# AN INVERSE PROBLEM: HISTOGRAM EQUALIZATION

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**ABSTRACT**

The well-known histogram equalization algorithm is not reversible, namely, given an equalized image and its initial histogram, the original image cannot be recovered. The paper proposes a solution to histogram inversion problem. An ordering on images, closely related to the human perception of brightness, is defined. The proposed ordering refines the normal ordering on graylevels up to a strict ordering. Based on the assumption that the ordering is conserved by histogram equalization algorithm, the inverse problem is further solved. The experimental results show a very good recovery of the original.

## 1 INTRODUCTION

Histogram equalization is a well-known image enhancement technique, [?]-[?], whose purpose is to transform an image to have an almost uniform histogram. The equalized image has a better appearance (looks more "equilibrated"), its contrast is improved and image details become visible.

Histogram equalization algorithm merges levels of gray and spreads the new values as uniform as possible over the graylevel scale. The values of the new graylevels are computed by a very simple procedure based on the histogram of the original image. Thus, if $H = [h_0, h_1, \ldots, h_{L-1}]$ represents the histogram of the original image, the new graylevels, $l_i$, $i = 0, \ldots, L-1$, are:

$$l_i = \left\lfloor (L-1)\frac{\sum_{j=0}^{i} h_j}{\sum_{j=0}^{L-1} h_j} \right\rfloor \tag{1}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. The histogram equalization is merely a pointwise transform on the graylevel range, $T : [0, L-1] \to [0, L-1]$, i.e., $l_i = T(i)$, where $i$ and $l_i$ are the initial and final graylevel, respectively. Due to truncation in equation (??), $T$ is not a mapping onto and therefore, the inverse transform $T^{-1}$ does not exist. In other words, the algorithm is not reversible; once the image has been equalized, one cannot recover the original.

The paper addresses the problem of original image recovery after histogram equalization. The basic idea is to define a strict ordering relation on images which is assumed to be invariant to histogram equalization algorithm. The original image is further recovered from the equalized one by using the initial histogram. The outline of the paper is as follows. In section 2, the proposed ordering relation is defined. Details on implementation and some quantitative evaluations are discussed in section 3. The problem of image recovery after equalization is presented in section 4. Experimental results are provided in section 5. Finally, in section 6, conclusions are drawn and the results of our research are summarized.

## 2 ORDERING: PRINCIPLES

In the sequel, we refer to discrete images of size $N \times N$ having $L$ graylevels, where $N^2 >> L$, i.e., the number of pixels is considerably larger than the number of graylevels. This is the usual case of images of size $512 \times 512$ or $256 \times 256$ having 256 graylevels. We intend to define a strict ordering relation which not only refines the usual ordering on graylevels but is also consistent with the human perception of brightness. Therefore we have to define a procedure such as pixels with similar levels of gray be differentiated. The proposed ordering takes into account the context of each pixel. Such an idea is not new; for instance, in histogram equalization [?], the local mean was considered to split between pixels having the same graylevel. While local mean append some more information, this is not enough for a complete discrimination among pixels. Our approach elaborates on this idea. Instead of a simple mean around each pixel, several mean-values are extracted at different scales and are further used for pixels ordering.

Let $f(x, y) : [1, N] \times [1, N]$ be a discrete image. Let $k$ be a fixed integer and let $W_i$, $i = 1, \ldots, k$ be a family of closed neighborhoods on $f$ such as:

$$W_1 \subset W_2 \subset \ldots \subset W_k \tag{2}$$

For any $(x, y)$, let $m_i(x, y)$ be the mean value of the gray levels of $f$ on the $W_i$ neighborhood of $(x, y)$:

$$m_i(x, y) = \frac{\sum_{(x-i, y-j) \in W_i} f(x-i, y-j)}{||W_i||} \tag{3}$$

where $||W_i||$ is the size (cardinal) of $||W_i||$. We assume the image is expanded with zero value pixels in order to accommodate $W_k$ on its borders.

Let $M(x,y)$ be the $k$-tuple of mean-values on $W_i$, $i = 1,\dots,k$:

$$M(x,y) = (m_1(x,y), m_2(x,y), \dots, m_2(x,y)) \quad (4)$$

Let us denote by $\mathcal{M}$ the set of $M(x,y)$ for all the $(x,y) \subset [1,N] \times [1,N]$. Let us further consider the lexicographic ordering, denoted by $\prec$, on $\mathcal{M}$. According to the lexicographic ordering, $M(a,b) \prec M(c,d)$ if either $m_1(a,b) < m_1(c,d)$ or there is a certain $j$, $1 < j < k$ such as $m_i(a,b) = m_i(c,d)$, $i = 1,\dots,j$, and $m_{j+1}(a,b) < m_{j+1}(c,d)$. The lexicographic ordering induces a complete ordering on $\mathcal{M}$. Since there is a one to one mapping, $M(x,y) \rightarrow (x,y)$, between $\mathcal{M}$ and $f$, the similar ordering is induced on $f$, too. We shall therefore write $f(x_1,y_1) \prec f(x_2,y_2)$ when $M(x_1,y_1) \prec M(x_2,y_2)$

