NEAR-OPTIMAL SENSOR PLACEMENT FOR SIGNALS LYING IN A UNION OF SUBSPACES

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ABSTRACT

Sensor networks are commonly deployed to measure data from the environment and accurately estimate certain parameters. However, the number of deployed sensors is often limited by several constraints, such as their cost. Therefore, their locations must be opportunely optimized to enhance the estimation of the parameters.

In a previous work, we considered a low-dimensional linear model for the measured data and proposed a near-optimal algorithm to optimize the sensor placement. In this paper, we propose to model the data as a union of subspaces to further reduce the amount of sensors without degrading the quality of the estimation. Moreover, we introduce a greedy algorithm for the sensor placement for such a model and show the near-optimality of its solution. Finally, we verify with numerical experiments the advantage of the proposed model in reducing the number of sensors while maintaining intact the estimation performance.

***Index Terms*—** Sensor placement, union of subspaces, frame potential

1. INTRODUCTION

The performance of a sensor network and the quality of the collected measurements mainly depend on the number and locations of the deployed sensors. In this paper, we consider a sensor network measuring a physical field y and assume that this field can be represented by a low-dimensional model. In previous works, the following model was considered

$$y = \Theta x, \tag{1}$$

where $y \in \mathbb{R}^N$ are the values of the physical field at N different locations, $\Theta \in \mathbb{R}^{N\times M}$ is the linear model, and $x \in \mathbb{R}^M$ is the low-dimensional parametrization of the field.

The linear model defined in (1) assumes that the signals y lie in one subspace of dimension M, that is $y \in \operatorname{span}(\Theta)$. However, such a model may not be sufficiently descriptive for certain applications. Therefore, in this paper we consider the following, more complex, model.

Definition 1 (Union of Subspaces (UoS) [1])**.**

$$\mathcal{Y} = \bigcup_{t\in\mathcal{T}} S_t, \quad S_t = \{y : y = \Psi_t x\} \tag{2}$$

where $\Psi_t \in \mathbb{R}^{N\times K}$ is the basis for subspace S_t of dimension $K \leq M$ and $\mathcal{T}$ is the set of indices.

That is to say that we consider the physical field y to lie in a union of lower-dimensional subspaces. More precisely, $y \in \mathcal{Y}$ if there exists a $t_0 \in \mathcal{T}$ such that $y \in S_{t_0}$.

Measuring y using N sensors is often too expensive or impractical. In reality, we would only place $L < N$ nodes, estimate the parameters x by solving the linear inverse problem, and then successively interpolate the measurements to reconstruct y. The accuracy of the estimation, and subsequently of the reconstruction, is directly related to the accuracy of the measurements. It is therefore crucial to place those L sensors in the locations where the most information can be collected.

When only L sensors are available and for the UoS model, the system of equations (1) becomes

$$y_{\mathcal{L}} = \Psi_{t,\mathcal{L}} x, \tag{3}$$

where t indicates the subspace and is generally unknown, $\mathcal{L}$ is the set of indices corresponding to the L chosen sensor locations, and $\Psi_{t,\mathcal{L}}$ is the matrix formed by the rows of Ψ_t indicated by $\mathcal{L}$. Note that if we use the model in (1), we would have $\Theta_{\mathcal{L}}$ instead of $\Psi_{t,\mathcal{L}}$.

As we require an accurate estimation of x, it is of interest to choose $\mathcal{L}$ such that the Mean Square Error (MSE) of the solution of the inverse problem is minimized:

$$MSE_{\mathcal{L}} = \mathbb{E}\left[\|\hat{x} - x\|^2\right], \tag{4}$$

where $\hat{x}$ are the estimated parameters from the measurements $y_{\mathcal{L}}$, see (3). However, since it is unknown beforehand which subspace produces the measurements in $y_{\mathcal{L}}$, the locations should be chosen such that the reconstruction error is minimized for all subspaces. Then given $y_{\mathcal{L}}$, we can estimate $\hat{x}$

This work was supported by an ERC Advanced Grant – Support for Frontier Research – SPARSAM Nr: 247006.

