A NEW METHOD FOR PARAMETER ESTIMATION OF AUTOREGRESSIVE SIGNALS IN COLORED NOISE

Md. Kamrul Hasan, A. K. M. Z. Rahim Chowdhury, Rubyet Adnan, M. M. Rahman Bhuiyan, and M. Rezwan Khan

Dept. of Electrical and Electronic Engg., Bangladesh University of Engineering and Technology, Dhaka-1000, Bangladesh

Tel: +88 02 8611594; fax: +88 02 8613046

e-mail: khasan@eee.buet.edu

ABSTRACT

This paper presents a new method for parameter estimation of autoregressive (AR) signals from colored noise-corrupted observations using a *damped sinusoidal* model of the autocorrelation function of the noise-free AR signal. Unlike conventional correlation-based techniques, the proposed scheme first estimates the *damped sinusoidal* model parameters from the given noisy observations using a least-squares (LS) based method. The AR parameters are then directly obtained from the *sinusoidal* model parameters. Simulation results show that the proposed method performs better at low SNRs as compared to other existing methods.

1 INTRODUCTION

Parameter estimation of stochastic signal model is an important issue in various fields of science and engineering, e.g., econometrics, geophysics, speech processing, image processing, biomedical signal processing, and communication [1], [2]. The most popular stochastic signal model is the Gaussian, minimum phase, AR model. In time series analysis and signal modeling, both noise-free and noisy autoregressive (AR) systems have been extensively studied by many researchers [3]-[5]. In the latter case, except very few exceptions for colored noise, research results mostly considered white additive noise.

Zhang and Takeda in [6] proposed a method for parameter estimation of ARMA systems corrupted by colored noise. In that work, a generalized least-squares (GLS) method was suggested for estimating the AR parameters using short and noisy data. Although it was claimed that the estimates converge to the true values within a few iterations, it was shown in [7] that they actually remain unchanged after the first iteration. Furthermore, the GLS method has limitations for certain AR systems and cannot be used to estimate the AR parameters, especially when the poles of the AR system lie relatively inside the unit circle and the noise is relatively strong, i.e., SNR is low. To alleviate this problem, a maximum likelihood method for identifying AR systems has been reported in [7]. The algorithm however utilizes a bootstrap technique where the initial values are obtained from the GLS method. The method is highly dependent on initial values and may fail to converge at a low SNR.

Recently, Zheng [8] has extended the improved least-squares (ILS) type method to the parameter estimation of AR processes corrupted by colored noise. Poor estimate of the initial values by the least-squares (LS) method impedes convergence of the iterative scheme particularly at low SNRs.

In this paper, we introduce a new AR parameter estimation scheme via *damped sinusoidal* modeling of the autocorrelation function of the noise-free AR signal. The model parameters, leading to AR parameters, are estimated from colored-noise corrupted observations by using a LS type algorithm.

2 PROBLEM FORMULATION

The input-output relationship of a p-th order AR process can be expressed by the difference equation as

$$A(z)x(n) = u(n) \tag{1}$$

where the unknown input $u(n)$ is a sequence of zero-mean white Gaussian noise with unknown variance σ_u^2, $x(n)$ denotes the output signal, and $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_p z^{-p}$. Here, z^{-1} is the unit delay operator, i.e., $z^{-1}x(n) = x(n-1)$ and a_k $(k = 1, 2, \cdots, p)$ are the unknown AR parameters. The order p of the AR system is assumed to be known.

In many practical situations, observation noise corrupts the data samples. In this work, we assume that the output signal $x(n)$ contains additive colored noise. Then the observed process $y(n)$ can be expressed as

$$y(n) = x(n) + w(n) \tag{2}$$

The additive colored noise $w(n)$ originates from an MA process given by

$$w(n) = B(z)v(n) \tag{3}$$

where $v(n)$ is a zero-mean white Gaussian noise with unknown variance σ_v^2 and $B(z) = 1 + b_1 z^{-1} + b_2 z^{-2} +$

$\cdots + b_{q_b} z^{-q_b}$. We consider that the colored noise $w(n)$ is finitely autocorrelated, i.e.,

$$R_{ww}(k) \equiv E[w(n)w(n-k)] = 0, \text{ for } |k| \geq L \tag{4}$$

where $E[\cdot]$ represents the expectation operator and L is a given positive integer. Moreover, $v(n)$ is statistically independent of $u(n)$, i.e., $E[v(n)u(n-t)] = 0$ for all t.

The objective of this paper is to propose a novel method using a *damped sinusoidal* model for autocorrelation function of the noise-free signal to estimate the AR parameters. The *damped sinusoidal* model parameters are estimated using $R_{yy}(m)$, calculated from a finite set of noisy observations. The desired AR parameters $\{a_k\}$ are then directly obtained from this model parameters.

