SOME THOUGHTS ON MORPHOLOGICAL PYRAMIDS AND WAVELETS

Henk J.A.M. Heijmans
CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

John Goutsias
Department of Electrical and Computer Engineering
The Johns Hopkins University, Baltimore, MD 21218, USA

ABSTRACT

The aim of this paper is to show how one can use tools from mathematical morphology in a systematic manner for the development of signal and image pyramids and wavelets. The abstract framework is illustrated by means of simple examples.

1 Introduction

It is an interesting phenomenon that change of resolution (or scale) may give rise to the creation, annihilation, and merging of features. It implies that the perception of a scene at various resolutions may give rise to different outcomes. This has led to an important paradigm in image processing and computer vision: for a comprehensive understanding of a scene, one has to analyse it at a broad range of resolution levels. This results in so-called multiresolution techniques, of which various examples can be found in the literature, e.g., quadtrees, pyramids, wavelets, fractal methods, scale-spaces, etc. Each of these techniques has its own merits and limitations.

The discovery of wavelets has greatly extended the utilisation of multiresolution approaches in signal processing applications [2, 8, 10]. Its use in image analysis, however, is somewhat limited by the fact that it hinges on the linearity assumption: the underlying spaces are linear spaces and the operations involved are linear (averaging, subtraction). Mathematical morphology [3, 6] is complementary to the linear approach in the sense that it considers images as geometric objects rather than as elements of a linear (Hilbert) space. The central idea of mathematical morphology is to examine the geometric structure of an image by probing it with small patterns, called "structuring elements," at various locations. One can extract useful shape information from the image by varying the size and shape of the structuring elements.

Many existing morphological tools, such as granulometries, skeletons, and alternating sequential filters, are essentially multiresolution techniques. Thus, one may wonder: What are the relationships between the existing linear (wavelets) and nonlinear (morphological) multiresolution approaches? The goal of this paper is to deal with such questions.

2 Linear pyramid transforms

In the earliest multiresolution approaches to signal and image processing, the most popular way was to obtain a coarse level signal by subsampling a fine resolution signal, after linear smoothing, in order to remove high frequencies. A *detail pyramid* can then be derived by subtracting from each level an interpolated version of the next coarser level; the best-known example is the Laplacian pyramid [1]. From a frequency point of view, the resulting difference signals (known as detail signals) form a signal decomposition in terms of bandpass-filtered copies of the original signal. Moreover, there is neurophysiological evidence that the human visual system indeed uses a similar kind of decomposition. This tool has been one of the most popular multiresolution schemes used in image processing and computer vision [5].

The development of wavelet theory during the past ten years [2, 8, 10] has resulted in signal and image decomposition schemes which avoid the redundancy inherent to the older schemes, such as the Laplacian pyramid, where the filter coefficients are chosen more or less arbitrarily.

3 Towards an axiomatic approach

3.1 Motivation

The linear pyramid scheme mentioned in the previous section however leaves a lot to be desired, obviously due to aliasing and use of non-ideal filters. In addition, a linear filtering approach may not be theoretically justified; in particular, the operators used for generating the various levels in a multiresolution pyramid must depend crucially on the application.

We seek to develop a general multiresolution scheme that represents a signal, or image, using a sequence of successively reduced volume signals, applying fixed rules that map one level to the next. In such a scheme, a level is *uniquely* determined by the level below it. Such an approach contains the following ingredients:

- No assumptions are made on the underlying signal/image space(s). It may be a linear space (Gaussian/Laplacian pyramid, wavelets), it may be a complete lattice (mathematical morphology), or any other set.
- The scheme is constituted by operators between different spaces (the levels of the pyramid). These operators are only required to satisfy some elementary properties and are decomposed into *analysis operators*, representing an upward step, and *synthesis operators*, representing a downward step.

In this paper, we only give a brief overview of some of our ideas; we refer to our future publications for a more comprehensive account.

