MAXIMUM LIKELIHOOD AND ROBUST G-MUSIC PERFORMANCE IN K-DISTRIBUTED NOISE

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ABSTRACT

For an antenna array input mixture of m point source signals in K-distributed noise, we compare DOA estimation delivered by Maximum Likelihood and the recently introduced Robust G-MUSIC (RG-MUSIC) technique. We demonstrate that similar to the Gaussian case, MLE is still superior to RG-MUSIC, especially within the so-called threshold region. This makes it possible to use the Expected Likelihood (EL) methodology to detect the presence of RG-MUSIC performance breakdown and "cure" those cases via an approach previously developed for the complex Gaussian circumstance.

Index Terms— Maximum Likelihood Estimation (MLE), G-MUSIC, Robust G-MUSIC, Expected Likelihood (EL)

1. INTRODUCTION

Under conditions of limited sample volume T and/or low signal-to-noise ratio (SNR), practical DOA algorithms begin to produce occasional wildly erroneous DOA estimates (outliers) while the computationally intensive (global) ML estimates continue to produce reasonable estimates (i.e. ones close to the Cramèr-Rao Bound (CRB)). For subspace-based DOA estimators, this threshold effect is associated with lack of full statistical separation between the noise and signal subspace of the sample covariance matrix estimate [1] (i.e. the noise subspace eigenvalues and the signal subspace ones do not belong to different "clusters"). Under Random Matrix Theory (RMT), distributions of these clusters of eigenvalues are described using alternative Kolmogorov G-asymptotic assumptions where both the array dimension M and sample volume T asymptotically grow without bound, while the ratio M/T tends to a finite constant, rather than classic asymptotic analysis (i.e. $T \to \infty$).

In [2], RMT is used to generate an alternative eigenvector weighting approach for subspace DOA estimation referred to as G-MUSIC, with threshold region performance superior to the classic MUSIC algorithm. Recently, this same technique was applied to the non-Gaussian case [3] using Maronna or Tyler fixed-point covariance matrix estimation [4] and delivered consistent (in the Kolmogorov sense) DOA estimates. Moreover, this consistency was proven under very mild i.i.d. requirements in the presence of differently distributed source signals and additive noise. This technique, termed RG-MUSIC, is explored in [5].

For the G-MUSIC technique, it has been shown that using the traditionally defined likelihood function, MLE performance in the threshold region is still superior [6] (albeit computationally impractical in many circumstances). To improve MUSIC/G-MUSIC performance, the Expected Likelihood (EL) technique has been applied to detect cases where the results contain erroneous DOA estimates. The EL technique is discussed in detail in [7], but briefly, the method exploits the fact that even though the likelihood ratio of the (unknown) true source parameters is unknown, the distribution of the likelihood ratio for certain source and noise models does not depend on those parameters and therefore is known. Likelihood ratios of MUSIC/G-MUSIC estimates which lie outside this LR p.d.f. support are viewed as erroneous and discarded. Furthermore, a search for an estimate with a likelihood ratio within this p.d.f. support can be conducted and that estimate used, even if the LR is not maximal. Thus the computationally efficient MUSIC and G-MUSIC technique can be used in a majority of cases, and where that technique fails, the EL technique can produce a superior estimate, albeit at higher computational cost, but lower than generation of the MLE.

In this study, we investigate whether a similar circumstance with RG-MUSIC performance in the threshold region exists and whether EL methodology can be used in essentially non-Gaussian scenarios with the fixed-point covariance matrix estimation approach.

2. STATISTICAL MODELS FOR MLE AND RG-MUSIC DOA ESTIMATION

2.1. Conditional Model

Let us consider an M element antenna array with M-variate snapshot $\mathbf{x}_t = A(\Theta)S_t + \eta_t,\ t = 1, \ldots, T$, where $A(\Theta)$ is the set of antenna manifolds specified by DOAs associated with the ensemble of m sources. Similarly, $S_t \in \mathcal{C}^{m\times 1}$ is the m-element vector of source signals and $\eta_t \sim \mathcal{C}^{M\times 1}$ is the se-

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quence of i.i.d. random vectors. Depending on the statistical description of vectors S_t and η_t (with $t = 1, \dots, T$), we get various models. For example, the classical "unconditional" or stochastic Gaussian model treats both vectors as i.i.d. Gaussian distributed vectors.

