A Review of the Invertibility of Frame Multipliers

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***Abstract*—In this paper we give a review of recent results on the invertibility of frame multipliers $M_{m,\Phi,\Psi}$. In particular, we give sufficient, necessary or equivalent conditions for the invertibility of such operators, depending on the properties of the sequences Ψ, Φ and m. We consider Bessel sequences, frames, and Riesz bases.**

I. Introduction

Certain mathematical objects appear in a lot of scientific disciplines, like physics [2], signal processing [16] and certainly mathematics [13]. In a general setting they can be described as frame multipliers, consisting of analysis, multiplication by a fixed sequence (called the symbol), and synthesis:

$$M_{m,\Phi,\Psi} f = \sum_k m_k \langle f, \psi_k \rangle \phi_k.$$

They are not only interesting mathematical objects [4], [7], [9], [12], [17], but also important for applications, for example for the realization of time-varying filters [8], [15], [24]. Therefore, for some applications it is important to find the inverse of a multiplier if it exists.

Here we collect results from [3], [20], [21], [23] about the invertibility of such operators.

II. Preliminaries and Notations

Throughout the paper, $\mathcal{H}$ denotes an infinite-dimensional Hilbert space, Φ (resp. Ψ) denotes a sequence $(\phi_n)_{n=1}^\infty$ (resp. $(\psi_n)_{n=1}^\infty$) with elements from $\mathcal{H}$, m denotes a complex scalar sequence $(m_n)_{n=1}^\infty$, $\overline{m} = (\overline{m}_n)_{n=1}^\infty$, and $m\Psi = (m_n\psi_n)_{n=1}^\infty$. The sequence m is called *semi-normalized* if $0 < \inf_n |m_n| \leq \sup_n |m_n| < \infty$. When the index set is omitted, $\mathbb{N}$ should be understood as the index set. A multiplier $M_{m,\Phi,\Psi}$ is the operator given by $M_{m,\Phi,\Psi} h = \sum_{n=1}^\infty m_n \langle h, \psi_n \rangle \phi_n$. If not stated otherwise, M denotes any one of the multipliers $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$. The identity operator on $\mathcal{H}$ is denoted by $I_\mathcal{H}$. An operator $T : \mathcal{H} \to \mathcal{H}$ is called *invertible* if it is a bounded bijection. Depending on the sequences Φ, Ψ, and m, the multiplier $M_{m,\Phi,\Psi}$ might be well defined or not well defined, as well as invertible or not invertible. In [22] an extensive collection of examples for all those cases can be found.

Recall that Φ is called a *Bessel sequence* (in short, *Bessel*) *in $\mathcal{H}$ with bound B_Φ* if $B_\Phi > 0$ and $\sum |\langle h, \phi_n \rangle|^2 \leq B_\Phi \|h\|^2$ for every $h \in \mathcal{H}$. A Bessel sequence Φ in $\mathcal{H}$ with bound B_Φ is called a *frame for $\mathcal{H}$ with bounds A_Φ, B_Φ*, if $A_\Phi > 0$ and $A_\Phi \|h\|^2 \leq \sum |\langle h, \phi_n \rangle|^2$ for every $h \in \mathcal{H}$. For a given frame Φ for $\mathcal{H}$, its *frame operator* S_Φ is given by $S_\Phi h = \sum \langle h, \phi_n \rangle \phi_n$, $h \in \mathcal{H}$. When Φ is a frame for $\mathcal{H}$, there exist frames $\Phi^d = (\phi_n^d)$ satisfying $h = \sum \langle h, \phi_n^d \rangle \phi_n = \sum \langle h, \phi_n \rangle \phi_n^d$ for every $h \in \mathcal{H}$. Such frames Φ^d are called *dual frames of* Φ.

The sequence Φ is called a *Riesz basis for $\mathcal{H}$ with bounds A_Φ, B_Φ*, if Φ is complete in $\mathcal{H}$, $A_\Phi > 0$, and $A_\Phi \sum |c_n|^2 \leq \| \sum c_n \phi_n \|^2 \leq B_\Phi \sum |c_n|^2$, $\forall (c_n) \in \ell^2$. Every Riesz basis for $\mathcal{H}$ with bounds A, B is a frame for $\mathcal{H}$ with bounds A, B. For standard references for frame theory and related topics see [10], [11], [14].

The notion of frame multipliers is naturally connected to the one of weighted frames [6], [19]. Frames can also be used for the representation of operators [5]. In this setting multipliers are those operators that can be represented with diagonal matrices.