3.2 Pyramid transform

Every level $j \in I$ of the pyramid corresponds to a domain V_j of signals. No particular assumption on V_j is made at this point; it is not necessarily true that V_j is a linear space, or that $V_{j+1} \subseteq V_j$. *Multiresolution signal analysis* consists of decomposing a signal in the direction of increasing j (bottom-up process). This task is accomplished by means of *analysis operators* $\psi_j^{\uparrow} : V_j \to V_{j+1}$; such operators reduce information. On the other hand, *multiresolution signal synthesis* (top-down process) proceeds in the direction of decreasing j, by means of *synthesis operators* $\psi_j^{\downarrow} : V_{j+1} \to V_j$. The crucial assumption to be made is

$$\psi_j^{\uparrow}\psi_j^{\downarrow} = \mathsf{id} \quad \text{on} \quad V_{j+1},$$

where id is the identity operator. We refer to this identity as the *pyramid condition.*

Although, as a direct consequence of this condition, the analysis operator $\psi_j^{\uparrow}$ is the left inverse of the synthesis operator $\psi_j^{\downarrow}$, it is not in general true that it is also the right inverse: $\hat{\psi}_j(x) = \psi_j^{\downarrow}\psi_j^{\uparrow}(x)$ is only a "coarse approximation" of x. Therefore, the analysis step cannot be used for signal representation; there is loss of information in the analysis step. This is not a problem however. In fact, this is in agreement with the inherent property of multiresolution signal decomposition of reducing information in the direction of increasing j.

Analysis followed by synthesis of a signal $x \in V_j$ yields an approximation $x' = \hat{\psi}_j(x) \in V_j'$. We assume that there exists a subtraction operator $(x, x') \mapsto x \dot{-} x'$ mapping $V_j \times V_j'$ into a set Y_j. Furthermore, we assume that there exists an addition operator $(x', y) \mapsto x \dot{+} y$ mapping $V_j' \times Y_j$ into V_j. The detail signal $y = x \dot{-} \hat{\psi}_j(x)$ contains the "high resolution" or "detail" information about x which is not present in x'. It is crucial that x can be recovered from its coarse approximation x' and the detail signal y. Towards this goal we introduce the following assumption:

$$x' \dot{+} (x \dot{-} x') = x \;\; \text{if} \;\; x \in V_j \;\; \text{and} \;\; x' = \hat{\psi}_j(x)$$

This leads to the following recursive signal analysis scheme:

$$x = x_0 \to (x_1, y_0) \to \cdots \to (x_{j+1}, y_j, y_{j-1}, \ldots, y_0) \to \cdots$$

where

$$\begin{cases} x_0 = x \in V_0 \\ x_{j+1} = \psi_j^{\uparrow}(x_j) \in V_{j+1}, \; j \geq 0 \\ y_j = x_j \dot{-} \hat{\psi}_j(x_j) \in Y_j \end{cases}$$

We refer to this scheme as the *pyramid analysis scheme.* The inverse scheme, called *pyramid synthesis scheme*, goes as follows:

$$x_j = \psi_j^{\downarrow}(x_{j+1}) \dot{+} y_j$$

In most applications, the subtraction and addition operators are the usual operations on $\mathbb{R}$, but this is not a prerequisite.

3.3 Some examples

The axiomatic pyramid approach described above encompasses several existing techniques.

Example (Burt-Adelson pyramid)
Consider the case when both the signal spaces and the operators are linear. Assume that for every $j \geq 0$, $V_j = \ell^2(\mathbb{Z})$, the space of real-valued sequences $(\ldots, x(-1), x(0), x(1), \ldots)$ with $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$. Assume, furthermore, that at every level j the same analysis and synthesis operator $\psi^{\uparrow}, \psi^{\downarrow}$ are used, namely,

$$\begin{aligned} \psi^{\uparrow}(x)(n) &= \frac{1}{8}\Big(-x(2n-2) + 2x(2n-1) + \\ &\quad +6x(2n) + 2x(2n+1) - x(2n+2)\Big) \end{aligned}$$

and

$$\begin{cases} \psi^{\downarrow}(x)(2n) = x(n) \\ \psi^{\downarrow}(x)(2n+1) = \frac{1}{2}(x(n) + x(n+1)) \end{cases}$$

A straightforward computation shows that $\psi^{\uparrow}\psi^{\downarrow} = \mathsf{id}$, i.e., the pyramid condition is satisfied. The resulting pyramid scheme is also known in the literature as the *Laplacian pyramid* [1]. □

The pyramid scheme to be discussed next has been proposed earlier by Heijmans and Toet in a paper on morphological sampling [4].

