric matrix extracted from a *Haar* distributed random unitary matrix : a random unitary matrix is said *Haar distributed* if its probability distribution is invariant by left multiplication by constant unitary matrices. Such a matrix can be generated the following way: let $\mathbf{X} = [x_{i,j}]_{1\leq i,j\leq N}$ be a $N\times N$ random matrix with independent complex Gaussian centered unit variance entries. Then the unitary matrix $\mathbf{X}(\mathbf{X}^H\mathbf{X})^{-1/2}$ is Haar distributed (see [7, 10]). In the sequel, it will be assumed that $\mathbf{W}_K$ is generated by extracting any K columns from a $N\times N$ Haar distributed unitary matrix independent of $\mathbf{H}_N$. We stress on the fact that in practical precoded OFDM systems, precoding matrices are not generated this way. Walsh-Hadamard matrices, whose entries are binary, cannot in particular be considered as realizations of Haar distributed random unitary matrices. However, simulations in the following section show that our theoretical evaluations fit very well with Walsh-Hadamard linear precoded systems.

The following result holds.

Theorem 1 *For all $\mathbf{W}_K$ matrices chosen as above, when $N\to\infty$ and $K/N \to \alpha \leq 1$, the SINR $\beta_{\mathbf{w}_i}$ at the output of the MMSE equalizer converges almost surely to a value $\beta(\alpha)$ that is the unique solution of the equation*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|h(f)|^2}{\alpha|h(f)|^2+\sigma^2(1-\alpha)\beta(\alpha)+\sigma^2}\,df = \frac{\beta(\alpha)}{\beta(\alpha)+1}\,. \qquad (2)$$

The detailed proof uses rather involved materials based on the so-called free probability theory (see [10]). For concision sake, only an outline is provided. Here are the main steps:

- Step 1: show that $\eta_{\mathbf{w}_i}$ converges almost surely to a value $\eta(\alpha)$ which does not depend on the choice of $\mathbf{w}_i$.
- Step 2: for a given N, there are K quantities $\eta_{\mathbf{w}_i}$ each corresponding to the choice of a particular column code in $\mathbf{W}_K$. Their sum over all the columns of this matrix is trace $\left((\mathbf{A}_{N,K}+\sigma^2\mathbf{I}_N)^{-1}\mathbf{A}_{N,K}\right)$ with $\mathbf{A}_{N,K} = \mathbf{H}_N\mathbf{W}_K\mathbf{W}_K^H\mathbf{H}_N^H$. Hence, the limit value $\eta(\alpha)$ is given by :

$$\begin{aligned}\eta(\alpha) &= \lim_{N\to\infty}\frac{1}{\alpha N}\text{tr}\left((\mathbf{A}_{N,K}+\sigma^2\mathbf{I}_N)^{-1}\mathbf{A}_{N,K}\right) \\ &= \frac{1}{\alpha}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_i}{\lambda_i+\sigma^2} = \frac{1}{\alpha}\lim_{N\to\infty}\int\frac{\lambda}{\lambda+\sigma^2}d\theta_N(\lambda)\end{aligned}$$

  where $(\lambda_i)_{i=1,\dots,N}$ are the eigenvalues of $\mathbf{A}_{N,K}$ and θ_N is the probability measure on $\mathbb{R}^+$ defined by $d\theta_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda_i-\lambda)$. In the context of random matrices theory, θ_N is called the empirical eigenvalue distribution of $\mathbf{A}_{N,K}$.
- step 3: by applying free probability results, we can show that θ_N converges almost surely to a compactly supported measure θ, which can be derived explicitly. Therefore, $\eta(\alpha)$ converges almost surely to: $\frac{1}{\alpha}\int\frac{t}{t+\sigma^2}\,d\theta(t)$. This also shows that $\beta_{\mathbf{w}_i}$ converges to a deterministic value $\frac{\eta(\alpha)}{1-\eta(\alpha)}$ denoted $\beta(\alpha)$ and solution of (2).

Interestingly, for a given channel, this result shows that the SINR converges to a deterministic value (depending on the channel realization) and that it is irrelevant to optimize the LP matrix.

