PARAMETER ESTIMATION OF CONICS: APPLICATION TO HANDWRITTEN DIGITS

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ABSTRACT

We expose a method for modeling handwriting thanks to conic sections described by Cartesian equations under an implicit form. The parameter estimation is processed by an extended Kalman filter, taking as minimization criterion, the squared orthogonal distance between a point and the conic. The state equation is here constant, and the observation is a system of two equations: the first one characterizes the minimization of the criterion, and the second one is a normalization constraint of the parameters. The method provides a robust and invariant estimation of parameters, and an unique solution allowing the classification of modeled patterns.

We apply this method to the coding of handwritten digits. A geometrical criterion allows to locate model changes. For a large interval of the used thresholds, we observe a great stability of the estimated parameters and of the instants of model changes. The method is evaluated in terms of accuracy, but equally by the data reduction rate, compared to other modeling techniques.

1 INTRODUCTION

The digitized curve description has two objectives: to decrease the mass of data and to allow processes of higher level (recognition, diagnosis...). A set of points in the 2D space representing handwriting can be described by one or several representations estimated sufficient to envisage ulterior processing. But choosing a technique of representation consists in finding a trade-off between the accuracy of the representation and the execution time necessary for its processing. In fact, an accurate representation is always computable but often not in a satisfactory time. A representation by sections of lines can be sufficient, and a method of segmentation based upon the use of an extended Kalman filter followed by an algorithm of detection of changes in drawing direction was already realized [1]. In the framework of the help in diagnosis of psychological illness in the child, the method was applied on the segmentation of the Rey's Complex Figure [2]. It provides on this geometrical drawing essentially composed of sections of lines, an important data reduction as well as a representation adapted to its analysis, by giving relevant information for the paediatrician [3]. Nevertheless, on the drawings presenting strong curvatures, it is legitimate to think that a representation by sections of lines becomes little efficient. A method of segmentation by bows of circles was thus realized [4] or [5], allowing to segment a drawing in a succession of circle bows. The method uses a technique of parameter estimation very close to the preceding one, always with the help of an extended Kalman filter. The objective of this paper is to generalize this method to all types of conic, what will allow a more accurate fitting of curves with less parameters.

In the next section, we develop the estimation technique using an extended Kalman filter. In the third section, we give some results in an application on real digits. The last section presents our strategy of model selection.

2 CONIC FITTING PROBLEM

In this work, the problem consists in estimating the parameters of a curve. We choose as general representation model a Cartesian equation in the implicit form [6]. The technique consists in estimating the vector of parameters of the form described by a vector of state X. The system of observation will be constituted of two observations: the first one is linked to the form to estimate, while the second one is a condition of parameter normalization. In a general manner, the curve C is described by a function Q: $\mathbb{R}^2 \rightarrow \mathbb{R}$ supposed continuous and having first and second derivatives in each point. We define $Z(Q) = \{Z: Q(Z,X)=0\}$, the set of zeroes of Q by the implicit function:

$$Q(Z,X)=0 \tag{1}$$

that represents the general form of the modelled feature. Generally, function Q is a non linear function between the vector of parameters X and the point Z. The problem is to formalize the recursive estimation of parameter X by a Kalman filter. The state model is described by an equation of state X characterizing the vector of parameters. Consequently, we pose the following stationary state equation:

$$X(t+1) = X(t) \tag{2}$$

The introduction of a noise $W(t)$ on the state would consequently lead to consider a variation of parameters [7], inducing a variation of the form, what would be opposite to our objective of segmentation. Due to the fact of the constancy of the state vector, calculations in each step only correspond to the equations of updating the Kalman filter.