3 AR PARAMETER ESTIMATION USING DAMPED SINUSOIDAL MODEL

The transfer function of a p-th order AR system in the z-domain can be expressed as

$$H(z) = \frac{1}{A(z)} = \sum_{k=1}^{p} \frac{C_k}{1 - z_k z^{-1}} \tag{5}$$

where z_k denotes the k-th pole of the AR system and C_k is the partial fraction coefficient corresponding to the k-th pole. The unit impulse response $h(n)$ of the causal AR system described in Eq. (5) can be expressed as

$$h(n) = \sum_{k=1}^{p} C_k (z_k)^n \tag{6}$$

If this AR system is excited by a white noise sequence $u(n)$, the response $x^M(n)$ is given by

$$x^M(n) = u(n) * h(n) = \sum_{l=0}^{n} u(l) h(n-l) \tag{7}$$

Using Eq. (6), Eq. (7) can be written as

$$x^M(n) = \sum_{k=1}^{p} \sum_{l=0}^{n} C_k u(l) (z_k)^{n-l} \tag{8}$$

Clearly, $x(n)$ and $x^M(n)$ are the same because Eq. (1) is the difference equation implementation of input-output using the system parameters and Eq. (8) is the convolution sum implementation of the same using the system roots. Using Eq. (8), the autocorrelation of the noise-free signal $x^M(n)$ can be obtained as

$$R_{xx}^M(m) = R_{xx}(m) = \sum_{k=1}^{p} \beta_k (z_k)^m \tag{9}$$

where

$$\beta_k = \sigma_u^2 \left[\frac{C_k^2}{1 - z_k^2} + \sum_{q=1, q \neq k}^{p} \frac{C_k C_q}{1 - z_k z_q} \right] \tag{10}$$

The coefficient β_k may be real or complex depending on whether the pole is real or complex. Since $x(n)$ is real, in the latter case, a complex pole will always be accompanied by its complex conjugate pole. Considering the effect of complex and real poles, Eq. (9) can be simplified as

$$R_{xx}(m) = \sum_{j=1}^{g} (r_j)^m \left[P_j \cos(\omega_j m) + Q_j \sin(\omega_j m) \right] \tag{11}$$

where $g =$ {number of complex conjugate pair of poles + number of real poles}, r_j is the magnitude of the j-th pole and P_j and Q_j are constants. In general, r_j governs the decay rate of the AR system response and ω_j determines the angular position of the pole of the AR system in the z-plane.

We estimate each of the damped sinusoidal function of the alternative representation of $R_{xx}(m)$ described in Eq. (11) in an iterative fashion. At first from the given set of noisy data points $y(n)$, the autocorrelation function of the noisy signal, $R_{yy}(m)$, is calculated as

$$R_{yy}(m) = \frac{1}{N} \sum_{n=0}^{N-1-|m|} y(n) y(n-m) \tag{12}$$

It is sufficient to consider only a few nonzero positive lags of $R_{yy}(m)$, where $m = L, L+1, \cdots, L+M-1$. The component function $\{(r_j)^m [P_j \cos(\omega_j m) + Q_j \sin(\omega_j m)]\}$ in Eq. (11) is then estimated by best fitting a finite sequence of this function with $R_{yy}(m)$ for $L \leq m \leq L+M-1$. The fitted parameters at the first step will give an estimate of r_j and ω_j, $j = 1$. The corresponding fitted function is then subtracted from $R_{yy}(m)$ to obtain the first residue function $\Re_1(m)$. In the second step, another function of the proposed model is fitted to this residue function to get the second set of r_j and ω_j, $j = 2$. Then a second residue function $\Re_2(m)$ is calculated by subtracting the second fitted function from the first residue function. The k-th residue function is thus defined as

$$\Re_k(m) = \begin{cases} R_{yy}(m), & k = 0 \\ \Re_{k-1}(m) - (r_k)^m T_k(m), & k = 1, 2, \cdots, g' \end{cases} \tag{13}$$

where $T_k(m) = P_k \cos(\omega_k m) + Q_k \sin(\omega_k m)$ and $g' = g - 1$. For $0 < \omega_k < \pi$, we obtain $r_k e^{(\pm j\omega_k)}$ as one pair of complex conjugate poles of the AR system. However, $\omega_k = 0$ or π represent a real pole given by r_k or $-r_k$, respectively. Proceeding this way when all the p poles are identified no further steps are required. As for example, in case of a fourth order system with two real poles and a pair of complex-conjugate poles we need three steps. Once the poles are estimated, the AR system parameters can be obtained from their unique relationship [5].