III. Sufficient and necessary conditions for invertibility of multipliers for Riesz bases

We start with the case of Riesz bases, where we can give equivalent conditions for the invertility of multipliers. We start with an easy result:

Proposition 3.1: [3] If Φ and Ψ are Riesz bases and m is semi-normalized, then $M_{m,\Phi,\Psi}$ is invertible and its inverse can be written as $M_{(1/m_n),\widetilde{\Psi},\widetilde{\Phi}}$, where $\widetilde{\Psi}$ and $\widetilde{\Phi}$ are the unique biorthogonal sequences to Ψ and Φ, respectively.

Even more, it can be shown, that if two of those three assumptions are assumed, the third one is equivalent to the invertibility of M:

Theorem 3.2: [20] Let Φ be a Riesz basis for $\mathcal{H}$. Then the following holds.

(i) If Ψ is a Riesz basis for $\mathcal{H}$, then M is invertible if and only if m is semi-normalized.
(ii) If m is semi-normalized, then M is invertible if and only if Ψ is a Riesz basis for $\mathcal{H}$.

More detailed, we can distinguish the cases of well-definedness, invertibility, injectivity, and surjectivity of multipliers for Riesz bases:

Proposition 3.3: [23] Let Φ be a Riesz basis for $\mathcal{H}$. The following equivalences hold.

(a) M is well defined if and only if $m\Psi$ is a Bessel sequence in $\mathcal{H}$.
(b) M is invertible if and only if $m\Psi$ is a Riesz basis for $\mathcal{H}$.
(c1) $M_{m,\Phi,\Psi}$ is injective if and only if $m\Psi$ is a complete Bessel sequence in $\mathcal{H}$.

(c2) $M_{m,\Psi,\Phi}$ is injective if and only if the synthesis operator $T_{m\Psi}$, given by $T_{m\Psi}(c_n) = \sum c_n m_n \psi_n$ for $(c_n) \in \ell^2$, is injective.
(d1) $M_{m,\Phi,\Psi}$ is surjective if and only if $\overline{m}\Psi$ is a Riesz basis for its closed linear span.
(d2) $M_{m,\Psi,\Phi}$ is surjective if and only if $m\Psi$ is a frame for $\mathcal{H}$.

As seen above, when at least one of Φ and Ψ is not a Riesz basis or m is not semi-normalized, then M does not need to be invertible. In such cases the multiplier M even does not need to be well defined (one can give simple examples, see e.g. [22]).

IV. Sufficient conditions for invertibility of multipliers for frames

In this section we consider the more general case, where it is assumed that Φ is a frame. We give four sufficient conditions for the invertibility of multipliers, give representations of the inverses as operator sums and give the corresponding n-term error:

Proposition 4.1: [20] Let Φ be a frame for $\mathcal{H}$. Assume that

$\mathcal{P}_1$: $\exists\, \mu \in [0, \frac{A_\Phi^2}{B_\Phi})$ such that $\sum |\langle h, \psi_n - \phi_n \rangle|^2 \le \mu \|h\|^2$, $\forall h \in \mathcal{H}$.

For every positive (or negative) semi-normalized sequence m, satisfying

$$\frac{\sup_n |m_n|}{\inf_n |m_n|}\sqrt{\mu} < \frac{A_\Phi}{\sqrt{B_\Phi}},$$

it follows that Ψ is a frame for $\mathcal{H}$, M is invertible,

$$M^{-1} = \sum_{k=0}^{\infty} [S^{-1}_{(\sqrt{m_n}\phi_n)}(S_{(\sqrt{m_n}\phi_n)} - M)]^k S^{-1}_{(\sqrt{m_n}\phi_n)},$$

if $m_n > 0, \forall n \in \mathbb{N}$, and

$$M^{-1} = -\sum_{k=0}^{\infty} [S^{-1}_{(\sqrt{|m_n|}\phi_n)}(S_{(\sqrt{|m_n|}\phi_n)} + M)]^k S^{-1}_{(\sqrt{|m_n|}\phi_n)},$$

if $m_n < 0, \forall n \in \mathbb{N}$,
where the n-term error is bounded by the constant $\frac{(b\sqrt{\mu B_\Phi})^{n+1}}{aA_\Phi - b\sqrt{\mu B_\Phi}}\left(\frac{1}{aA_\Phi}\right)^{n+1}$.

