NON-UNIFORMLY DOWNSAMPLED FILTER BANKS

B.E. Sarroukh and A.C. den Brinker
Dept. of Electrical Engineering,
Eindhoven University of Technology,
Eindhoven, NETHERLANDS
Tel: +31 40 2473628; fax: +31 40 2455674
e-mail: b.e.sarroukh@ele.tue.nl

ABSTRACT
The problem of perfect reconstructing non-uniformly downsampled filter banks is considered using frame theory and filter bank theory. This problem can be reformulated to a uniformly downsampled filter bank thus allowing the usual analysis. Of special interest are those filter banks where the output of the analysis bank has a direct interpretation e.g., the sliding-window Fourier transform or the wavelet transform. The concept of the sliding-window Fourier transform can be extended by replacing the Fourier transformation by an arbitrary unitary transformation. As an example the sliding-window Kautz transformation is considered.

1 INTRODUCTION

Filter banks can be used for many applications, e.g. signal coding and compression, multi-resolution and wavelet analysis [1]. In order to be able to tailor the filter bank to the application at hand, it is desirable to have a theory as general as possible. Therefore, we first consider rather arbitrary filter banks (causal and stable) followed by arbitrary downsamplers. In this way we have the desired flexibility in choice of the filter bank and the possibility to deal with issues as signal quantization noise and coefficient quantization.

While on the one hand one requires maximal flexibility in design, it is often desired to have insight in the representation of the signal as is given by the output of the analysis bank. This is for instance the case if the filter bank implements a sliding-window Fourier transform or the analysis agrees with a wavelet representation. We argue that next to the concept of a sliding-window Fourier transformations, other sliding-window unitary transformation can be valuable as well. This is illustrated by an example, the sliding-window Kautz transformation.

2 ANALYSIS BANK

Consider the analysis bank shown in Fig. 1 consisting of the L filters with impulse responses $f_l(n)$ $(l = 1, \cdots, L)$ cascaded with the downsamplers M_l. The input signal is called x and the output signals are called y_l $(l = 1, \cdots, L)$ and these output signals each have their specific sample frequency f_s/M_l with f_s the sample frequency of the input signal x.

The filters f_l are assumed to be stable, i.e. $f_l \in \ell_1$, and causal. The filters are of the FIR or IIR type: the z-transform of f_l is a rational function of z. Furthermore, we take $L < \infty$ and M_l a positive integer.

For the output signals we can write

$$y_l(k) = \sum_n x(n) f_l(kM_l - n) = \langle x, \sigma^{kM_l} \check{\bar{f}}_l \rangle \qquad (1)$$

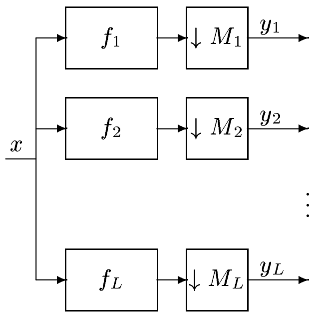

Figure 1: Non-uniformly downsampled analysis bank

where $\bar{\ }$ denotes conjugation, $\check{\ }$ denotes time reversal and σ is the shift operator.

From frame theory [2] we know that the reconstruction of the signal x is possible if and only if the set

$$\left\{ \sigma^{kM_l} \check{\bar{f}}_l | l = 1, \cdots, L, \; k \in \mathbb{Z} \right\} \qquad (2)$$

constitutes a frame in ℓ_2, i.e.,

$$\exists_{A>0,B>0} \forall_{f \in \ell_2} \; A||f||^2 \leq \sum_{l,k} |\langle f, \sigma^{kM_l} \check{\bar{f}}_l|^2 \leq B||f||^2. \qquad (3)$$

It is in general not easy to check whether a given filter bank constitutes a frame using the previous condition. We will derive a more direct way to evaluate this later on.

A necessary condition to obtain a frame from this analysis bank is that $\sum_l 1/M_l \geq 1$. This is intuitively clear since otherwise the data stream per unit of time at the output of the analysis bank is less than that at its input. This condition will be proved more rigorously later.

In general, the reconstruction is not unique. We have a unique reconstruction if and only if the set (2) constitutes a Riesz basis in ℓ_2 in which case we have $\sum_l 1/M_l = 1$.