In many practical applications, the Gaussian assumption on $S_t \in C^{m\times 1}$ is not accurate and in those cases, the so-called "conditional" or "deterministic" model may be used, where the vector S_t is treated as an *a priori* unknown deterministic vector embedded in Gaussian noise described by covariance R_n, which results in the following model:

$$\mathbf{x}_t \sim \mathcal{CN}(A(\Theta)S_t, R_n), \;\; t = 1, \dots, T. \tag{1}$$

While RG-MUSIC is applicable to scenarios with arbitrary and differently distributed S_t and η_t, in our study we wish to consider tractable statistical models with non-Gaussian data that allow for likelihood function calculations. For this reason, we consider use of a complex elliptically symmetric (CES) contoured distribution with the following statistical representation [3]:

$$\mathbf{x}_t \overset{d}{=} \mu_t + \sqrt{Q_t} C \mathbf{u}_t. \tag{2}$$

Here $\overset{d}{=}$ means "has the same distribution as", while μ_t represents the deterministic mean value. The non-negative real random value $\sqrt{Q_t}$ (called the modular variate), is independent of the complex random vector $\mathbf{u}_t$. That complex vector $\mathbf{u}_t$ possesses a uniform distribution on the complex sphere $\mathcal{CS}^M = \{z \in \mathcal{C}, \; ||z|| = 1\}$, which we denote as $\mathbf{u}_t \to U(\mathcal{CS}^M)$. The full-rank matrix $C \in \mathcal{C}^{M\times M}$ is such that $CC^H = \Sigma_0$, where Σ_0 is the so-called scatter matrix. In this paper we limit ourselves to the absolutely continuous case where Σ_0 is positive definite. In such a case, the p.d.f. of $\mathbf{x}_t$ can be defined and we wish to consider a specific CES distributed p.d.f. $p(Q_t)$ for the so-called K-distribution [3]:

$$p(Q_t) = \frac{2\nu^{\frac{M+\nu}{2}}}{\Gamma(\nu)\Gamma(M)} Q_t^{[\frac{M+\nu}{2}-1]} K_{\nu-M}(2\sqrt{\nu Q_t}) \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function while $K_l(\cdot)$ denotes the modified Bessel function of the second kind of order l. This distribution is based on a "density generator" $g(\cdot)$ (which for the Gaussian case is just $g(t) = \exp(-t)$) of the form [3]:

$$g(Q_t) = Q_t^{\frac{\nu-M}{2}} K_{\nu-M}(2\sqrt{\nu Q_t}). \tag{4}$$

In the limit when $\nu \to \infty$, the K-distribution reduces to the $\mathcal{CN}$ distribution, while $0 < \nu < 1$ corresponds to "heavy-tailed" distributions. Therefore, analysis of DOA estimation performance for various ν covers a broad class of distributions.

Similarly to the Gaussian case, one can also consider a variation on the "conditional" model, when the signal is given by $\mu_t = A(\Theta)S_t$ and $\Sigma_0 = \sigma_n I_M$, where the noise power σ_n can be *a priori* known or unknown and the deterministic unknown source waveform S_t is drawn from a Gaussian distribution. This conditional model considers Gaussian source signals in essentially non-Gaussian additive white noise.