Proposition 4.2: [20] Let Φ be a frame for $\mathcal{H}$ and $\mathcal{P}_1$ hold. Let m satisfy

$$|m_n - 1| \le \lambda < \frac{A_\Phi - \sqrt{\mu B_\Phi}}{B_\Phi + \sqrt{\mu B_\Phi}},\ \forall n \in \mathbb{N}, \qquad (1)$$

for some λ. Then Ψ is a frame for $\mathcal{H}$, $M_{m,\Phi,\Phi}$ and M are invertible,

$$M^{-1}_{m,\Phi,\Phi} = \sum_{k=0}^{\infty} [S_\Phi^{-1}(S_\Phi - M_{m,\Phi,\Phi})]^k S_\Phi^{-1},$$

where the n-term error is bounded by $\frac{1}{A_\Phi - \lambda B_\Phi}\left(\frac{\lambda B_\Phi}{A_\Phi}\right)^{n+1}$, and

$$M^{-1} = \sum_{k=0}^{\infty} [M^{-1}_{m,\Phi,\Phi}(M_{m,\Phi,\Phi} - M)]^k M^{-1}_{m,\Phi,\Phi},$$

where the n-term error is bounded by the constant $\frac{1}{A_\Phi - \lambda B_\Phi - (\lambda+1)\sqrt{\mu B_\Phi}}\left(\frac{(\lambda+1)\sqrt{\mu B_\Phi}}{A_\Phi - \lambda B_\Phi}\right)^{n+1}$.

Proposition 4.3: [20] Let Φ be a frame for $\mathcal{H}$ and $\Phi^d = (\phi_n^d)$ be a dual frame of Φ. Let M denote any one of M_{m,Φ,Φ^d} and $M_{m,\Phi^d,\Phi}$, and m be such that $|m_n - 1| \le \lambda < \frac{1}{\sqrt{B_\Phi B_{\Phi^d}}}$, $\forall n \in \mathbb{N}$, for some λ. Then M is invertible,

$$M^{-1}_{m,\Phi^d,\Phi} = \sum_{k=0}^{\infty} (M_{(1-m_n),\Phi^d,\Phi})^k,$$

and the n-term error is bounded by $\dfrac{\left(\lambda \cdot \sqrt{B_\Phi B_{\Phi^d}}\right)^{n+1}}{1 - \lambda \cdot \sqrt{B_\Phi B_{\Phi^d}}}$.

Proposition 4.4: [20] Let Φ be a frame for $\mathcal{H}$. Assume that

$\mathcal{P}_2$: $\exists\, \mu \in [0, \frac{1}{B_\Phi})$ such that $\sum |\langle h, m_n\psi_n - \phi_n^d \rangle|^2 \le \mu \|h\|^2$, $\forall\, h \in \mathcal{H}$,

for some dual frame $\Phi^d = (\phi_n^d)$ of Φ. Let M denote any one of $M_{\overline{m},\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$. Then $m\Psi$ is a frame for $\mathcal{H}$, M is invertible,

$$M^{-1} = \sum_{k=0}^{\infty} (I_{\mathcal{H}} - M)^k,$$

and the n-term error is bounded by $\frac{(\sqrt{\mu B_\Phi})^{n+1}}{1 - \sqrt{\mu B_\Phi}}$.

V. Invertibility of Multipliers for equivalent frames

Two frames Φ and Ψ are called equivalent [1], [10], if there exists a bounded bijective operator G, such that $\psi_n = G\phi_n$ for every $n \in \mathbb{N}$.

Proposition 5.1: [20] Let Φ and Ψ be equivalent frames for $\mathcal{H}$. Let m be semi-normalized and satisfy one of the following three conditions:

- $m_n > 0$ for every $n \in \mathbb{N}$;
- $m_n < 0$ for every $n \in \mathbb{N}$; or
- there exists λ with $|m_n - 1| \le \lambda < A_\Phi / B_\Phi$, $\forall n \in \mathbb{N}$.

Then M and $M_{m,\Phi,\Phi}$ are invertible, $M^{-1}_{m,\Phi,\Psi} = (G^{-1})^* M^{-1}_{m,\Phi,\Phi}$, and $M^{-1}_{m,\Psi,\Phi} = M^{-1}_{m,\Phi,\Phi} G^{-1}$.

VI. Necessary conditions for invertibility of multipliers for Bessel sequences

In this section we generalize the assumptions, considering the invertibility of multipliers for Bessel sequences.