If the system $\{\sigma^{kM_l} \check{\bar{f}}_l | l = 1, \cdots, L, \; k \in \mathbb{Z}\}$ is a frame, we can reconstruct the signal x using the frame operator $\mathcal{S}$

$$\mathcal{S}x = \sum_{l,k} \langle x, \sigma^{kM_l} \check{\bar{f}}_l \rangle \sigma^{kM_l} \check{\bar{f}}_l = \sum_{l,k} y_i(k) \sigma^{kM_l} \check{\bar{f}}_l \qquad (4)$$

which is invertible and the reconstruction becomes

$$x = \sum_{l,k} y_l(k) \mathcal{S}^{-1} \sigma^{kM_l} \check{\bar{f}}_l. \qquad (5)$$

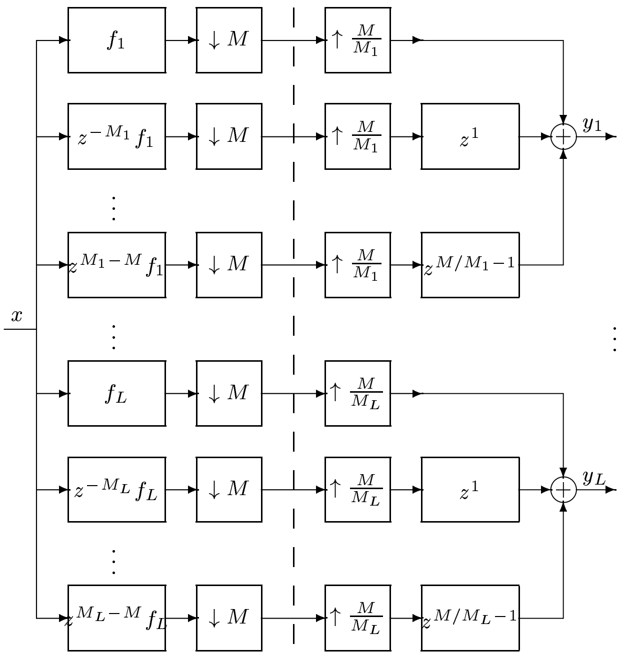

Figure 2: Equivalent uniformly downsampled filter bank

In general, the computation of $\mathcal{S}^{-1}$ is not simple. To solve this problem, one uses the Neumann series [2] to approximate $\mathcal{S}^{-1}$. This series is given by

$$\mathcal{S}^{-1} = \frac{2}{B+A} \sum_{k=0}^{\infty} (I - \frac{2}{B+A} S)^k, \tag{6}$$

where A, B are the frame bounds. A good approximation of $\mathcal{S}^{-1}$ is obtained if $A \approx B$ and taking many terms in the Neumann series.

The stability in the frame concept means that a frame yields a well-conditioned reconstruction. This is reflected in the fact that the frame operator is bounded and bounded from below.

What we really would like is that we can reconstruct the signal x (or a delayed version thereof) using causal and stable filters. This is by no way guaranteed if (2) is a frame: if the reconstruction exists this means that we can do the reconstruction using pattern functions in ℓ_2 but this is insufficient for our goals.

3 EQUIVALENT UNIFORMLY DOWN-SAMPLED FILTER BANK

From (18) it is clear that, given the analysis bank, there is only freedom of choice in the reconstruction in the form of K_0 if we have oversampling. We can redraw the analysis bank of Fig. 1 as is shown in Fig. 2 where M is the smallest integer having M_l as its divisors. If we draw a time-channel grid indicating the sampling instants at the output of the analysis bank, M is the number on the time-axis after which the sampling pattern repeats itself. Note that the part left of the dashed line in Fig. 2 consists of causal filters. We consider now the reconstruction of this uniformly downsampled analysis bank.

First we introduce the necessary variables. We have the outputs of the analysis filter by $y_{l,j}(k) = \sum_n x(n) f_l(nM - (j-1)M_l - n)$ for $l = 1, \cdots, L$, $j = 1, \cdots, M/M_l$. Next we have the input and output vector signals denoted by

$$\underline{x}(n) = (x(nM), x(nM-1), \cdots, x(nM-M+1))^t \tag{7}$$
$$\underline{y}(n) = (y_{1,1}(nM), y_{1,2}(nM), \cdots, y_{L,M/M_L}(nM))^t \tag{8}$$

for $n \in \mathbb{Z}$ (^{t} denotes transposition) and, lastly, the analysis matrices defined by

$$\{A(m)\}_{n,i} = f_l(mM - (j-1)M_l + i - 1), \tag{9}$$

where $m \in \mathbb{Z}$, $i = 1, \cdots, M$, $n = j + \sum_{k=1}^{l-1} M/M_k$, $l = 1, \cdots, L$ and $j = 1, \cdots, M/M_l$. Note that $A(m)$ is causal: $A(m) = 0$ for $m < 0$. The matrix sequence $A(m)$ is no more than a stack containing the impulse responses of the filters of Fig. 2. The matrix A has dimensions $\tilde{M} \times M$ with $\tilde{M} = \sum_{l=1}^{L} M/M_l$. From these definitions we can write the processing in the analysis bank as a convolution

$$\underline{y}(n) = \sum_{k \le n} A(n-k) \underline{x}(k). \tag{10}$$

Note that we used the causality of the analysis bank.