Example (Morphological pyramid)
Consider the complete lattice $\mathcal{L} = \mathrm{Fun}(\mathbb{Z}^2, \overline{\mathbb{R}})$ comprising all functions mapping $\mathbb{Z}^2$ into the extended reals $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. Let $A \subseteq \mathbb{Z}^2$ be a structuring element, and define $A[n] = \{k \in A \mid k - n \text{ is even}\}$. Note that $m \in \mathbb{Z}^2$ is called even if both its coordinates are even integers. Define the analysis and synthesis operator in the following way:

$$\begin{aligned} \psi^{\uparrow}(x)(n) &= \bigwedge_{k \in A} x(2n + k) \\ \psi^{\downarrow}(x)(n) &= \bigvee_{k \in A[n]} x(\frac{n-k}{2}) \end{aligned}$$

where $\vee$, $\wedge$ denote maximum and minimum, respectively. This pair of morphological operators forms a so-called *adjunction* [3], which means that $\psi^{\downarrow}(x') \leq x$ if and only if $x' \leq \psi^{\uparrow}(x)$ (pointwise) for every pair $x, x' \in \mathcal{L}$. It satisfies the pyramid condition if and only if there exists an $n \in \mathbb{Z}^2$ for which $A[n]$ contains exactly one element.

We give an explicit example. Let $A = \{(0,0), (1,0), (0,1), (1,1)\}$. It is evident that $A[n] = \{n\}$ for $n \in A$, hence the previous condition is satisfied. The operators $\psi^{\uparrow}$ and $\psi^{\downarrow}$ are given by (writing (m,n) instead of n):

$$\psi^{\uparrow}(x)(m,n) = \bigwedge\{x(m,n), x(m+1,n), x(m,n+1), x(m+1,n+1)\}$$

and

$$\begin{aligned}\psi^{\downarrow}(x)(2m,2n) &= \psi^{\downarrow}(x)(2m+1,2n) \\ &= \psi^{\downarrow}(x)(2m,2n+1) \\ &= \psi^{\downarrow}(x)(2m+1,2n+1) \\ &= x(m,n)\end{aligned}$$

for $(m,n) \in \mathbb{Z}^2$. □

Some other morphological concepts that can be conveniently described in terms of our axiomatic scheme are:

- *Skeleton*: The (morphological) skeleton representation [7] that can be expressed in terms of morphological operations (dilation, erosion, opening, closing) is a special case of the pyramid transform; here the underlying signal spaces are complete lattices, and the analysis and synthesis operators are constituted by adjunctions [3].

- *Granulometries*: Granulometries and size distributions form one of the most practical concepts in mathematical morphology [6]. They fit, in a very natural way, into our general scheme.

A signal representation obtained by means of a pyramid scheme (coarsest signal along with the detail signals y_j) is overcomplete, in the sense that it produces more samples than the original finest resolution signal. This is a direct consequence of the fact that the detail signal y_j lives at level j.

4 Wavelet decomposition

4.1 Axiomatics

Suppose that we are able to find a second analysis operator $\omega_j^{\uparrow}$ mapping V_j into some space W_{j+1} (with smaller cardinality than V_j) and a synthesis operator $\omega_j^{\downarrow} : W_{j+1} \to Y_j$ such that

$$\psi_j^{\downarrow}\psi_j^{\uparrow}(x) \dot{+} \omega_j^{\downarrow}\omega_j^{\uparrow}(x) = x, \text{ for } x \in V_j.$$

In this case, $\omega_j^{\downarrow}\omega_j^{\uparrow}(x)$ replaces the detail signal $x \dot{-} x' = x \dot{-} \psi_j^{\downarrow}\psi_j^{\uparrow}(x)$. Here $\dot{+}$ is addition from $V_j' \times Y_j$ into V_j; however, there does not need to exist a subtraction.