In the proposed method, the parameters ω_k, r_k, P_k, and Q_k of the k-th component function are chosen such that the sum-squared error, between the $(k-1)$-th residue function and the k-th component function, defined by

$$J_k^{(i)} = \sum_m \left| \Re_{k-1}(m) - (r_k^{(i)})^m T_k^{(i)}(m) \right|^2, \; k = 1, 2, \cdots, g'$$
$$m = L, L+1, \cdots, L+M-1 \quad (14)$$

is minimized, where $T_k^{(i)}(m) = P_k^{(i)} \cos(\omega_k^{(i)} m) + Q_k^{(i)} \sin(\omega_k^{(i)} m)$. Since the proposed method is iterative, the superscript '(i)' denotes the iteration index, i.e., $\omega_k^{(i)}$ denotes the angle of the k-th pole at iteration i. The optimum parameters are found as $P_k = P_k^{(i)}$, $Q_k = Q_k^{(i)}$, $r_k = r_k^{(i)}$, and $\omega_k = \omega_k^{(i)}$ for the value of i at which $J_k^{(i)}$ is minimum. For arbitrary values of $r_k^{(i)}$ and $\omega_k^{(i)}$, $P_k^{(i)}$ and $Q_k^{(i)}$ can be obtained by minimizing $J_k^{(i)}$ in the least-squares sense as

$$\mathbf{DU} = \mathbf{V} \quad (15)$$

where the elements of (2×2) matrix $\mathbf{D}$ are defined by $D_{11} = \sum_m (r_k^{(i)})^{2m} \cos^2(\omega_k^{(i)} m)$, $D_{22} = \sum_m (r_k^{(i)})^{2m} \sin^2(\omega_k^{(i)} m)$, $D_{12} = D_{21} = \sum_m (r_k^{(i)})^{2m} \cos(\omega_k^{(i)} m) \sin(\omega_k^{(i)} m)$, $\mathbf{U}^T = \left[P_k^{(i)} \; Q_k^{(i)} \right]$, and $\mathbf{V}^T = [V_1 \; V_2]$ with $V_1 = \sum_m \Re_{k-1}(m)(r_k^{(i)})^m \cos(\omega_k^{(i)} m)$, and $V_2 = \sum_m \Re_{k-1}(m)(r_k^{(i)})^m \sin(\omega_k^{(i)} m)$.

4 RESULTS

In this section, we examine and compare the performance of the proposed method with that of reported in [8] using two numerical examples. First, the noisy sequence $y(n) = x(n) + w(n)$ is generated using the AR(3) process and noise model expressed by

$$\begin{aligned} x(n) &= 2.299x(n-1) - 2.1262x(n-2) \\ &\quad +0.7604x(n-3) + u(n) \quad (16) \\ w(n) &= v(n) - v(n-1) + 0.2v(n-2) \quad (17) \end{aligned}$$

The variance of the input signal is fixed at $\sigma_u^2 = 1$ and the variance σ_v^2 of the noise process $v(n)$ is selected to give different SNRs defined as

$$\text{SNR} = 10 \log_{10} \frac{\sum_{n=1}^{N} x^2(n)}{\sum_{n=1}^{N} w^2(n)} \text{ dB} \quad (18)$$

In all the simulations $N = 4000$ data samples from noisy observations were used. For determining the *damped sinusoidal* model parameters we have used $R_{yy}(m)$ for $m = L, L+1, L+2, \cdots, L+M-1$. In simulations $M = 10p$ was used, where p is the AR system order and L is chosen to be equal to p as also assumed in [8].

In order to estimate ω and r, a domain of ω from 0 to π was scanned at a resolution of 5×10^{-3} for different values of r. Scanning interval of r was taken to be 0 to 1 and scanning resolutions were chosen to be 1×10^{-3}.

The estimated AR parameters using the proposed method and ILS-CN method reported in [8] are presented in Table 1 for different SNRs. The entries denote arithmetic means and standard deviations of the estimated a_1, a_2, and a_3 based on 10 independent runs. As can be seen, the accuracy of estimation of both the methods are comparable at SNR=10 dB and SNR=5 dB. But at SNR=0 dB, the ILS-CN method completely fails to estimate the AR parameters while no noticeable deterioration in performance of the proposed method is observed. Also the standard deviations of estimation using the proposed method are significantly lower than the ILS-CN method demonstrating better consistency of the proposed scheme.