4. PERFORMANCE RESULTS

This section analyzes the effect of the diversity order on the performance of LP-OFDM schemes.

Diversity Considerations: it is extremely difficult to derive the probability density function of β. Indeed, β is a random variable linked to the number of paths of the time channel impulse response by the non explicit equation (2). The evolution of the SINR with respect to the number of independent taps is an interesting diversity related topic and we illustrate numerically the effect of the number of independent taps with $\alpha = 1$. Figure 5 shows that by increasing the channel order, the SINR converges to a deterministic value thus converting the Rayleigh channel (L=1) into a gaussian channel. Similar conclusions were drawn in [5] but the analysis only focused on the ML detector behavior of multidimensional QAM constellations systems. An interesting feature is that even though the variance of the SINR distribution decreases, the mean value also decreases. However, this phenomenon will have no impact on the performance as highlighted in the next section

BER Considerations: this section aims at showing that our analysis fits with usual precoding matrices. Figure 3 and 4 plot the theoretical and simulated (**with Walsh Hadamard matrices**) BER respectively for $\alpha = \frac{1}{2}$ and $\alpha = 1$ considering different channel orders. By considering QPSK constellations, the asymptotic BER is given by $\text{Q}\left(\sqrt{\beta}\right)$ where $\text{Q}(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty}e^{-\frac{t^2}{2}}dt$. Since the SINR is a random variable depending on the channel realization, we consider the mean BER: $\mathbb{E}\left(\mathrm{Q}\left(\sqrt{\beta}\right)\right)$. One can observe that the simulated and theoretical curves closely match for a realistic number of carriers (N=128). As the channel order increases, the BER improves considerably. As expected when α decreases, the performance also increases for a given channel order. However, a loss in terms of spectral efficiency incurs.

Spectral efficiency Considerations: the purpose of this section is to design the LP matrix with respect to the ratio $K/N = \alpha$ in order to maximize the spectral efficiency . This spectral efficiency (in bits per symbol) corresponds to the maximal average value of the mutual information between the transmitted and the equalized received signal. Each carrier k is assumed to be encoded independently according to fig.1. Indeed, the encoder cannot

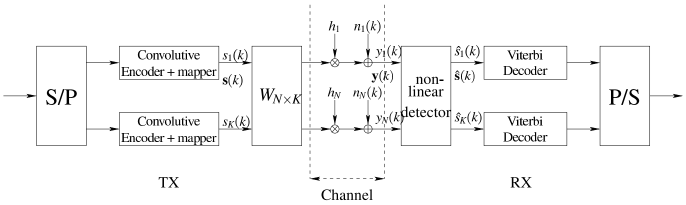

Fig. 1. Classical frequency domain LP-OFDM transmitter

be applied prior to the carrier symbol allocation (as in COFDM) due to the inter-carrier noise correlations introduced by the despreading of the received symbols. For this reason, the same coding is applied on each of the carriers independently. The spectral efficiency with MMSE equalization is defined as (see [11]: $\gamma(K,N) = \frac{1}{N}\sum_{i=1}^{K} log_2(1+\beta_{\mathbf{w}_i}) \rightarrow \alpha log_2(1+\beta(\alpha))$.

In this case, since the SINR is a random variable depending on the channel realization, the spectral efficiency will be random as well. In our simulations, only the mean spectral efficiency is considered. Figure 2 plots the optimum α of the spectral efficiency for different channel orders. Nearly no redundancy ($\alpha > 0.92$) should be spent on Linear Precoding **whatever the diversity order** L. In other words, coding in the Galois field is spectrally more efficient then redundant linear precoding. Based on this result and using square matrices, notice that the filtering matrix has a very simple implementation structure (scalar channel equalization followed by a matrix multiplication):

$$\mathbf{G} \quad = \quad \mathbf{W}_N^H \mathrm{diag}\left(\frac{h_1^*}{|h_1|^2+\sigma^2}, \ldots, \frac{h_N^*}{|h_N|^2+\sigma^2}\right)$$