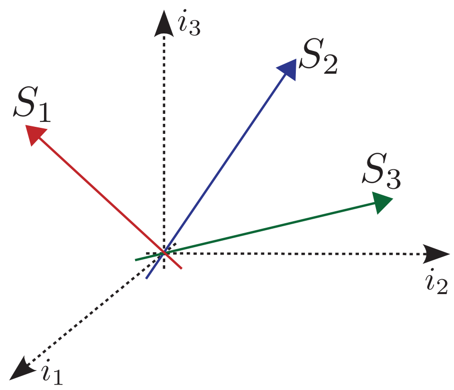

Fig. 1. A union of three 1D subspaces embedded in a 3D space. The data is generated from either S_1, S_2 or S_3. Note that if we model the data as coming from the entire space ($i_1 \oplus i_2 \oplus i_3$) we need three parameters, while only two are necessary for the union of subspaces ($S_1 \cup S_2 \cup S_3$).

by solving the least squares problem

$$\min_{\hat{x}} \|y_{\mathcal{L}} - \Psi_{t,\mathcal{L}}\hat{x}\|_2^2, \tag{5}$$

for every $t \in \mathcal{T}$ and selecting the minimizer.

Note that for a UoS model represented by T K-dimensional matrices, there is an equivalent model using only one matrix. We refer to the latter as the Span of Subspaces (SoS) model since it represents the TK-dimensional space spanned by all T subspaces.

It is interesting to compare the two models in terms of the minimum number of sensors necessary to have a unique solution to the inverse problem. For the SoS model, we can easily show using standard linear algebra results that $L \geq \min(N, TK)$. On the other hand, $L = K$ sensors are not enough to distinguish between the $T > 1$ subspaces of the UoS model, i.e. to ensure a one-to-one mapping between the observed measurements $y_{\mathcal{L}}$ and the parameters x. Actually, if we define $H_{i,j}$ as the convex hull of subspaces S_i and S_j, then the minimum number of sensors L is the maximum dimension of $H_{i,j}$ over every pair of subspaces [1, 2]:

$$L \geq \sup_{(i,j)\in\mathcal{T}\times\mathcal{T}} \dim(H_{i,j}).$$

Moreover, if the subspaces are independent and of the same dimensionality K, we obtain $L = 2K$.

Here, we underline one of the main motivations inspiring this paper: if the physical field is well modeled by a UoS, then we may be able to reduce the number of sensors needed to solve the inverse problem. We can see an example of such a scenario in Figure 1, where we depicted a 3D space and three 1D subspaces, representing the SoS and the UoS model respectively. In this case, only two parameters are needed to uniquely determine the signal in the UoS model: one to select the subspace and one to describe the signal in the selected subspace, whereas we need three parameters for the SoS model.

One possible practical scenario where the UoS model (2) applies is in the placement of thermal sensors for multicore microprocessors. In [3], the temperature field y was modeled as in (1); however, since different workloads produce different temperature patterns, y could be modeled as a UoS with subspaces corresponding to those workloads and thus gain a reduction in the required number of sensors as discussed.

So far different techniques have been developed to optimize the sensor locations when considering the SoS model. There exist methods based on convex optimization, such as [4]. The authors of [5] proposed to place the sensors on a uniform grid whose granularity is optimized. Then there exist greedy algorithms based on different cost functions like entropy [6] or mutual information [7]. Finally, we proposed [8] a greedy algorithm based on the frame potential (8), that shows superior performance to [6, 7], reduced computational time and is also near-optimal with respect to both the frame potential and the MSE.

In this work, we extend the methods presented in [8] to physical fields modeled as UoS and demonstrate the possibility of using fewer sensors while still maintaining the reconstruction accuracy. More precisely, we propose a cost function and a greedy algorithm to choose the optimal L rows from the matrices Ψ_t such that the MSE of the reconstruction for all matrices is jointly minimized. Then, building upon the results obtained in [8] for the SoS model, we derive the theoretical bounds on the performance of the algorithm showing that it is near-optimal with respect to the joint MSE. Finally, we present numerical simulations highlighting the advantages of the UoS model and of the proposed sensor placement algorithm.

2. NEAR-OPTIMAL SENSOR PLACEMENT

In this section, we first introduce two cost functions used to select the optimal rows from the matrices for the UoS model and to evaluate the performance of the sensor placement. Then, we describe a greedy sensor placement algorithm and prove its near-optimality in terms of the two cost functions.

We assume throughout the section that the measured samples of the physical field are corrupted by i.i.d. Gaussian noise with zero-mean and variance σ. For the chosen noise model, the MSE of the linear inverse problem defined in (3) is equal to

$$\mathrm{MSE}(\Psi_{t,\mathcal{L}}) = \sigma^2 \sum_{k=1}^{K} \frac{1}{\lambda_{t,k}}, \tag{6}$$

where $\lambda_{t,k}$ are the eigenvalues of the matrix $T_{t,\mathcal{L}} = \Psi_{t,\mathcal{L}}^* \Psi_{t,\mathcal{L}}$, see [9].