Now we give the definition of finite-order filtering in the multidimensional case.

Definition An input-output relation of the form (10) is said to be of finite order with order N if there exist $B_0, B_1, \cdots, B_N$ in $\mathbb{C}^{\tilde{M} \times M}$ and $C_1, \cdots, C_N$ in $\mathbb{C}^{\tilde{M} \times \tilde{M}}$ such that

$$\underline{y}(n) = \sum_{k=0}^{N} B_k \underline{x}(n-k) - \sum_{k=1}^{N} C_k \underline{y}(n-k). \tag{11}$$

The following theorem tells us when A leads to finite-order filtering.

Theorem An input-output relation of the form (10) leads to finite order filtering with order N if and only if there exist $(D_m)_{m=1,\cdots,N}$ such that

$$A(m) = \sum_{k=1}^{N} D_k A(m-k), \qquad \text{for } m > N. \tag{12}$$

Expressed in the notation of (11) we have

$$B_m = A(m) - \sum_{k=1}^{m} D_k A(m-k), \tag{13}$$
$$C_m = -D_m. \tag{14}$$

Remark: one can prove that we have a finite-order filtering if and only if the filters f_l are of the FIR or IIR type.

For a finite-order filtering we have the relation (11). In the z-domain this can be written as

$$\left(I + \sum_{k=1}^{N} z^{-k} C_k \right) \hat{\underline{y}}(z) = \left(\sum_{k=0}^{N} z^{-k} B_k \right) \hat{\underline{x}}(z), \tag{15}$$

where $\hat{}$ denotes the z-transform of the pertinent variable. The analysis bank is stable if and only if the zeros of $det(I + \sum_{k=1}^{N} z^{-k} C_k)$ are within the unit circle. This last holds if and only if the filters f_l are stable which was assumed from the outset.

To test if the system $\{\sigma^{kM_l} \check{\overline{f}}_l | l = 1, \cdots, L,\ k \in \mathbb{Z}\}$ constitutes a frame in $\ell_2(\mathbb{Z})$ we can use the polyphase matrix $H(z)$ of the analysis bank [1] defined by

$$\{H(z)\}_{n,i} = (\Omega_M^{i-1} z)^{(j-1)M_l} \hat{f}_l(\Omega_M^{i-1} z) \tag{16}$$

where $n = j + \sum_{k=1}^{l-1} M/M_l$, $\Omega_M = e^{j2\pi/M}$, $i = 1, \cdots, M$, and $j = 1, \cdots, M/M_l$. The system $\{\sigma^{kM_l}\check{\tilde{f}}_l | l = 1, \cdots, L,\ k \in \mathbb{Z}\}$ is a frame in $\ell_2(\mathbb{Z})$ if and only if there exist $A > 0$, $B > 0$ such that

$$AI \leq H^h(z)H(z) \leq BI, \tag{17}$$

for all z on the unit circle with $0 \leq \arg(z) \leq 2\pi/M$ (^{h} denotes Hermitian transposition). From this we can immediately infer that a necessary condition is that $\tilde{M} \geq M$ and thus $\sum_l 1/M_l \geq 1$.

4 SYNTHESIS BANK

Consider (15). It is clear that there is a perfect reconstruction if B_0 is injective. For a stable perfect reconstruction it is required that the zeros of $det(I + \sum_{k=1}^{N} z^{-k} K_0 B_k)$ are within the unit circle, where $K_0 B_0 = I$. If both requirements are met we have a causal stable perfect reconstruction by

$$\underline{x}(n) = K_0 \left[\underline{y}(n) + \sum_{k=1}^{N} C_k \underline{y}(n-k)\right] - \sum_{k=1}^{N} K_0 B_k \underline{x}(n-k) \tag{18}$$

or, in the z-domain, the synthesis filter is given by

$$\hat{S}(z) = \left(I + \sum_{k=1}^{N} z^{-k} K_0 B_k\right)^{-1} K_0 \left(I + \sum_{k=1}^{N} z^{-k} C_k\right).$$

In general, including the non-causal case, K_0 is non-unique. Only in the critically sampled case the reconstruction is unique. In the oversampled case we can use e.g. the pseudo-inverse of $\sum_{k=0}^{N} z^{-k} B_k$ which agrees with the minimal dual frame of the frame associated with the analysis bank. However, the pseudo-inverse is not necessarily causal.