Note also that the gain in spectral efficiency by optimizing α increases with Eb/No as shown in fig.6 but little can be gained when $0.7 < \alpha < 1$ and $Eb/No < 6dB$

Convolutive Coding schemes: The performance of a system where Linear Precoding of rate α is combined with classical Convolutional Coding of rate R is studied in order to confirm the spectral efficiency analysis. In the context of coded LP-OFDM, one Viterbi decoder is applied on each subband k and processes the real and imaginary parts of the signal output of the Wiener filter. In order to evaluate upper bounds on the BER, one usually first evaluates the probability $P_{\text{coded LP-OFDM}}(d)$ of deciding P_1 instead of P_0 where P_0 is the path of the Viterbi algorithm trellis associated to the transmitted sequence and P_1 is a path which differs by d bits from P_0. Following [12, 4], the probability of that event, is given by

$$P_{\text{coded LP-OFDM}}(d) \quad = \quad \mathbb{E}\left(\mathrm{Q}\left(\sqrt{d\beta(\alpha)}\right)\right)$$

The overall error probability is thus bounded by:

$$\boxed{P_{\text{coded LP-OFDM}} \leq \sum_{d=d_{min}}^{\infty} \frac{\gamma_d}{M} P_{\text{coded LP-OFDM}}(d)} \qquad (3)$$

Here, d_{min} is the minimal distance of the code, M is the number of input bits in the encoder and γ_d is the number of incorrectly decoded information bits, for each possible incorrect path differing from the correct one by d bits.

The goal is to determine the optimum balance between α and R, assuming a constant overall spectral efficiency of 1 bit (In the case of QPSK constellations, we have $2\times\alpha R = 1$), The role of α and R is revealed by eq.(3) and shows that the performance is deeply related to the transfer function of the code. Indeed, when α increases, $\beta(\alpha)$ decreases but R also decreases yielding a greater minimum distance d_{min} through the transfer function. Thus, the trade-off depends on the transfer function of the code (through γ_d and d_{min}). The analytical expression of the transfer function can only be drawn for codes with a limited number of states. Since theoretical study of general codes is extremely difficult, we will conduct performance simulations of various convolutional codes. Fig.7 and Fig.8 plot respectively the performance of various Convolutional Coding rate schemes $(\alpha, R) = \left((1,\frac{1}{2});(\frac{8}{10},\frac{10}{16});(\frac{3}{4},\frac{2}{3});(\frac{2}{3},\frac{3}{4})\right)$ with diversity order 4 and 8. In that purpose, the convolutional code specified in the HIPERLAN/2 standard is used. The mother code has a constraint length of 7 (133o, 171o) and rate $\frac{1}{2}$. The other rates are achieved by puncturing the coded symbols. These figures suggest that the optimum α is 1 and are in accordance with the spectral efficiency analysis of fig.6 and fig.2

5. CONCLUSION

In this paper, we have provided a theoretical framework for the design of LP-OFDM schemes with **MMSE equalization** under finite channel diversity constraint . We have modeled the precoding matrix as an isometric matrix extracted from a Haar unitary random matrix, and proved that the performance is independent of the particular matrix realization. In this case, we have shown that coding in the Galois field is spectrally more efficient then redundant linear precoding. These theoretical results fit with realistic parameters ($N = 128$, $L = 8$) and usual precoders such as Walsh-Hadamard precoding matrices. These results were confirmed by numerical simulations using convolutional codes.

6. ACKNOWLEDGMENTS

The authors would like to thank Patrick Maillé for useful discussions and his contribution to the simulations.