For the joint optimization of the MSE for all subspaces in the UoS model, we propose to use a weighted sum of the MSE for each matrix $\Psi_{t,\mathcal{L}}, t \in \mathcal{T}$:

$$\mathrm{MSE_u}(\Psi_{\mathcal{L}}) = \sum_{t=1}^{T} w_t \ \mathrm{MSE}(\Psi_{t,\mathcal{L}}), \tag{7}$$

where $w_t > 0$ and $\sum_{t=1}^{T} w_t = 1$. The weights represent the probability that the measured signal lies in the corresponding subspace. If we assume that the signals are uniformly generated by the subspaces, then $w_t = \frac{1}{T}$. A higher weight for a specific subspace will bias the sensor placement towards better estimates for that subspace.

However, directly minimizing the MSE can lead to a placement with an arbitrarily bad MSE [10] and a proxy should be used instead [8]. Recently, we showed [8] that minimizing the frame potential for a SoS model provides a solution that is near-optimal in terms of MSE. The frame potential (FP) is defined as:

$$\mathrm{FP}(\Psi_{t,\mathcal{L}}) = \sum_{i,j \in \mathcal{L}} |\langle \psi_{t,i}, \psi_{t,j} \rangle|^2, \tag{8}$$

and measures how close the rows of $\Psi_{t,\mathcal{L}}$ are to being orthogonal. Indeed, the minimum FP is achieved by orthonormal bases ($L = K$) or unit norm tight frames ($L > K$) [11].

To extend the concept described in [8] to the UoS case, we consider a weighted sum of the FP of the matrices describing the different subspaces:

$$\mathrm{FP_u}(\Psi_{\mathcal{L}}) = \sum_{t=1}^{T} w_t \ \mathrm{FP}(\Psi_{t,\mathcal{L}}), \tag{9}$$

where the weights are equal to the ones introduced in (7).

We can then extend the algorithm FrameSense to the UoS model. FrameSense was introduced in [8] and chooses $\mathcal{L}$ such that the FP (8) is greedily minimized. The extension considers the $\mathrm{FP_u}$ (9) in place of the FP and its pseudocode is given in Algorithm 1. The chosen L sensor locations are selected by optimizing the following cost function

$$\mathrm{F}(S) = \mathrm{FP_u}(\Psi) - \mathrm{FP_u}(\Psi_{\mathcal{N}\setminus\mathcal{S}}), \tag{10}$$

where $\Psi_{\mathcal{N}\setminus\mathcal{S}}$ is the matrix formed by the remaining rows after removing the set of rows in $\mathcal{S}$ which correspond to eliminated sensor locations.

It should be noted that the chosen cost function (10) is submodular and that the algorithm implements a greedy worst-out strategy. Using these two characteristics jointly with a classic result of Nemhauser et al [12] regarding the greedy optimization of submodular cost functions, we are able to bound the worst case performance of Algorithm 1 with respect to both the $\mathrm{FP_u}$ and the $\mathrm{MSE_u}$. The proofs of the two following theorems follow the strategy used in [8] and are omitted for sake of brevity.

Algorithm 1 FrameSense for UoS

Require: T subspaces Ψ, Number of sensors L

Ensure: Sensor locations $\mathcal{L}$

1. Initialize the set of available locations $\mathcal{N} = 1, 2, ..., N$.
2. Initialize the set of eliminated locations $\mathcal{S} = \emptyset$.
3. Repeat until L locations are found:
   (a) Find the worst row: $i^* = \underset{i \in \mathcal{N}\setminus\mathcal{S}}{\arg\min}\ F(\mathcal{S} \cup i)$.
   (b) Update $\mathcal{S} = \mathcal{S} \cup i^*$.
   (c) If $|\mathcal{S}| = N - L$, stop.
4. Set $\mathcal{L} = \mathcal{N} \setminus \mathcal{S}$.

Before stating the results, we need to introduce the following quantities related to the sensing energies of the selected rows. For a given matrix, we define $L_{t,\mathcal{L}}$ as the sum of the L row norms for the matrix Ψ_t

$$L_{t,\mathcal{L}} = \sum_{i \in \mathcal{L}} \|\psi_{t,i}\|^2. \tag{11}$$

We also define $L_{t,min}$ and $L_{t,max}$ as the minimizers and maximizers of (11); finally, we define L_{min} and L_{max} to be their minimum and maximum over all the matrices in the UoS model.

In the first theorem, we prove that Algorithm 1 finds a solution that is always close, in terms of $\mathrm{FP_u}$, to the optimal one. We define the optimal solution as the one obtained by an exhaustive search over all possible selections of sensor locations.