2.2. Comparison with Unconditional Model

The unconditional model can be introduced with

$$\mu_t = 0, \; \Sigma_0 = A(\Theta_0)BA^H(\Theta_0) + R_n \tag{5}$$

where B is the $m \times m$ source powers matrix and R_n is the noise covariance matrix. According to (2), in this case CES representation of the input data as $\mathbf{x}_t \sim \sqrt{Q_t} C u_t$, i.e. a compound Gaussian representation [3], with u_t a Gaussian random vector, and the modular variate $\sqrt{Q_t}$ random values represented by $\sqrt{\tau}$ drawn from a Gamma distribution:

$$\mathbf{x}_t \sim \sqrt{\tau} C u_t \quad ; \quad u_t \sim \mathcal{CN}(0, \Sigma_0) \tag{6}$$

$$\tau \sim \text{Gamma}[\nu, \nu^{-1}] \qquad \eta_t \in \mathcal{CN}(0, R_n) \tag{7}$$

These K-distributed scenarios demonstrates significant difference between conditional (1) and unconditional (5) models, especially for low ν (i.e. ≤ 1). Within the conditional model, use of such "heavy-tailed" noise distributions with fixed second moments means that while in some instances (in t), source signals are contaminated by very strong impulsive noise events, the majority of noise samples will have very low power (to keep the average power fixed). Therefore, if the estimation processing can somehow ignore the strong noise events and concentrate on the low noise samples, one would expect that DOA estimation accuracy could be significantly superior to estimation on models with Gaussian noise of the same average power.

In contrast, use of the unconditional representation (6) means that both the Gaussian source and noise vectors are equally scaled by the same (random) scalar $\sqrt{\tau}$. Since the SNR is not changed by this scaling, it is clear that even a straightforward normalization of $\mathbf{z}_t = \frac{\mathbf{x}_t}{||\mathbf{x}_t||}$ makes $\mathbf{z}_t$ indistinguishable from a similarly normalized Gaussian vector $\mathbf{x}_t \sim \mathcal{CN}(0, \Sigma)$, which means that the optimal (MLE) estimation accuracy should only weakly depend on the $\sqrt{Q_t}$ distribution and cannot be worse than for the data $\mathbf{z}_t$, distributed as [3]

$$p(\mathbf{z}_t) = \frac{(M-1)!}{\pi^M} \frac{1}{|\Sigma_0| [\mathbf{z}_t^H \Sigma_0^{-1} \mathbf{z}_t]^M}. \tag{8}$$

While validation of the anticipated unconditional MLE properties is required, we focus our attention here on the more challenging conditional MLE case.

2.3. DOA Estimation Accuracy - Conditional Model

Based on the above examination, we wish to compare the estimation accuracy delivered by RG-MUSIC with the MLE global search, as well as examining the potential for application of EL. We start with specification of the analytic expressions associated with the fixed-point covariance matrix estimation used for RG-MUSIC, which is defined by iteration of:

$$\hat{\Sigma}_{\text{ML}} = \lim_{k\to\infty} \hat{\Sigma}_k;\ \hat{\Sigma}_k = \frac{1}{T}\sum_{t=1}^{T} \varphi(\mathbf{x}_t^H \hat{\Sigma}_{k-1}^{-1}\mathbf{x}_t) x_t x_t^H \quad (9)$$

where $\varphi(\cdot) = -\frac{g'(\cdot)}{g(\cdot)}$. Using the density generator $g(\cdot)$ defined for K-distributed noise in (4), we get

$$\varphi(y) = \sqrt{\frac{\nu}{y}}\frac{K_{M+1-\nu}(2\sqrt{\nu y})}{K_{M-\nu}(2\sqrt{\nu y})} \quad (10)$$

which leads to

$$\hat{\Sigma}_k = \frac{\sqrt{\nu}}{T}\sum_{t=1}^{T}\frac{1}{\sqrt{\mathbf{x}_t^H\Sigma_{k-1}^{-1}\mathbf{x}_t}}\frac{K_{M+1-\nu}(2\sqrt{\mathbf{x}_t^H\Sigma_{k-1}^{-1}\mathbf{x}_t})}{K_{M-\nu}(2\sqrt{\mathbf{x}_t^H\Sigma_{k-1}^{-1}\mathbf{x}_t})}\mathbf{x}_t\mathbf{x}_t^H. \quad (11)$$

