DECOMPOSITION AND ORDER STATISTICS IN FILTERING

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ABSTRACT

The paper investigates a three stage filtering scheme, namely: 1) signal decomposition, 2) filtering and 3) signal reconstruction. If marginal rank order filtering is used in step 2, the derived filtering scheme generalizes the classical order statistics one. Based on this idea, a new family of nonlinear filters, called *decomposition filters*, is proposed and investigated. The most interesting feature of the new filters is their dependency on signal decomposition. Three decomposition procedures that exhibit certain minimum and symmetry properties are investigated. They are the canonical decomposition of functions, the Jordan decomposition of bounded variation functions and the parity decomposition, respectively. The properties of the derived filters are discussed.

1 INTRODUCTION

Since the median filter has been proposed, the class of order statistics filters has been developed extensively. This paper proposes a framework which generalizes the classical order statistic filters. The outline of the paper is as follows. In section 2, the basic principles and definitions of decomposition filters are introduced. Based on the formalism of section 2, three classes of filters are proposed in section 3, 4 and 5. They are: canonical decomposition filters, Jordan decomposition filters and parity decomposition filters. Finally, conclusions are drawn in section 6.

2 DECOMPOSITION FILTERS

Let $\mathcal{F}$ be a set of functions defined on a finite domain $D \subset S^p$ ranging in a bounded interval $M \subset S$, where S stands either for R or for Z; $f \in \mathcal{F}$ if $f : D \to M$. The set $\mathcal{F}$ is meant to represent signals: real signals when S stands for R, discrete ones when S stands for Z; 1D signals for $p = 1$ and 2D signals for $p = 2$. For the moment, we consider f to be measurable functions. According to the particular signals to deal with, some other constraints can be imposed on f, e.g., bounded variation functions, class L^k functions, etc.

Let m be a fixed integer and let $\mathcal{D}$ be an operator that maps $\mathcal{F}$ into $\mathcal{F}^m$:

$$\mathcal{D}(f) = (f^1, f^2, \ldots, f^m) \tag{1}$$

where $f^i : D \to M$ for $i = 1, 2, \ldots, m$ and

$$f = \sum_{i=1}^{m} f^i \tag{2}$$

The operator $\mathcal{D}$ can be thought as:

$$\mathcal{D} = (\mathcal{D}^1, \mathcal{D}^2, \ldots, \mathcal{D}^m) \tag{3}$$

where

$$\mathcal{D}^i(f) = f^i, \; i = 1, \ldots, m \tag{4}$$

The operator $\mathcal{D}$ provides a decomposition of a signal f in a sum of exactly m components. The decomposition is a representation of f as an ordered set of components. The ordering should be understood as strictly referring to the position of the components within the m-tuple. No other ordering on the components is assumed so far.

With the above notations the new filters are defined as follows:

Definition 1 *Given an operator $\mathcal{D}$ and a set of m ranks, $(r_1, r_2, \ldots, r_m)$, the output $DF(f)$ of the decomposition filter of a signal f for a window w is*

$$DF(f) = \sum_{i=1}^{m} f^i_{(r_i)} \tag{5}$$

where $f^i_{(r_i)}$ is the r_i-th order statistic of f^i within the filter window w.

In other words, the new scheme of filtering takes the representation of the signal f provided by using the operator $\mathcal{D}$, it executes marginal rank order filtering (with m fixed ranks) within the filter window w, and finally, it delivers the output as the sum of the filtered components. If the size of the filter window is n, for each component, the ordering within the filter window is performed on n samples. The ranks r_i should obey $1 \le r_i \le n$ for $i = 1, \ldots, m$.

While rank order filters can be defined for only n ranks, where n is the size of the window filter, the new scheme increases the range up to n^m. Furthermore, a fundamental observation should be made: while the rank order filters are completely specified by the considered rank and the filter window, the new scheme introduces the dependence on the operator $\mathcal{D}$ that becomes crucial for the behavior of the filter. Thus, much care should be taken in the selection of $\mathcal{D}$. First of all, it is desirable to yield filters with interesting properties for signal processing, namely good noise suppression or an ability to emphasize some features like edges, etc. The operator $\mathcal{D}$ should be well-defined, i.e., to yield an unique representation of the signal. For computational complexity reasons, $\mathcal{D}$ should be as simple as possible and m should have moderate values. Condition (**??**) has been explicitly enforced in the definition of $\mathcal{D}$ in order to reduce the computational complexity of the implementation of the new filters.