Let $W_1$ be reduced at pixel size. We observe that for $k = 1$ one recovers the usual ordering on $f$. Let $k$ be greater than 1, e.g., $k = 2$. The mean values over $W_2$ do count in the induced ordering and thus, one can discriminate between certain pixels having the same graylevel. The greater $k$, the finer the ordering.

Let us suppose that for a certain $k$ and a family of $W_i$ one obtains a strict ordering on $\mathcal{M}$:

$$M(x_1,y_1) \prec M(x_2,y_2) \prec \dots \prec M(x_{N^2},y_{N^2}) \quad (5)$$

Due to the mapping between $\mathcal{M}$ and $f$, the same strict ordering appears on the corresponding image:

$$f(x_1,y_1) \prec f(x_2,y_2) \prec \dots \prec f(x_{N^2},y_{N^2}) \quad (6)$$

The proposed ordering is consistent with the usual ordering on $f$. Thus, the strict classical ordering is conserved by the new one:

$$f(x_1,y_1) < f(x_2,y_2) \;\Rightarrow\; f(x_1,y_1) \prec f(x_2,y_2) \quad (7)$$

Besides, the new ordering refines the classical one, i.e., equal pixels by the classical ordering become strictly ordered. Thus, the converse of (??) appears as:

$$f(x_1,y_1) \prec f(x_2,y_2) \;\Rightarrow\; f(x_1,y_1) \leq f(x_2,y_2) \quad (8)$$

The proposed ordering relation corresponds to the intuitive idea of brightness; at the same graylevel, a pixel appears brighter than another, when its local mean is brighter than the local mean of the other one. Besides, if $k$ is large enough, for any two pixels, $(x_1,y_1)$ and $(x_2,y_2)$ one gets either $M(x_1,y_1) \prec M(x_2,y_2)$ or $M(x_2,y_2) \prec M(x_1,y_1)$. However, such an ordering cannot be possible for any image. If we consider a constant image, $f(x,y) = a$, regardless $k$ and $W_i$, equation (??) fails (except for some pixels near the borders). Therefore, we assume images we deal with have a good resolution, namely "enough" graylevels. Also, in order to avoid large uniform areas, we assume natural (real) images. Furthermore, since $k$ and the size of $W_i$ should be connected, in a certain sense, to the human perception on brightness, we should consider moderate values for $k$ as well as for the size of $W_k$. Otherwise stated, a certain correlation being assumed among image pixels, it does not make sense to deal with pixels far apart from each others. On the other hand, the requirement to have a strict ordering among the entire image could be relaxed, i.e., we accept a strict ordering *almost everywhere*. This means that, we allow some equalities in equation (??), meantime keeping $k$ as small as possible

Table 1: Ordering with respect to $k$.

| $k$ | Separability [%] | Max-length | Mean-length |
|---|---|---|---|
| 1 | 0.082 | 2,742 | 1,225 |
| 2 | 9.17 | 139 | 10.9 |
| 3 | 62.44 | 26 | 1.6 |
| 4 | 96.85 | 6 | 1.0325 |
| 5 | 99.87 | 2 | 1.00013 |
| 6 | 99.98 | 2 | 1.00002 |

## 3 ORDERING: RESULTS

The previous section states an ambitious goal, namely to induce a strict ordering almost everywhere and concludes with some comments on images and on the dimension of the windows family. In order to give some meaning to the rather ambiguous figures of merit "moderate values for $k$" and "enough graylevels" we define first a family $W_i$ and afterwards we evaluate the induced ordering with respect to $k$.

### 3.1 $W_i$ Family

The only constraint imposed on $W_i$ family is given by equation (??) which imposes a strict inclusion relation for $W_i$ family. Meantime, some geometrical symmetry is assumed. Therefore, starting with a pixel size element, we designed a symmetric family where the increase from successive elements was kept to a minimum.