Given the analysis bank and the downsampling scheme (thus $B(m)$ and $C(m)$) and requiring causal reconstruction, we only have freedom to choose K_0 under the constraint $K_0 B_0 = I$.

5 OVERSAMPLING

That the reconstruction is non-unique in the oversampled case has several advantages. We can exploit this non-uniqueness for instance to obtain simple reconstruction filters in terms of the number of operations required to do the reconstruction, for minimizing (or shaping) the noise in the reconstructed signal caused by the noise usually introduced on the signal $\underline{y}(k)$ or to reduce the coefficient sensitivity [3, 4, 8].

We can write the reconstructed signal r by the causal reconstruction formula

$$\underline{r}(n) = \sum_{k=0}^{\infty} S_k \underline{y}(n-k) \tag{19}$$

with the definition of $\underline{r}(n)$ similar to (7) and where S_k are the reconstruction matrices. The noise power σ_r^2 in the reconstructed signal r caused by additive white noise with variance σ_0^2 introduced at the output of the analysis bank (i.e., on $y_{l,j}$) can be simply written in terms of these matrices as

$$\sigma_r^2 = \frac{\sigma_0^2}{M} \sum_{k=0}^{\infty} \text{ trace } \{S_k S_k^h\}. \tag{20}$$

This form is especially suitable if the reconstruction is of the FIR-type.

6 EXAMPLES

A special and attractive case of filter banks is formed if the behaviour of the analysis bank can be easily interpreted. For instance, this is the case if the outputs $\underline{y}$ is the sliding-window Fourier transform (SWFT) of x or a wavelet transform (WT). In that case the index l ($l = 1, \cdots, L$) associated with the filters immediately takes on a physical interpretation in terms of a center frequency (SWFT) or a scale (WT). Thus the time-channel grid becomes a time-frequency grid (SWFT) or a time-scale grid (WT).

The idea of a sliding-window Fourier transform can be extended to a sliding-window unitary transformation [5, 7]. As a specific example we consider the Kautz transformation.

Definition [Kautz system] [6] Let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence within the unit circle where the λ_j's are not necessarily distinct. Define the natural number n_j by $n_j = \sum_{l=1}^{j-1} \delta_{\lambda_j,\lambda_l}$ (δ is the Kronecker delta). Let $f_j \in \ell_2(\mathbb{N}_0)$ be defined as $f_j(k) = \binom{k}{n_j}\lambda_j^k$, $k \in \mathbb{N}_0$, $j \in \mathbb{N}$. The Kautz system $\{g_j | j \in \mathbb{N}\}$ is the result of the Gram-Schmidt orthonormalization procedure applied to $\{f_j | j \in \mathbb{N}\}$.

The Kautz system is complete in $\ell_2(\mathbb{N}_0)$ under the Szász condition $\sum_{j=1}^{\infty}(1 - |\lambda_j|) = \infty$.

Definition [7] The causal sliding-window Kautz transform (SWKT) is defined by

$$(G_w x)(k, l) = \sum_{j\leq 0} x(j+k) w(-j) g_l(-j), \quad k \in \mathbb{Z}, l \in \mathbb{N},$$

with $w \in \ell_1(\mathbb{Z})$ and causal.

The simplest case of a sliding-window Kautz transformation occurs if the window function is a causal exponential sequence. Furthermore, we take the first L basis functions of the Kautz system with arbitrary poles λ_l.

Theorem For the SWKT based on a causal exponential window, the first L basis functions of the Kautz system with arbitrary poles λ_l, and an arbitrary non-undersampled and non-uniform downsampling, there exists a causal stable synthesis bank which is of first-order FIR (in the downsampled domain).

In terms of the previous notation we have $N = 1$ and $A(m) = C_1 A(m-1)$ for $m \geq 1$.