For $M-\nu \gg 1$ (a "large array" approximation), we may approximate this scatter matrix estimate by using the asymptotic formula for the modified Bessel function $K_l(x)$, getting the Tyler fixed-point covariance matrix estimate

$$\hat{\Sigma_k} = \frac{M}{T}\sum_{t=1}^{T}\frac{\mathbf{x}_t\mathbf{x}_t^H}{\mathbf{x}_t^H\Sigma_{k-1}^{-1}\mathbf{x}_t}. \quad (12)$$

Now, given $\hat{\Sigma}_{\text{ML}}$ via accurate (9) or Tyler recursions (12), we may use the eigendecomposition with $\hat{\lambda}_j$ eigenvalues and $\hat{U}_j$ eigenvectors directly for MUSIC pseudo-spectrum calculations (R-MUSIC). Similarly [5], one can use the G-asymptotic transformations to $(\hat{\lambda}_j, \hat{U}_j)$ to provide for a given number of point sources m the G-estimate of the MUSIC pseudo-spectrum (RG-MUSIC). For the details on these transformations, see [8].

Let us now consider the MLE for the conditional model of interest, assuming the additive noise power is known and scaled to unity ($\sigma_n = 1$). Again, using the $M-\nu \gg 1$ large arrray approximation, we will use the asymptotic formula for the modified Bessel function $K_l(x)$. Following the methodology in [9] and deriving the conditional ML criterion that is searching for the minimum trace of the "projected" covariance matrix, we arrive at

$$\ln LF(\Theta|X_T) \approx \frac{1}{T}\sum_{t=1}^{T}\ln x_t^H P_\perp(\Theta)x_t \quad (13)$$

$$P_\perp(\Theta) = I_M - A(\Theta)[A^H(\Theta)A(\Theta)]^{-1}A^H(\Theta). \quad (14)$$

This expression can now be used for a direct (albeit computationally inefficient) global search over $(\theta_1,\ldots,\theta_m)$ and evaluated to see if the LR distribution is independent of specific scenario parameters (other than those known *a priori* such as M, T, and σ_n) and therefore suitable for use in the EL methodology [7]. Recall that it is a basic property of the MLE technique (both for conditional and unconditional cases) that the likelihood evaluated at the MLE is greater than the likelihood of the (unknown) true parameters (otherwise MLE would be error-free):

$$\max_{\Theta}\ln LF(\Theta|X_T) \ge \ln LF(\Theta_0|X_T) \quad (15)$$

where $\Theta_0 = [\theta_1,\ldots,\theta_m]$ are the actual (true) DOAs. Yet according to the compound-Gaussian representation of the K-distributed random vector [3]

$$\sqrt{Q_t}Cu_t \sim \sqrt{\tau}u_t, u_t \sim \mathcal{CN}(0,\Sigma_0), \tau \sim \text{Gamma}[\nu,\nu^{-1}], \quad (16)$$

we get

$$x_t^H P_\perp(\Theta_0)x_t \sim \tau u_t^H P_\perp(\Theta_0)u_t \sim \tau(u_t^{M-m})^H(u_t^{M-m}) \quad (17)$$

where $u_t^{M-m} \sim \mathcal{CN}(0, I_{M-m})$ and $u_t^{M-m} \in \mathcal{C}^{M-m}$.

For any given (T,m), the p.d.f. for $\ln LF(\Theta_0|X_T)$ may be accurately evaluated using Monte-Carlo simulations.