A rather trivial example for $\mathcal{D}$ ($m = 2$), is when $\mathcal{D}^1$ is the identity operator on $\mathcal{F}$, $\mathcal{D}^1(f) = f$, and $\mathcal{D}^2$ is the null operator on $\mathcal{F}$, $\mathcal{D}^2(f) = 0$. The representation of f by using this very simple operator is $(f, 0)$. It is easy to see that, in this case, Definition 1 recovers the classical rank order filters. Since decomposition filters are strongly dependent on the decomposition, we look for operators that have some properties that are expected to generate interesting filters. Thus, we investigate operators that obey some minimum or symmetry properties. They yield representations with $m \leq 3$ components.

2.1 L-Decomposition Filters

An extension of decomposition filters can be easily defined by following the approach of L-filters, which are linear combinations of order statistics [**?**]:

$$y_i = \sum_{j=1}^{n} a_j x_{(j)} \tag{6}$$

The well-known moving average, median, r-th rank order, α-trimmed mean and midpoint filters are special cases of L-filters when appropriate sets of coefficients $a_j,\ j = 1, \ldots, n$ are used. The ability of the L-filter to have optimal coefficients for a variety of input distributions makes it suitable for a large number of applications.

The L-decomposition filters definition is as follows:

Definition 2 *Given a decomposition operator $\mathcal{D}$, and an array of $m \times n$ coefficients, $A = [a_{i,j}]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, the output LD of the L-decomposition filter of a signal f for a window w is*

$$LD = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} f^i_{(j)} \tag{7}$$

where $f^i_{(r)}$ is the r-th order statistic of f^i within the filter window w.

L-decomposition filter computes marginal L-filtering on the components of the decomposition and delivers their sum as the output of the filter. For a window filter of size n and a decomposition in m components, $m \times n$ coefficients are needed. This suggests that L-decomposition filters are more difficult to be analyzed than the classical L-filters. A simplified definition of L-decomposition filters can be given by considering, for each component, only one non-zero coefficient.

Definition 3 *Given an operator $\mathcal{D}$, a set of m ranks, $(r_1, r_2, \ldots, r_m)$ and a set of m coefficients, $(a_1, a_2, \ldots, a_m)$, the output LD of the L-decomposition filter of a signal f is*

$$LD = \sum_{i=1}^{m} a_i f^i_{(r_i)} \tag{8}$$

where $f^i_{(r_i)}$ is the r_i-th order statistic of f^i within the filter window w.

Equations (**??**) and (**??**) look very similar. However, it should be noted that (**??**) is a linear combination of the ordered samples within the window filter, while (**??**) is a linear combination of marginal rank order statistics. Thus, each rank r_i is computed for its corresponding coefficient f^i within the filter window and only one sample for each component contributes to the output of the filter. The contribution of the components in equation (**??**) can be very different and thus, the selection of the weights a_i can be used to scale the output, emphasize the effect of some components, design adaptive filters, etc.

3 CANONICAL FILTERS

The canonical decomposition of a function f is

$$f = f^+ - f^- \tag{9}$$

where

$$f^+ = \max\{f, 0\},\ f^- = -\min\{f, 0\} \tag{10}$$

f^+ and f^- are called the positive and negative parts of f. They are positively defined and measurable functions. Equation (**??**) gives a standard representation of f as a difference of two non-negative functions. A certain minimum property of the representation exists, since given two functions g and h such that $f = g - h$ and $g, h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.

Let T be a suitable value within the range of the signal f and let us consider the operator $\mathcal{CD}$ defined as follows:

$$\mathcal{CD} = (\mathcal{CD}^1, \mathcal{CD}^2, \mathcal{CD}^3) \tag{11}$$

where

$$\mathcal{CD}^1(f) = f^1 = \max\{f - T, 0\} \tag{12}$$

$$\mathcal{CD}^2(f) = f^2 = \min\{f - T, 0\} \tag{13}$$

Figure 1: Noisy image.

Figure 2: Filtered image.

$$\mathcal{CD}^3(f) = f^3 = T \tag{14}$$

By using the operator $\mathcal{CD}$ defined above, we have:

$$f = f^1 + f^2 + f^3 \tag{15}$$

When $T = 0$, (**??**) is merely the canonical decomposition of the signal f, with the minor difference that $f^2 = -f^-$. Equation (**??**) can be seen as the approximation of the original signal f by a constant signal T. The approximation error is split in two components: f^1 the positive error and f^2 the negative one. Convenient selections for T are: the arithmetic mean of f, the median of f, etc.

According to Definition 1, the output of the canonical decomposition filter is:

$$\mathcal{CD}_{r_1,r_2} f = f^1_{(r_1)} + f^2_{(r_2)} + f^3 \tag{16}$$

Canonical decomposition filters are closely related to rank order filters. Their output range is bounded by the classical max and min filters within the same window.