Next, consider the AR(4) process given by

$$\begin{aligned} x(n) &= 1.0427x(n-1) + 0.1871x(n-2) \\ &\quad -0.9704x(n-3) + 0.6177x(n-4) + u(n) \quad (19) \end{aligned}$$

The noise model is assumed to be the same as in Eq. (17). Fig. 1 depicts the true and estimated average roots of the AR system at SNR=0 dB calculated from 10 independent runs. It is evident from this figure that the ILS-CN method completely fails to identify the AR system. Moreover, the solution is unstable. On the contrary, the proposed method estimates the AR system roots quite accurately. Notice that the solution obtained using the proposed method is always guaranteed to be stable. This is because the search space of r, the magnitude of a root, is 0 to 1. It was also observed that at a relatively low SNR, the ILS-CN method faces non-convergence problem and there was an average of 3 flop tests out of 10 simulations. For non-convergence within 2500 iterations was considered a 'flop test'.

Finally, to illustrate the effect of reducing the SNR on the performances of the proposed and ILS-CN methods, we calculate the normalized error, NE, defined as

$$NE = \frac{\sum_{k=1}^{p} (a_k - \widehat{a}_k)^2}{\sum_{k=1}^{p} a_k^2} \times 100\% \quad (20)$$

where a_k and $\widehat{a}_k$ are the true and estimated parameters of the unknown AR process. The results are shown in Table 2. It can be seen that in contrast to the ILS-CN method the proposed one consistently shows good performance from high to low SNRs.

5 CONCLUSIONS

In this paper, a new method has been presented for estimating the parameters of autoregresive (AR) signals

Table 1: Estimated AR parameters of third-order AR process using the ILS-CN and proposed methods.

True $\{a_k\}$	Estimated AR parameters		
	SNR, dB	ILS-CN	Proposed
$a_1 =$ -2.2990	10	-2.2963 (± 0.1080)	-2.3343 (± 0.0229)
	5	-2.2322 (± 0.3983)	-2.3336 (± 0.0233)
	0	2.1740 (± 3.0840)	-2.3348 (± 0.0370)
$a_2 =$ 2.1262	10	2.1246 (± 0.1691)	2.1963 (± 0.0358)
	5	2.1804 (± 0.6221)	2.1953 (± 0.0365)
	0	15.9248 (± 22.0143)	2.1975 (± 0.0356)
$a_3 =$ -0.7604	10	-0.7613 (± 0.0848)	-0.8038 (± 0.0188)
	5	-0.7985 (± 0.3171)	-0.8040 (± 0.0190)
	0	106.8487 (± 174.9853)	-0.8050 (± 0.0184)

Table 2: Normalized error NE with SNR.

$NE(\%)$ for third-order AR process				
SNR	20 dB	10 dB	5 dB	0 dB
ILS-CN	0.0003	0.0001	0.0852	1.1353×10^5
Proposed	0.0774	0.0775	0.0758	0.0805
$NE(\%)$ for fourth-order AR process				
SNR	20 dB	10 dB	5 dB	0 dB
ILS-CN	0.0139	0.0039	0.0933	2.8344×10^3
Proposed	0.9947	1.1447	0.9420	1.0109

corrupted by colored noise. The AR parameters are computed from the *damped sinusoidal* model parameters introduced in this paper as a novel model for the autocorrelation sequence of the noise-free AR signal. A least-squares type algorithm is used for estimating the *sinusoidal* model parameters iteratively from the noisy data. Compared with the extended improved least-squares technique reported in [8], the proposed one consistently gives more accurate results particularly at low SNRs. Using the proposed method, the stability of the estimated AR system is always guaranteed. The major shortcoming is that it is computationally expensive.

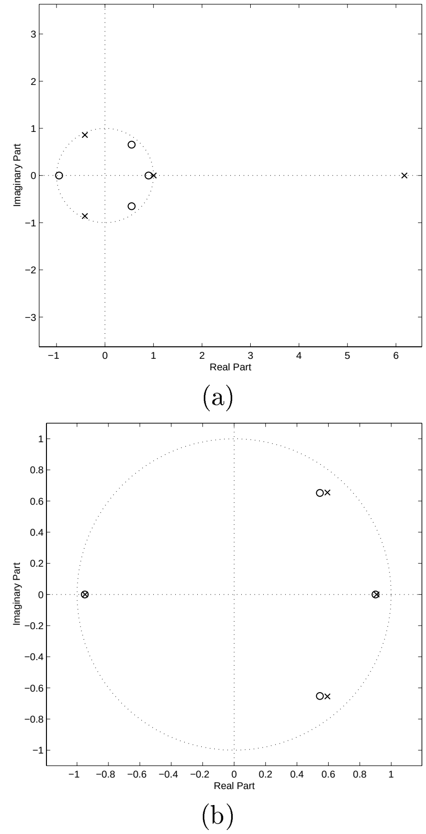

Fig. 1 Estimated roots of the AR(4) system at SNR=0 dB using the ILS-CN and proposed methods: (a) ILS-CN method; (b) Proposed method; (o: true value, x: estimate).

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