Theorem 1 ($\mathrm{FP_u}$ bound)**.** *Let $\mathcal{L}$ be the output of Algorithm 1 and $OPT = \underset{\mathcal{A} \in \mathcal{N}, |\mathcal{A}| = L}{\arg\min} \mathrm{FP_u}(\Psi_{\mathcal{A}})$ be the optimal sensor placement that minimizes $\mathrm{FP_u}$, then Algorithm 1 is near-optimal w.r.t. the $\mathrm{FP_u}$:*

$$\mathrm{FP_u}(\Psi_{\mathcal{L}}) \leq \gamma \, \mathrm{FP_u}(\Psi_{OPT}),$$

where $\gamma = \left(1 + \frac{1}{e}\left(\mathrm{FP_u}(\Psi)\frac{K}{L_{min}^2} - 1\right)\right)$.

Note how the quality of $\mathrm{FP_u}(\Psi_{\mathcal{L}})$ depends on the $\mathrm{FP_u}$ of the given set of matrices as well as L_{min}, that is the smallest sum of L row norms. In the literature, γ is also known as the approximation factor of the algorithm.

Since the main goal is to have a sensor placement such that $\mathrm{MSE_u}$ of the solution to the inverse problem (3) is minimized, we now show that under some conditions on the spectra of the initial matrices, the obtained $\mathrm{MSE_u}$ is near-optimal. Leveraging the concept of (δ, L)-bounded frames [8] and considering the maximizer and minimizer of the eigenvalues over all matrices in the UoS model, the following theorem establishes the bound on the obtained $\mathrm{MSE_u}$.

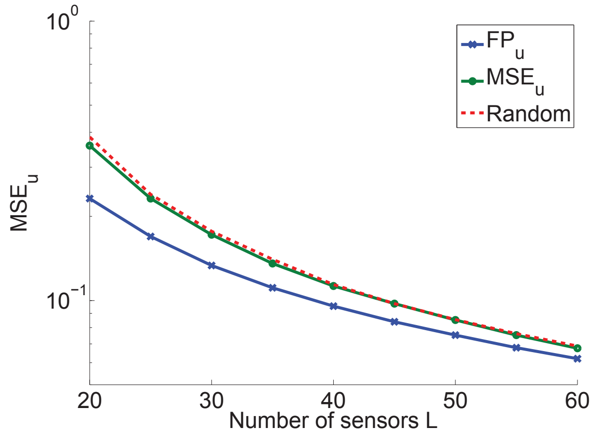

Fig. 2. Performance evaluation of the three sensor placement algorithms on randomly generated subspaces. We considered $\Psi \in \mathbb{R}^{100\times 10}$ and varied the number of sensors placed while measuring the obtained $\mathrm{MSE_u}$. Note how the algorithm based on the FP outperforms the other two.

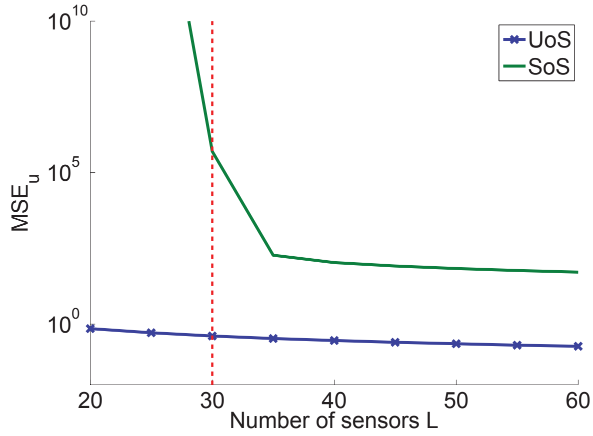

Fig. 3. Performance comparison for the linear inverse problem when the physical field y is modeled as a UoS or a SoS. For each realization of the experiment, we randomly generated three matrices $\Psi_t \in \mathbb{R}^{100\times 10}$ and Θ is a basis for the SoS model. For both models, we considered the greedy algorithm minimizing the $\mathrm{FP_u}$ and we compare the obtained $\mathrm{MSE_u}$. Note that the $\mathrm{MSE_u}$ achieved by the UoS model is significantly better for all L. For $L < 30$ the performance of the SoS is not measurable since the solution of its inverse problem is not unique. If we consider $L \geq 30$, the average improvement of the MSE given by the UoS model is 48.5 dB.