Since r_1 and r_2 range in $[1, \ldots, n]$, equation (**??**) allows n^2 combinations of the ranks. Not all the combinations yield distinct outputs; at most $n^2/4$ ones. We mention two aspects regarding ranks selection: 1) when ranks are equal, $r_1 = r_2 = r$, $1 \leq r \leq n$, the canonical decomposition filters become the r-th rank order statistic filters; 2) $r_1 = r$ and $r_2 = n - r$, the output variance of the derived canonical filters is smaller than the output variance of the classical rank order filters (the filter has the tendency of approximating T).

In smoothing, the smaller the variance, the better the filter performance is. Thus, the most appropriate selection of ranks is as in case 2. An example of image filtering, for a 5×5 window and with $r_1 = 12$ $r_2 = 14$, is shown in Fig. 2. The noisy girl image shown in Fig. 1 was heavily corrupted with a mixture of Gaussian and impulsive noise.

4 JORDAN FILTERS

Jordan decomposition theorem asserts that any bounded variation 1D function can be decomposed in a difference of two non-decreasing functions [**?**]. By considering a decomposition operator

$$\mathcal{JD}(f) = (f^1, f^2) \tag{17}$$

where f^1 and $-f^2$ obey Jordan decomposition of f, the definition of Jordan decomposition filters is:

$$\mathcal{JD}_{r_1,r_2} f = f^1_{(r_1)} + f^2_{(r_2)} \tag{18}$$

Jordan decomposition filters (1D and 2D) have been introduced in [**?**, **?**] for the case of Jordan max and min filters, namely for the case $r_1 = n$ $r_2 = 1$ and $r_1 = 1$ $r_2 = n$, respectively. Jordan max and min filters bound the classical max/min filters; their output increases (decreases, respectively) with n. Opposite to max/min filters, they do not smooth the signal; by the contrary, they are well-suited to emphasize signal variation. Jordan filters have been used in low level image processing tasks [**?**], and recently, Jordan features have been derived for texture segmentation [**?**].

The general definition of Jordan filters given here increases their flexibility and preserves their attractive computational features, namely low computational complexity, constant number of operations per sample regardless the window size and decreasing computational cost with respect to the number of filtering steps in multiple filtering applications.

5 PARITY FILTERS

The decomposition operator is locally defined for symmetric windows as $\mathcal{PD}(f) = (f^1, f^2)$, where

$$f^1(x) = \frac{f(x) - f(-x)}{2}, \; f^2(x) = \frac{f(x) + f(-x)}{2} \tag{19}$$

Figure 3: Original image.

Figure 4: Filtered image.

The filter output is:

$$\mathcal{PD}_{r_1,r_2} f = f^1_{(r_1)} + f^2_{(r_2)} \tag{20}$$

f^1 is odd and f^2 is even with respect to the center of the window filter. The odd and even parity components capture information on the local shape of the signal: thus, symmetrical signals within the filter window generate null odd components, and anti-symmetrical signals generate null even components, respectively. Several possible selections of the ranks follow: 1) When r_1 is close to the median and r_2 is close to the minimum or maximum, the corresponding filter is expected to enhance the signal. Its output is the superposition of the maximum variation on a local mean of the signal. Such an example is given in Fig. 4 by using a filter on a 3×3 window with $r_1 = 5$ and $r_2 = 9$. As it can be seen, the filtered image is biased to white. 2) When r_2 is the median, f^2 does not influence the filter. If r_1 is close to the median, the filter smoothes the signal. If r_1 is close to 1 or n, the filter behavior is similar to the one of max or min filters. The case when r_2 is the median reminds the Wilcoxon filter.

Some more flexibility exhibits L-parity filters. The selection of the filter coefficients controls the effect of f^2 and f^1. For instance, a small value of a_1 and a large value for a_2 reduces the local mean of the signal and accentuates the variation of the signal within the filter window.

6 CONCLUSIONS

A theoretical framework for the definition of new classes of nonlinear filters, called decomposition filters, was developed. The proposed scheme consists of the decomposition of the signal in a sum of components, marginal rank order filtering on the components followed by signal reconstruction. The scheme generalizes classical rank order filters; rank order filters are retrieved for particular decompositions as well as for certain rank selections. The combinations of the ranks for marginal rank order filtering of the components yield up to n^m possible filters (n is the size of the filter window). The decomposition operator is essential for the behavior of the filters. Three decomposition operators are investigated. They are based on the canonical decomposition of functions, the odd-even decomposition and the Jordan decomposition of bounded variation functions. The derived filters are discussed and applications in image smoothing and enhancement are investigated. The paper is mainly an introductory one. The proposed examples are far from exhausting the topic. Further researches are in progress.

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