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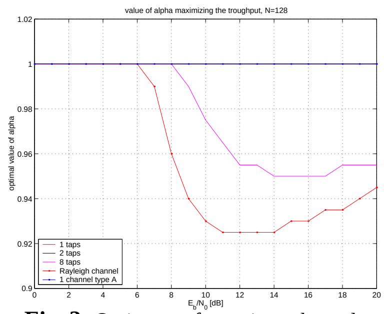

Fig. 2. Optimun α for various channels

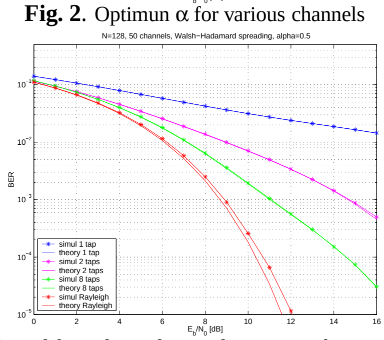

Fig. 3. Walsh-Hadamard precoding versus theory $\alpha = 0.5$

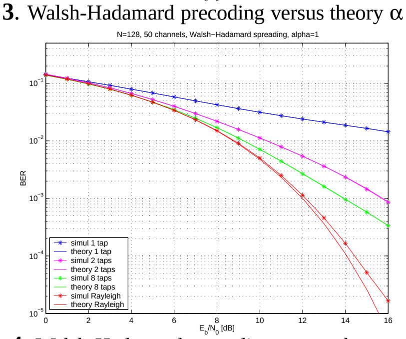

Fig. 4. Walsh-Hadamard precoding versus theory $\alpha = 1$

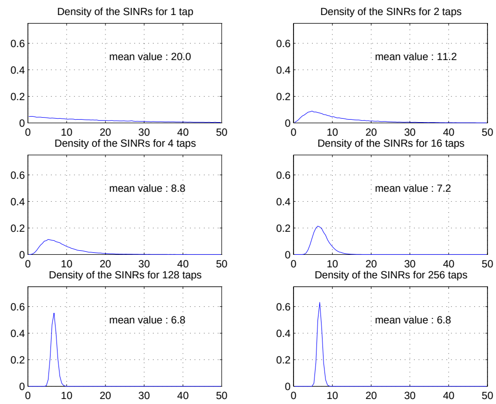

Fig. 5. MMSE SINR density

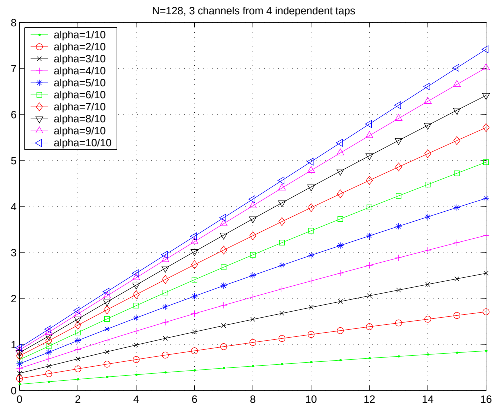

Fig. 6. Spectral efficiency for different α, L=4

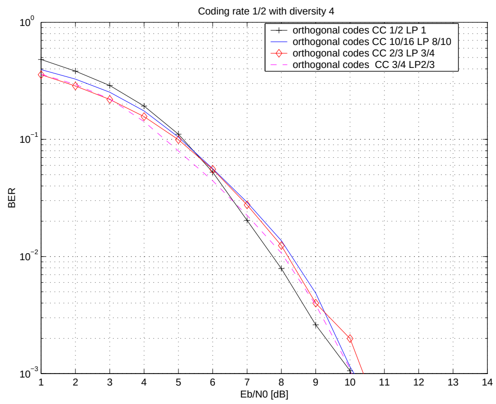

Fig. 7. different α, L=4

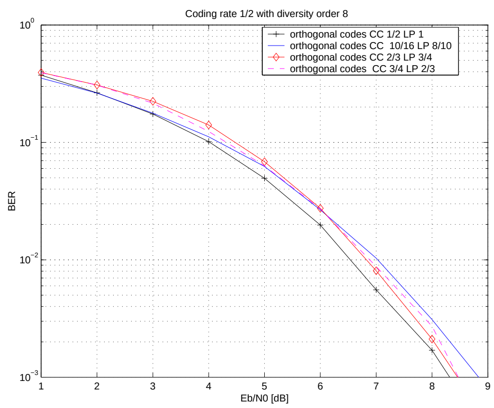

Fig. 8. different α, L=8