Theorem 2 ($\mathrm{MSE_u}$ bound for (δ, L)-bounded frames)**.** *For a set of T (δ, L)-bounded frames Ψ, let $\mathcal{L}$ be the output of Algorithm 1 and $OPT = \underset{\mathcal{A}\in\mathcal{N},|\mathcal{A}|=L}{\arg\min}\ \mathrm{MSE_u}(\Psi_{\mathcal{A}})$ be the optimal sensor placement that minimizes $\mathrm{MSE_u}$, then Algorithm 1 is near-optimal w.r.t. the $\mathrm{MSE_u}$:*

$$\mathrm{MSE_u}(\Psi_{\mathcal{L}}) \leq \eta\, \mathrm{MSE_u}(\Psi_{OPT}),$$

with $\eta = \gamma \frac{L_{max}}{L_{min}} \frac{(d+\delta)^2}{(d-\delta)^2}$ where $d = \frac{L_{mean}}{K}$ and L_{mean} is the average value of $L_{t,\mathcal{L}}$ and γ is the approximation factor for the FP_u introduced in Theorem 1.

Thus, minimizing $\mathrm{FP_u}$ also results in a placement that is near-optimal with respect to $\mathrm{MSE_u}$ as desired.

3. NUMERICAL RESULTS

In this section we aim at evaluating two factors: how well Algorithm 1 performs with regards to other sensor placement algorithms for the UoS model and the advantages of using a UoS model over a standard approach based on the SoS one. For both experiments, we considered three subspaces spanned by Gaussian random matrices $\Psi_t \in \mathbb{R}^{100\times 10}$. Therefore, we assume the physical field $y \in \mathbb{R}^{100}$ to lie in three subspaces of dimension ten.

In the first experiment, we compared Algorithm 1 with a greedy algorithm optimizing directly the $\mathrm{MSE_u}$ by iteratively adding to $\mathcal{L}$ the row with the best $\mathrm{MSE_u}$ and a random algorithm choosing uniformly among all the possible locations. We varied the number of placed sensors and measured the $\mathrm{MSE_u}$ obtained by each algorithm. The results are shown in Figure 2, where we note that the algorithm based on $\mathrm{FP_u}$ outperforms the other two. Furthermore, we underline that a direct greedy optimization of the MSE is not effective and does not show any performance gains compared to a random placement.

In the second experiment, we compared the quality of the inverse problem solution when the physical field y is modeled as a UoS or a SoS. As mentioned in Section 2, it is possible to represent a UoS modeled by T matrices with a single matrix spanning the SoS. For each model, the $\mathrm{FP_u}$ is used to select the rows and we measure the $\mathrm{MSE_u}$ on the output matrices. For the UoS, we assign equal weights to the different subspaces. The results are shown in Figure 3. Notice that the SoS model is 30-dimensional since the three subspaces are with probability one orthogonal to each other. Therefore, the solution of the inverse problem for the SoS model when $L < 30$ is not unique and the obtained MSE values for the SoS model under this threshold cannot be fairly compared to the UoS case. However, when we compare the performance for $L \geq 30$, we highlight that UoS has on average a 48.5 dB advantage. This considerable difference is attributed to the fact that the union in this case consists of three independent uncorrelated subspaces, an optimal scenario; in a real world setting, it could be less pronounced. Nonetheless, we underline two significant advantages of the UoS model over the SoS model:

1. It is possible to use less sensors while obtaining a higher

precision.

2. The achieved $\mathrm{MSE_u}$ for UoS is significantly better even for the same number of sensors.

While this result was expected, it is not trivial. In fact, finding a sensor placement that achieves a good MSE for all the three matrices is a harder problem and could have counterbalanced the reduced number of parameters to estimate.

4. CONCLUSION

We studied the sensor placement problem as defined in [8] to consider the case when the signals of interest lie in a union of subspaces, each represented by a linear model Ψ_t. More precisely, we proposed a greedy algorithm that uses a cost function based on the weighted sum of the frame potentials to choose the rows that jointly minimize the reconstruction MSE for all matrices. Further, we showed that the proposed algorithm is near-optimal w.r.t. to the joint MSE, an important aspect for the practical applications of such an algorithm.

With two numerical experiments, we showed that the use of the union of subspaces model allows to i) use a lower number of sensors while maintaining the reconstruction performance, or ii) achieve, using the same number of sensors, a lower reconstruction MSE. For example, with random Gaussian matrices we improve the MSE by 48.5 dB on average.

Future work is focused on the testing of the UoS model and the sensor placement algorithm for real-world applications. Note that not every physical field may be successfully modeled as a UoS. Additionally, the estimation of such a model is more challenging than that of the usual SoS model. In fact, while optimal algorithms exist for the latter, see principal components analysis, the estimation of the former relies on the solution of an optimization problem with many non-optimal local minima.

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