Notice that the detail signal $x \dot{-} \psi_j^{\downarrow}\psi_j^{\uparrow}(x)$ at level j can now be obtained from the detail signal $\omega_j^{\uparrow}(x)$, that lives at level $j+1$, by means of the second synthesis operator $\omega_j^{\downarrow}$. A signal decomposition scheme obtained by means of this modification is complete, in the sense that it (approximately) produces the same number of samples as the original finest resolution signal. This is a direct consequence of the fact that the detail signal y_j can be now calculated from the detail signal $\omega_j^{\uparrow}(x_j)$ that lives at level $j+1$, as opposed to level j in the case of the pyramid transform.

The analysis and synthesis operators $\psi_j^{\uparrow}$, $\omega_j^{\uparrow}$, $\psi_j^{\downarrow}$, $\omega_j^{\downarrow}$, have to satisfy conditions which are very similar in nature to the *biorthogonality conditions* known from the theory of wavelets (note, however, that these conditions are formulated in operator terms only, and do not require any sort of linearity assumption or inner product). In the sequel, we refer to $\omega_j^{\uparrow}$ and $\omega_j^{\downarrow}$ as the *highpass* analysis and synthesis operators.

In this case, analysis may proceed recursively in the direction of increasing j by means of:

$$\begin{cases} x_{j+1} = \psi_j^{\uparrow}(x_j) \\ w_{j+1} = \omega_j^{\uparrow}(x_j) \end{cases}$$

Synthesis proceeds recursively in the direction of decreasing j by means of:

$$x_j = \psi_j^{\downarrow}(x_{j+1}) \dot{+} \omega_j^{\downarrow}(w_{j+1})$$

We assume that the operators satisfy the following conditions.

(1) $\psi_j^{\downarrow}\psi_j^{\uparrow}(x) \dot{+} \omega_j^{\downarrow}\omega_j^{\uparrow}(x) = x$, for $x \in V_j$.

(2) $\psi_j^{\uparrow}(\psi_j^{\downarrow}(x) \dot{+} \omega_j^{\downarrow}(w)) = x$, for $x \in V_{j+1}$, $w \in W_{j+1}$.

(3) $\omega_j^{\uparrow}(\psi_j^{\downarrow}(x) \dot{+} \omega_j^{\downarrow}(w)) = w$, for $x \in V_{j+1}$, $w \in W_{j+1}$.

The decomposition of a signal $x_0 \in V_0$ into the collection $w_1, w_2, \ldots, w_N, x_N$, where N is the final possible step is called the *wavelet transform*.

4.2 A simple example

Obviously, existing linear wavelets [2] or filter banks [8, 10] fit into our abstract wavelet scheme. But our framework also encompasses several nonlinear examples. Here we present a very simple 1-dimensional example, which can be considered as the morphological analogue of the Haar wavelet.

Example (Morphological Haar wavelet)
Let $V_j = \mathrm{Fun}(\mathbb{Z}, \overline{\mathbb{R}})$, for every j, and use the same analysis operator $\psi^{\uparrow}$ and synthesis operator $\psi^{\downarrow}$ at every level j, namely:

$$\psi^{\uparrow}(x)(n) \quad = \quad x(2n) \wedge x(2n+1)$$

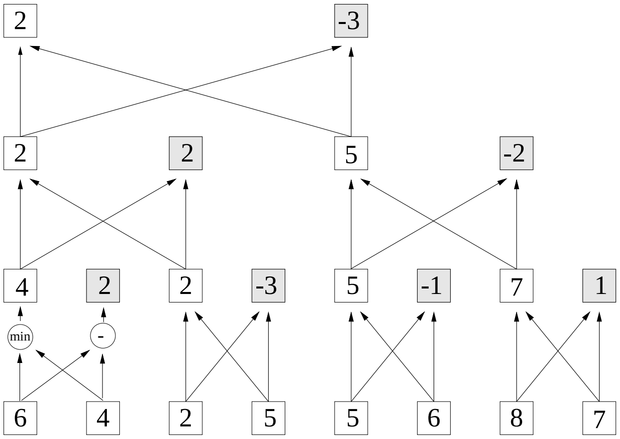

Figure 1: Signal analysis by the morphological Haar wavelet. The grey boxes contain the detail signals.