A conic curve is generally represented by a 2^{nd} degree polynomial put under the next implicit form:

$$Q(Z,X) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{3}$$

with $Z = (x, y)^{tr}$. The discriminant $B^2 - 4AC$ will be negative if the conic is an ellipse, positive if it is an hyperbole and null if the conic is a parabola. One shows that $Q(Z,X)$ is a function of the distance to center, from the point Z to the conic $Q(Z,X)$. If the point (x,y) is out of the ellipse, the value of $Q(Z,X)$ will be negative, and positive in the opposite case. The fitting of a general conic, from a set of T points, can be realized by minimizing the following quadratic sum:

$$\sum_{i=1}^{T} Q(Z(t), X(t))^2 \tag{4}$$

The minimization of a quadratic error function of this distance would present the following disadvantage: a point would differently contribute to the estimation of parameters according to its position of the conic, what is easily understood if one considers the case of an ellipse. We propose to use an orthogonal distance to the conic, but the expression of this distance is complicated and necessitate a step of iterative optimization [8]. We have developed an approximation of the square d_t^2 of this distance, based on the limited development of the quadratic error between an estimated measure and the true measure, pondered by the covariance of the measure noise. To minimise the orthogonal distance, the minimisation criterion must satisfy:

$$J = \sum_{t=1}^{T} d_t^2 \tag{5}$$

We define the norm d_t^2 by the following scalar product:

$$d_t^2 = \left(Z(t) - \hat{Z}(t)\right)^{tr} L(t)^{-1} \left(Z(t) - \hat{Z}(t)\right) \tag{6}$$

The factor $L(t)^{-1}$ means that the distance is independent of the measure noise. Let us note $L^{-1} = S^{tr}S$. We can show that d_t^2 can take the following form, with Q and ∂Q, taken in $\hat{X}(t/t-1)$ and $S\hat{Z}(t)$:

$$d_t^2 = \frac{1}{v(t)^2}\left[Q + \left(\frac{\partial Q^{tr}}{\partial X} - \frac{Q}{v(t)}\frac{\partial v(t)^{tr}}{\partial X}\right)\left(X(t) - \hat{X}(t/t-1)\right) + V_Q(t)\right]^2 \tag{7}$$

where $v(t)$ is defined by:

$$v(t)^2 = \frac{\partial Q(S\hat{Z}(t), X(t))^{tr}}{\partial Z} L(t) \frac{\partial Q(S\hat{Z}(t), X(t))}{\partial Z} \tag{8}$$

and where $V_Q(t)$ is a noise due to the quadratic term. The form of this approximation allows it to be integrated in the Kalman filter.

Fitting the conic consists in minimizing eq.(7), by writing:

$$Y_Q(t) = -Q\left(S\hat{Z}(t), \hat{X}(t/t-1)\right) + \left[\frac{\partial Q^{tr}}{\partial X} - \frac{Q}{v(t)}\frac{\partial v(t)^{tr}}{\partial X}\right]\hat{X}(t/t-1) \tag{9}$$

With the purpose to avoid the trivial solution, all parameters null, or a proportional solution, the vector X of parameters is rendered unique with the help of a constraint. We therefore define a vectorial observation equation, and the following system has to be satisfied:

$$Y(t) \hat{=} \begin{pmatrix} Y_Q(t) \\ Y_N(t) \end{pmatrix} = 0 \tag{10}$$

Several published algorithms differ only by the form of the constraint applied on parameters [9]. For necessary reasons of invariance to geometrical transformations in Pattern Recognition, we use the constraint

$$Y_N(t) = N[X(t)] = A^2 + ½ B^2 + C^2 - 1 = 0$$

proposed by Bookstein [10]. The chosen constraint Y_N is non linear according to parameters of vector X; it is linearized under the hypothesis of weak variation of parameters between two successive instants. The second component of the observation may is written by:

$$Y_N(t) = N(\hat{X}(t/t-1)) + \frac{\partial N(\hat{X}(t/t-1))^{tr}}{\partial X}\left(X(t) - \hat{X}(t/t-1)\right) + V_N(t) \tag{11}$$

where $V_N(t)$ is also a noise due to the quadratic term.

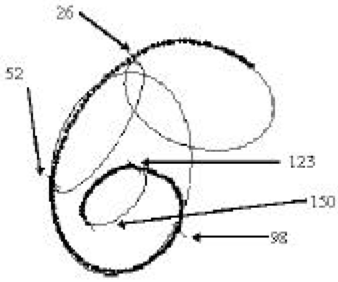
The initialization of parameters is based upon a least squares method applied on the 5 first points of the drawing. The obtained values are normalized by $(Y_N+1)^{-½}$, giving the fist vector of parameters.