For the unconditional model, we may use standard EL methodology [7] whereby

$$LR(\Theta|X_T) = |\hat{\Sigma}_{\text{ML}}\Sigma^{-1}(\Theta)|^T\prod_{t=1}^{T}\frac{g(\mathbf{x}_t^H\Sigma^{-1}(\Theta)\mathbf{x}_t)}{g(\mathbf{x}_t^H\hat{\Sigma}_{\text{ML}}^{-1}(\Theta)\mathbf{x}_t)} \le 1 \quad (18)$$

with a p.d.f. for $LR(\Theta_0|X_T)$ not dependent on the individual true DOAs Θ_0. Naturally, we may again adopt the asymptotic approximation for the modified Bessel function and get

$$LR^{\frac{1}{T}}(\Theta|X_T) = |\hat{\Sigma}_{\text{ML}}\Sigma^{-1}(\Theta)|^T\left[\prod_{t=1}^{T}\frac{\mathbf{x}_t^H\Sigma^{-1}(\Theta)\mathbf{x}_t}{\mathbf{x}_t^H\hat{\Sigma}_{\text{ML}}^{-1}(\Theta)\mathbf{x}_t}\right]^{\frac{\nu-M}{T}}. \quad (19)$$

As expected, $LR(\Theta_0|X_T)$ does not for practical purposes depend on the distribution of the Q_t.

3. SIMULATION RESULTS

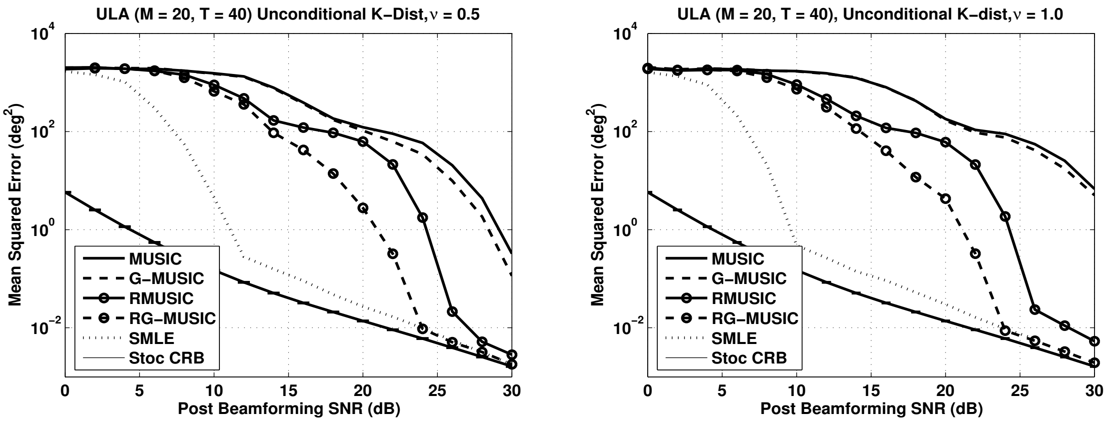
To keep the global search implementation manageable, we simulate a two-source scenario ($m=2$) in a $M=20$ element uniform linear antenna array with half-wavelength element spacing and $T=40$ snapshots. To allow for comparison with previous investigations [10], we use DOAs of $\theta_1 = 16^o$ and $\theta_2 = 18^o$.

Let us start from the unconditional scenario where (5)-(6)

$$\mathbf{x}_t \sim \sqrt{\tau}Cu_t;\ CC^H = A(\Theta_0)BA^H(\Theta_0) + \sigma_n^2 I_M, \quad (20)$$

where $B = diag[p_1,p_2]$. For powers $\hat{p}_1,\hat{p}_2$ estimation given $\Theta = [\theta_1,\theta_2]$, we apply the usual matrix fitting approach using the MLE (Tyler) fixed-point covariance matrix estimate instead of the direct sample covariance matrix used in the Gaussian case.