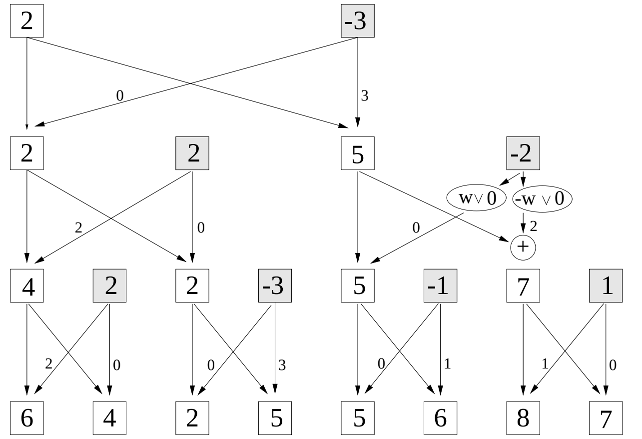

Figure 2: Signal synthesis by the morphological Haar wavelet. The grey boxes contain the detail signals.

$$
\begin{aligned}
\omega^{\uparrow}(x)(n) &= x(2n) - x(2n+1) \\
\psi^{\downarrow}(x)(2n) &= \psi^{\downarrow}(x)(2n+1) = x(n) \\
\omega^{\downarrow}(w)(2n) &= w(n) \vee 0 \\
\omega^{\downarrow}(w)(2n+1) &= (-w(n)) \vee 0
\end{aligned}
$$

It is easy to verify that the aforementioned conditions (1)–(3) are satisfied. The analysis scheme is illustrated in Fig. 1, the synthesis scheme in Fig. 2.

5 Conclusions and perspectives

We believe that the multiresolution schemes introduced in this paper may be of great interest in image processing applications, in particular at those instances where shape plays a role, e.g., in image coding and compression, texture analysis, and image fusion. The underlying mathematical theory, however, is much harder than in the linear case: the computational framework of Fourier or z-transform techniques is no longer at our disposal. However, the lifting scheme, recently developed by Sweldens [9], can be used to construct interesting morphological wavelet transforms.

Acknowledgments

This work was supported in part by NATO Collaborative Research Grant CRG.971503. John Goutsias was also supported by the Office of Naval Research (U.S.A.), Mathematical, Computer, and Information Sciences Division, under ONR Grant N00014-90-1345.

References

[1] P. J. Burt and E. H. Adelson. The Laplacian pyramid as a compact image code. *IEEE Transactions on Communications*, 31:532–540, 1983.

[2] I. Daubechies. *Ten Lectures on Wavelets.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[3] H. J. A. M. Heijmans. *Morphological Image Operators.* Academic Press, Boston, 1994.

[4] H. J. A. M. Heijmans and A. Toet. Morphological sampling. *Computer Vision, Graphics and Image Processing: Image Understanding*, 54:384–400, 1991.

[5] A. Rosenfeld, editor. *Multiresolution Image Processing and Analysis.* Springer Verlag, Berlin, Germany, 1984.

[6] J. Serra. *Image Analysis and Mathematical Morphology.* Academic Press, London, 1982.

[7] J. Serra, editor. *Image Analysis and Mathematical Morphology. II: Theoretical Advances.* Academic Press, London, 1988.

[8] G. Strang and T. Nguyen. *Wavelets and Filter Banks.* Wellesley-Cambridge Press, Wellesley, MA, 1996.

[9] W. Sweldens. The lifting scheme: A custom-design construction of biorthogonal wavelets. *Appl. Comput. Harmon. Anal.*, 3(2):186–200, 1996.

[10] M. Vetterli and J. Kovacevic. *Wavelets and Subband Coding.* Prentice Hall, Englewood Cliffs, NJ, 1995.