We thus have that the SWKT has an local energy interpretation, and that the poles λ_l and the subsampling grid can be chosen freely. Furthermore, the noise analysis is quite simple as consequence of the first-order reconstruction, namely

$$\sigma_r^2 = \frac{\sigma_0^2}{M} \text{ trace } \{K_0(I + C_1 C_1^h) K_0^h\}. \tag{21}$$

The causal reconstruction for which σ_r^2 is minimal is called the minimal causal dual frame and is attained for $K_0 = (B_0^h(I + C_1 C_1^h)^{-1} B_0)^{-1} B_0^h (I + C_1 C_1^h)^{-1}$ [9].

Consider the Kautz transformation with poles as shown in Fig. 3. The ordering of these poles is taken in order of increasing radius. Consequently, the first filter in the windowed Kautz analysis bank has the broadest bandwidth and complex-conjugated poles occur sequentially. The parameter of the exponential window sequence is indicated by the circle.

In Fig. 4 we have plotted the amplitude transfers of the filters performing the windowed Kautz transformation as defined by the poles and window parameter of Fig. 3.

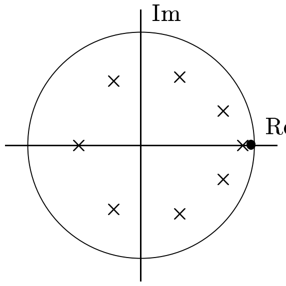

Figure 3: The poles of the Kautz transformation (crosses) and window parameter (circle).

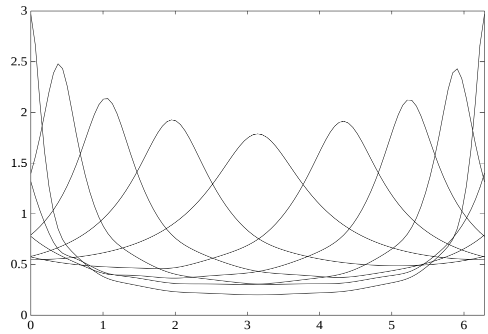

Figure 4: Amplitude transfers of the analysis bank.

In principle, we can select a quite arbitrary downsampling scheme in cascade with the analysis bank. For perfect reconstruction we require critical sampling or oversampling.

As an example we take the downsampling scheme as shown in Fig. 5 in the form of a time-frequency grid. Horizontally we have the time-axis and vertically the center frequencies of our analysis filters. The channel index number is shown on the right. The crosses indicate the samples taken after the analysis. The downsamplers per channel are taken (nearly) inverse proportional to the bandwidth with the restriction of a repeating sampling pattern after 24 samples ($M = 24$). There are 25 samples within the fundamental cell of length 24, which means there is a slight oversampling with a factor 25/24. Note that the time-frequency grid is similar to that in the wavelet case.

As stated earlier, the reconstruction is of first-order FIR in the downsampled domain. This means we have to determine two matrices (S_1 and S_2) of size 24×25 to do the reconstruction. These matrices were determined numerically.

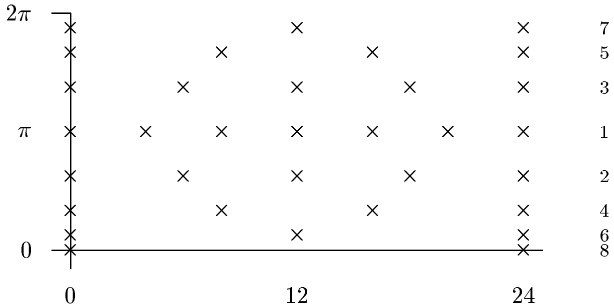

Figure 5: Time-frequency grid.

Other time-frequency grids are possible as well. We could use for instance a uniform downsampling scheme with decimators with factor 8 after each analysis filter. We still have perfect reconstruction by first-order FIR, but now the two reconstruction matrices are of size 8×8. Uniform downsampling gives the minimum possible delay. However, such scheme is not attractive from the point of view of additive noise. Noise (e.g. signal quantization noise) introduced at the output of the analysis bank results in more noise power (nearly a factor 5 in this example) in the reconstruction for the uniform downsampling scheme than for the downsampling scheme of Fig. 5.

7 DISCUSSION

We have reviewed concepts from frame theory and filter bank theory. By an example we have shown that the sliding window Fourier transformation can be extended to arbitrary sliding-window unitary transformation. The elegance of the considered unitary transformation is its simple implementation of both the analysis and synthesis bank and its large degree of freedom of choosing a time-frequency grid.

For good numerical properties, i.e., immunity to signal quantization noise and coefficient quantization, the proper choices of the poles in the Kautz transformation and the time-frequency grid are undoubtedly interconnected. This issue has to be explored further.

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