The description of the first 6 elements in the family follows:

- $W_1$ - pixel size $(1 \times 1)$;
- $W_2$ - classical $V_4$ unit ball;
- $W_3$ - classical $V_8$ unit ball $(3 \times 3)$;
- $W_4$ - $V_4$ for radius 2;
- $W_5$ - the $5 \times 5$ window without corner pixels;
- $W_6$ - $V_8$ for radius 2 (the full $5 \times 5$ window);
- ...

| | | | | | | |
|---|---|---|---|---|---|---|
| 80 | 84 | 95 | 97 | 97 | 105 | 100 |
| 81 | 87 | 90 | 95 | 95 | 96 | 97 |
| 84 | 92 | 97 | 96 | **96** | 99 | 100 |
| 85 | 93 | 93 | 94 | 104 | 99 | 99 |
| 87 | 89 | 91 | 94 | **96** | 103 | 100 |
| 83 | 85 | 92 | 94 | 92 | 94 | 97 |
| 88 | 90 | 91 | 96 | 101 | 97 | 102 |

$\Downarrow$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 5 | 25 | 34 | 35 | 49 | 44 |
| 2 | 9 | 13 | 26 | 27 | 32 | 38 |
| 4 | 16 | 33 | 29 | **31** | 39 | 42 |
| 6 | 19 | 20 | 23 | 48 | 41 | 40 |
| 8 | 11 | 15 | 22 | **30** | 47 | 43 |
| 3 | 7 | 17 | 21 | 18 | 24 | 36 |
| 10 | 12 | 14 | 28 | 45 | 37 | 46 |

Figure 1: Original image (up) and the induced ordering (down).

| | | | | | | |
|---|---|---|---|---|---|---|
| 55 | 59 | 71 | 74 | 74 | 89 | 80 |
| 56 | 61 | 65 | 71 | 71 | 73 | 74 |
| 59 | 67 | 74 | 73 | **73** | 78 | 80 |
| 60 | 68 | 68 | 70 | 88 | 78 | 78 |
| 61 | 64 | 66 | 70 | **73** | 86 | 80 |
| 58 | 60 | 67 | 70 | 67 | 70 | 74 |
| 63 | 65 | 66 | 73 | 82 | 74 | 84 |

$\Downarrow$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 5 | 25 | 34 | 35 | 49 | 44 |
| 2 | 9 | 13 | 26 | 27 | 32 | 38 |
| 4 | 16 | 33 | 29 | **30** | 39 | 42 |
| 6 | 19 | 20 | 23 | 48 | 41 | 40 |
| 8 | 11 | 15 | 22 | **31** | 47 | 43 |
| 3 | 7 | 17 | 21 | 18 | 24 | 36 |
| 10 | 12 | 14 | 28 | 45 | 37 | 46 |

Figure 2: Equalized image (up) and the induced ordering (down).

Obviously, the development of the family can continue, on the same idea, with $W_7$, and so on.

### 3.2 Quantitative Aspects

The evaluation of ordering regards how the proposed ordering applied on real images satisfies (??). A strict ordering means that all inequalities in (??) are strict inequalities, i.e., for a $N \times N$ image, we should have $N^2$ distinct pixels. Since the equation holds only approximately, we expect to find groups of "equal" pixels in the string. We are interested to evaluate how often such groups appears and how large they are. A global measure of the quality to be evaluated is the percentage of equal pixels, i.e., nonseparable, in the string. The evaluation is performed with respect to the size of $W_i$ family, being meant to guide the choice of $k$.

In Table 1 we present the results of our evaluation on the test image lena of size $512 \times 512$. Besides the percentage of separable pixels and the maximum length of a group of equal pixels we display also the medium length of the group of equal pixels (in the ideal case, 1). The first row of the table recovers the usual ordering on images; there are 215 graylevels which hardly can separate 262,144 pixels. As $k$ increases, the quality of the order increases as well. Thus, for $k > 4$ the pixels are almost strictly ordered. For example, for $k = 5$, the ordered string contains only 351 pairs of nonseparable pixels and, for $k = 6$, only 54 pairs are present.

We found quite similar results in all our tests (mandrill baboon, sailboat, cameraman, tree, peppers, etc.). Therefore, the information present in a $5 \times 5$ window can be used to induce by the procedure we described an almost strict ordering on real images.

## 4 THE INVERSE ALGORITHM

Let us suppose that the image $f$, having original histogram $H$, has been equalized. The equalized image is:

$$f^* = T(f) \tag{9}$$

The inverse algorithm is based on the assumption that the proposed ordering is conserved by the histogram equalization algorithm. This means that:

$$f(x_1, y_1) \prec f(x_2, y_2) \Rightarrow f^*(x_1, y_1) \prec f^*(x_2, y_2) \tag{10}$$

Ordering $f^*$ one gets a string of pixels

$$f^*(x_1^*, y_1^*) \prec f^*(x_2^*, y_2^*) \prec \ldots \prec f^*(x_{N^2}^*, y_{N^2}^*) \tag{11}$$

If the ordering is invariant to histogram equalization, the strings in (??) and (??) should be identical. This hypothesis states that there is no need for re-indexing in (??), i.e., for each $i$, $(x_i^*, y_i^*) \equiv (x_i, y_i)$, and consequently:

$$f^*(x_1, y_1) \prec f^*(x_2, y_2) \prec \ldots \prec f^*(x_{N^2}, y_{N^2}) \tag{12}$$

Furthermore, if $H = [h_0, h_1, \ldots, h_{L-1}]$ is the original histogram of $f$, the leftmost group of $h_0$ pixels in (??) have the graylevel 0, the next group of $h_1$ pixels have the graylevel 1, and so on.