3 APPLICATION TO THE SEGMENTATION OF REAL DIGITS

The segmentation by conic sections necessitates a detector of end of model. The realized detector is based on an angular deviation between the model and the acquired point. However, due to the fact of local perturbations of the layout, the crossing of this threshold has to be confirmed by a second test. This test is based upon a crossing of the threshold on a window of observation fixed to three samples. We illustrate results of segmentation on handwritten figures, on-line acquired thanks to a digitizer. In figures 1 and 2, we show a figure six written by an adult from the top to the bottom. This figure is composed of 150 points. We overlap on this figure the result of the segmentation by conic sections for two values of the threshold α_S (0,20 and 0,30 radians). Points of model changes are represented by segments of lines at each end of conic section. There is no important difference between the two figures, except the nature of the fourth section: with the threshold α_S =0,20, it concerns a section of hyperbole, while with the threshold α_S =0,30, we find an ellipse section. We give in tables 1 and 2, the position of bow changes, the value of each bow parameter, and the type of conic.

Pos.	A	B	C	D	E	F	Type
26	0.457	-0.295	0.864	-4952	-3488	20893914	Ellipse
52	0.739	0.808	0.356	-10324	-6604	37645955	Ellipse
98	0.893	0.176	0.431	-10305	-4557	36299681	Ellipse
123	0.015	1.101	0.627	-4406	-12665	40319321	Hyperb
150	0.564	0.677	0.671	-8991	-9608	45190671	Ellipse

Table 1: segmentation of a 6 with α_S= 0.2



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101	0.893	0.176	0.431	-10305	-4557	36299680	Ellipse
125	0.359	0.602	0.830	-6282	-11496	43279595	Ellipse
150	0.536	0.630	0.716	-8536	-9796	44613413	Ellipse

Table 2: segmentation of a 6 with α_S = 0.3

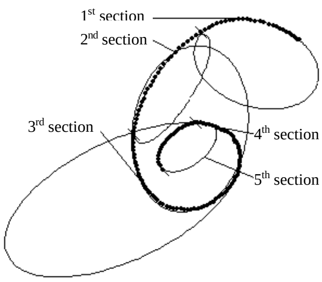

Figure 2: segmentation of a 6 with α_S= 0.3

We can observe that in areas of weak curvature, changes of sections append on the same points, despite the strong differences of threshold values, and obtained parameters are therefore the same. In regions of strong curvatures, the variation of the value of the threshold delays the change of segment of a maximum of three points without modifying in a perceptible manner the value of parameters, except if a change of type of conic appears. Note that these changes of type of conic do not happen in a very large interval of variation of the threshold (on this example, only from α_S=0,29). Thus, we observe, on the totality of the analyzed drawings, a great stability of parameters and the obtained segmentation. The segmentation depends on the wished precision but we note a large robustness of thresholds. The same thresholds have been retained for all the tested drawings.

4 MODEL SELECTION

The quality of the segmentation depends on a trade-off between the obtained reduction rate and the precision. An aspect of the quality of our method of segmentation can be evaluated by the reduction rate. This rate is computed from the ratio between the number of elements not memorized by the algorithm and the number of initial elements represented by the number of points given by the digitizer.

Files of drawings are constituted by the points acquired with the digitizer. Since each point is known by its abscissa and its ordinate, n points are described by $2n$ pieces of information. We here give the number of items retained by different segmentation algorithms :

- for a linear modeling (for example, [1]), the first point and the point of section change. For each segment, only one point must be retained, that is to say 2 values. The first segment needs two supplementary values for the start point. Finally, for l segments, the drawing is described by $(2l+2)$ values.
- for a circular modeling [4], we also have the center of the circle, then the drawing will be described using $(4c_i+2)$ values for c_i bows of circles.
- In the same manner, we obtain $(8c_o+2)$ values to retain for c conic sections.