First, at Fig. 1 we introduce simulation results on MUSIC, G-MUSIC, R-MUSIC, and RG-MUSIC estimation for



(a) $\nu = 0.5$

(b) $\nu = 1.0$

Fig. 1. Unconditional K-distributed signal/noise case, shown with Gaussian CRB

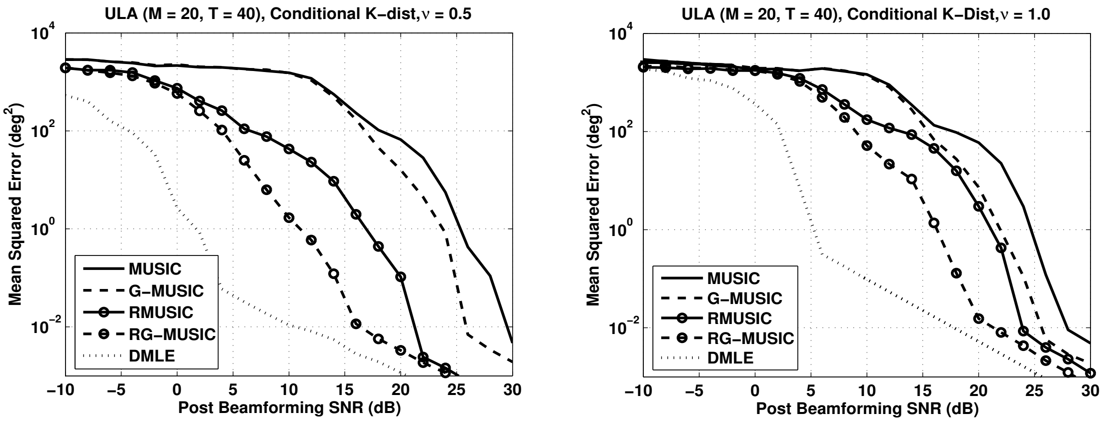
$\nu = 0.5$ and $\nu = 1.0$. The results shown for R-MUSIC and RG-MUSIC at either value of ν match closely with the MUSIC and G-MUSIC results shown in [10] (Fig. 1 in that reference) with the same signal parameters but computed in that case for a Gaussian noise and signal model. This quite expected result allows us to retain unmodified for the unconditional case all the conclusions from that earlier study regarding correspondence between MUSIC, G-MUSIC and MLE threshold conditions, but now extended to the non-Gaussian unconditional case and R-MUSIC/RG-MUSIC. Furthermore, the results from [10] which showed that MUSIC and G-MUSIC outlier generation could be detected and "cured" using EL over a wide range of SNR values can also be extended to this non-Gaussian unconditional case for R-MUSIC and RG-MUSIC. The significant gains of R-MUSIC and RG-MUSIC (or conversely the significant degradation of classic MUSIC and G-MUSIC) in the presence of non-Gaussian signal and noise can be attributed to the large dynamic range between the scaled by $\sqrt{\tau}$ training snapshots leading to an effective reduction of the actual sample support within the direct sample covariance matrix, while the fixed-point covariance estimate accommodates the scaling, allowing the effective sample support to be maintained.

Yet for $m < M$ point sources in additive noise, this unconditional scenario which results in equal scaling of both the source signal and the noise by the same random number $\sqrt{Q_t}$ does not in practice model common array or environment behavior. Therefore the expected results leading to performance which is indistinguishable from the Gaussian results after replacement of the direct sample covariance matrix by its fixed point version is less useful than consideration of the conditional case below. In that conditional case, we simulate two Gaussian distributed signal sources of equal power embedded in K-distributed additive noise (with $\nu = 0.5, 1.0$).

For comparison, we once again refer the reader to [10], which analyzed both conditional and unconditional models in the Gaussian case for MUSIC and G-MUSIC. In that study, there was not much observable difference between unconditional and conditional model performance. But as shown below for the K-distributed case, the situation is much different (see Fig. 2(a) ($\nu = .5$) and Fig. 2(b) ($\nu = 1.0$)).