Due to the equivalence of the ordered strings (??) and (??) the invertion algorithm immediately follows: for each $j$, $j = 0, \ldots, L-1$ and starting with $j = 0$, we assign to the group of $h_j$ successive pixels from left to right in (??) the corresponding graylevel in the original histogram. After the final pixel in (??), $(x_{N^2}, y_{N^2})$, has been assigned to its corresponding graylevel, the original image has been recovered.

$$
\begin{array}{ccc}
 & -1 & \\
0 & \mathbf{0} & 3 \\
 & 8 & \\
-2 & \mathbf{0} & 7 \\
 & -4 &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & -2 & \\
0 & \mathbf{0} & 5 \\
 & 15 & \\
-3 & \mathbf{0} & 13 \\
 & -6 &
\end{array}
$$

Figure 3: Graylevel variation before (left) and after (right) histogram equalization.

**Comment.** The recovered image has exactly the original histogram. This makes the procedure very suitable for exact histogram specification tasks, including exact histogram equalization.

### 4.1 Order Conservation

The algorithm would perform exactly if the order should be strictly conserved by histogram equalization. Obviously a linear transform on $f$ conserves ordering. Histogram equalization is a nonlinear transform which, however, does not destroy natural ordering. For example, if $f(x_1, y_1) < f(x_2, y_2)$ one should have $f^*(x_1, y_1) \leq f^*(x_2, y_2)$ and never $f^*(x_1, y_1) > f^*(x_2, y_2)$. On the other hand, due to the intrinsic nonliniarity, there is no proportionality between differences of graylevels before and after histogram equalization. This can, in certain conditions, violate the conservation of the proposed ordering.

Such an example is given next, namely for a $7 \times 7$ block of a real image. In Fig.1 and Fig. 2 are shown the original - equalized block, respectively, together with the corresponding induced ordering. As it can be seen, an error appears for the positions 30 and 31, namely for the pixels $(3,5)$ and $(5,5)$, respectively. For the original image $f(3,5) \prec f(5,5)$ ($m_1(3,5) = m_1(5,5)$ and $m_2(3,5) > m_2(5,5)$). For the equalized one, the order is reversed, i.e., $f^*(5,5) \prec f^*(3,5)$, since $m_1(3,5) = m_1(5,5)$ but $m_2(3,5) < m_2(5,5)$. This is due to the nonlinear assignment of graylevels by the histogram equalization procedure. To make things more visible, Fig. 3 presents only the graylevel variation over $W_2$ with respect to the central pixels $(3,5)$ and $(5,5)$ before and after histogram equalization, i.e., the variation with respect to graylevel 96 (original image) and 73 (equalized image), respectively.

## 5 EXPERIMENTAL RESULTS

Since the hypothesis of order conservation does not completely hold, we expect that a certain reconstruction error does exists. In order to evaluate the results, we have computed the Peak-to-peak Signal to Noise Ratio (PSNR) of the reconstructed images. The experimental results for some well-known test images are presented in Table 2. As it can be seen, the quality of the reconstruction is very good.

Table 2: Image recovery, experimental results.

| Image | Size | PSNR [dB] |
|---|---|---|
| peppers | $512 \times 512$ | 60.35 |
| girl | $256 \times 256$ | 58.52 |
| lena | $512 \times 512$ | 58.45 |
| tree | $256 \times 256$ | 57.60 |
| sailboat | $512 \times 512$ | 56.60 |

After the first restoration, we noticed that the ordering induced on the reconstructed image becomes stable to a further application of the histogram equalization algorithm and the inversion becomes error free.

## 6 CONCLUSIONS

A method for recovering images after histogram equalization has been presented. An ordering relation, strict almost everywhere, is defined first on graylevel images. The ordering is supposed to be invariant to histogram equalization algorithm and the inversion take place by ordering the equalized image and considering its original histogram. The experimental results obtained so far show a very good recovery of the original, namely PSNR's of 50 to 60 dB.

The resulted image has exactly the original histogram. The inverse problem we stated can be seen as an exact histogram specification plus a restoration problem. Another interesting aspect of our approach is the fact that the recovered image and the equalized one form an invariant pair of images to the classical histogram equalization algorithm and to the inverse algorithm we proposed.

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