In the case of a description using several models, the reduction rate may be written by the following ratio :

$$R = \frac{2l + 4c_i + 8c_o + 2}{2n}$$

With the maximal precision, the description of 5 points gives the same ratio R by 4 successive linear segments or by an unique conic section. From the 6th point, a comparison becomes possible between these two models. For a same accuracy, a description by conic sections gives a data reduction R_{co} better than R_l obtained with linear segments if the number c_o is smaller than $l/4$ segments. The model M_i , $i = l$, c_i, c_o will be chosen such than M_i=Arg min $(4l, 2c_i, c_o)$.

On the treated examples, we have computed the factor R_i for each type i of model. It varies between 6% and 20% for the linear modeling and between 7 and 18% for the conic model.

From this fact, we propose an on line segmentation criterion giving the best model to adopt in each point by:

$$K_i = \lambda R_i + (1-\lambda)\varepsilon_i$$

where λ is a trade-off factor between the data reduction rate and the segmentation quality.

This criterion of segmentation can be used as detector of model end and the proposed modeling can be on-line processed at the same time than other circular or linear filters. The criterion has for objective to maximize the length of the model live while minimizing the modeling error since the reduction rate depends on the duration of validity of the running model. In figures 3 and 4, we give results of segmentation by the linear model compared to the conic model on an example of numeral 2.

In the case of the modeling by conics (figure 3, with a_S=0,25), the algorithm here retained c_o= 4 conic sections, that is to say 34 items of information to retain. We had to decrease thresholds in the case of the linear modeling until to obtain the same information to memorize, 34, that is to say 16 line segments (some segments are then very small). The result is visible in figure 4 and does not reach the same precision.

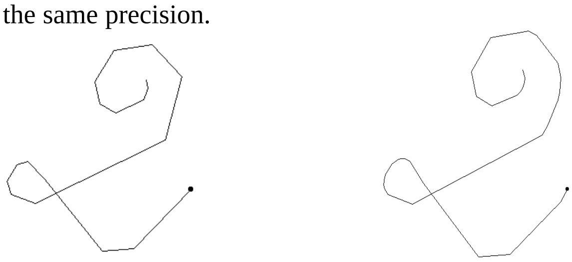

Fig. 3.a: a *=15°* *Fig. 3.b:* a*=6°*

Figure 3: linear modeling for two values of thresholds

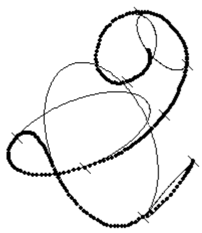

Figure 4: modeling by conics, $a_S = 0,29$

However, the best representation depends on the manuscript layout type, for example of the type of figure. The choice of a model among the three possible ones is already an element of classification. For example, with drawings presenting large portions to very weak curvature, the length of initialization does not allow the conics model to converge, but they are these cases where the linear model becomes very good, such as on numerals 1, 4 or 7. But generally, two or three models must be used for the best modeling in the sense of the criterion K_i.

To compute the error e_i in K_i, we can adopt the same calculation of a sum of squared orthogonal distances, for each point of the drawing, multiplied by the distance from the previous point, to give the same weight to all points, whatever the speed of the drawing.

So, we propose a strategy of segmentation by the three models in parallel. This strategy will use the principle of the Minimum Description Length (MDL, [11]) of Rissanen, or more precisely the Minimum Message Length principle (MML, [12],[13]), because of the little amount of data, to decide the best model in each instant of changes. The initial priority is given to the linear model.

Conclusion

We have exposed how curves can be modeled by sections of conics thanks to an extended Kalman filter. Curves are described by equations under an implicit form providing a first equation of observations, the second one being a condition of normalization that leads to a unique solution allowing classification and providing an invariant estimation by geometrical transformations. The conic estimation integrates an approximation of the orthogonal distance between a point of the space and the conic and gives a robust parameter estimation allowing the coding of handwritten drawings. For a large interval of the used thresholds, we observe a great stability of the parameters and the instants of changes. The modeling is evaluated in terms of accuracy, but equally by the obtained rate of data reduction. The results on real drawings show that the best compromise is in the use of several kinds of models and a segmentation criterion based upon this trade-off.

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