One can not only observe the large improvement in DOA performance delivered by R-MUSIC, RG-MUSIC, and MLE using the fixed-point covariance estimate relative to MUSIC/G-MUSIC, but the significant (and expected) improvement of this accuracy relative to the Gaussian case with the same SNR (averaged across all T array snapshots). The reason can be seen by comparing the minimized likelihood function (13), which is a sum of logarithms (i.e. a product) of the individual residuals (some with high impulsive noise, many with low noise), i.e. $\min \sum_{t=1}^{T} \ln \mathbf{x}_t^H P_\perp(\Theta)\mathbf{x}_t$ to the Gaussian equivalent [10], which is a sum of those same residuals i.e. $\min \sum_{t=1}^{T} \mathbf{x}_t^H P_\perp(\Theta)\mathbf{x}_t$. The fixed second moment for the "heavy-tailed" distribution explored in this paper means that extremely powerful impulse-like realizations of $\sqrt{Q_t}$ are augmented by a large number of snapshots with very low noise amplitudes. The logarithm in (13) strongly emphasizes the samples with "small" Q_t, concentrating on the samples $\mathbf{x}_t$ with the maximal instantaneous signal-to-noise ratio, in contrast to the Gaussian formulation.

As one would expect, for $\nu = 0.5$ with the heavier tails, this improvement is much more significant than for $\nu = 1.0$. In terms of MLE versus RG-MUSIC performance improvement, one can once again observe significant differences in the threshold SNR between RG-MUSIC and MLE, making the estimator performance a candidate for EL improvement. To use EL, it is necessary for the LF distribution to be essentially independent of scenario parameters such as number of sources and SNR of those sources. In Fig. 3, we show the LF distribution for an SNR which happens to have about 15% RG-MUSIC outliers, with separate curves for the trials with and without outliers. The LF offset between outlier and non-outlier cases can be used to detect some (but in this case, not all) the outliers. The figure also shows the distribution for an



(a) $\nu = 0.5$ (b) $\nu = 1.0$

Fig. 2. Conditional K-distributed noise case.

LF determined clairvoyantly, and at a different SNR. The LF distribution is stable across scenario parameters such as SNR for the non-outlier and true DOA cases, allowing us to pre-compute the LF threshold to use for outlier detection, and thus use the EL methodology. Therefore, in the more challenging (and physically relevant) conditional case, some outliers from R-MUSIC and RG-MUSIC can be reliably detected by EL, supporting further DOA estimate improvement.

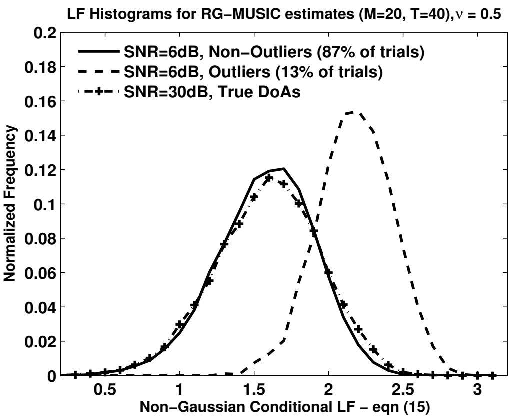


Fig. 3. Distribution of conditional LF for trials with different SNRs and different DOA estimates (outlier, non-outlier, "true" DOAs)

4. SUMMARY AND CONCLUSIONS

Conducted performance comparison of the recently introduced RMT-based RG-MUSIC DOA estimation technique against the appropriate ML estimation for conditional and unconditional models and K-distributed noise demonstrated MLE superiority over RG-MUSIC in the threshold region, as previously shown in the Gaussian case. We also showed that for trials in the SNR range where RG-MUSIC sometimes produced large outliers, but MLE provided accurate estimates, we could reliably identify those trials using the Expected Likelihood (EL) method.

We also demonstrated for a (quite artificial for this application) unconditional ML problem formulation, the potential DOA estimation accuracy approaches that of the Gaussian case, described by the Gaussian unconditional CRB. For the conditional ML problem with Gaussian signal and non-Gaussian K-distributed noise, the RG-MUSIC and MLE DOA estimation accuracy improves with progressively heavier tails for the noise distribution.

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