Let one of the sequences Φ and Ψ be a Bessel sequence in $\mathcal{H}$. If the multiplier $M_{m,\Phi,\Psi}$ is invertible, then the other sequence does not need to be a Bessel sequence. The next statement contains necessary conditions for invertibility concerning the other sequence. In particular, it shows that if Φ and Ψ are Bessel sequences and m is bounded, then the multiplier $M_{m,\Phi,\Psi}$ can be invertible only if Φ and Ψ are frames for $\mathcal{H}$.

Proposition 6.1: [20] Let $M_{m,\Phi,\Psi}$ be invertible.

(i) If Ψ (resp. Φ) is a Bessel sequence in $\mathcal{H}$ with bound B, then $m\Phi$ (resp. $m\Psi$) satisfies the lower frame condition for $\mathcal{H}$ with bound $\frac{1}{B\,\|M^{-1}_{m,\Phi,\Psi}\|^2}$.
(ii) If Ψ (resp. Φ) and $m\Phi$ (resp. $m\Psi$) are Bessel sequences in $\mathcal{H}$, then they are frames for $\mathcal{H}$.

(iii) If Ψ (resp. Φ) is a Bessel sequence in $\mathcal{H}$ and $m \in \ell^\infty$, then Φ (resp. Ψ) satisfies the lower frame condition for $\mathcal{H}$.

(iv) If Ψ and Φ are Bessel sequences in $\mathcal{H}$ and $m \in \ell^\infty$, then Ψ, Φ, $m\Phi$, and $m\Psi$ are frames for $\mathcal{H}$.

VII. Unconditionally convergent invertible multipliers

The previous results contain conclusions for both of the multipliers $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$ (under the same assumptions in each statement). This leads naturally to the question for a connection between the multipliers $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$. Note that $M_{m,\Phi,\Psi}$ being invertible is not equivalent to $M_{m,\Psi,\Phi}$ being invertible, see Example 2.2 in [21]. The next statement gives an equivalence of the invertibility of those operators when unconditionally convergent multipliers $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$ are considered.

Proposition 7.1: [21] For any Φ, Ψ and m, the following holds.

(i) Let $M_{m,\Phi,\Psi}$ be invertible and let $M_{\overline{m},\Psi,\Phi}$ be well defined. Then $M_{\overline{m},\Psi,\Phi}$ is invertible and $M^{-1}_{\overline{m},\Psi,\Phi} = (M^{-1}_{m,\Phi,\Psi})^*$.

(ii) $M_{m,\Phi,\Psi}$ is unconditionally convergent and invertible $\Leftrightarrow$ $M_{\overline{m},\Psi,\Phi}$ is unconditionally convergent and invertible.

Below we consider a necessary condition for certain classes of multipliers to be both unconditionally convergent and invertible. Among these multipliers we consider the cases when Gabor or Wavelet frames are used. Consider the condition

$\mathcal{P}_3$: $\exists$ (c_n) and (d_n) so that $M_{m,\Phi,\Psi}$ can be written as $M_{(1),(c_n\phi_n),(d_n\psi_n)}$, where the summands are kept and $(c_n\phi_n)$ and $(d_n\psi_n)$ are frames for $\mathcal{H}$.

Proposition 7.2: [21] Let Φ and Ψ be Gabor (or wavelet) systems and let m satisfy $\inf_n |m_n| > 0$. If $M_{m,\Phi,\Psi}$ is unconditionally convergent and invertible, then $\mathcal{P}_3$ holds.

Proposition 7.3: [21] Let Φ be minimal. If $M_{m,\Phi,\Psi}$ is unconditionally convergent and invertible, then $\mathcal{P}_3$ holds.

Proposition 7.4: If $M_{m,\Phi,\Phi}$ is unconditionally convergent and invertible, then $(\sqrt{m_n}\phi_n)$ is a frame for $\mathcal{H}$ (where $\sqrt{m_n}$ denotes one (any one) of the two complex square roots of m_n, $n \in \mathbb{N}$) and $\mathcal{P}_3$ holds.

VIII. Conclusion

In this paper we have considered the invertibility of frame multipliers and reviewed analytical results.

In the future we will investigate the numerics of the inversion of multipliers (using the LTFAT toolbox [18]) and classify the cases when the inverse of a multiplier is again a multiplier, i.e. generalizations of Prop. 3.1.

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References

[1] S. T. Ali, J. P. Antoine, and J. P. Gazeau, *Continuous frames in Hilbert space*, Ann. Phys. **222** (1993), no. 1, 1–37.

[2] ———, *Coherent States, Wavelets and Their Generalization*, Graduate Texts in Contemporary Physics, Springer New York, 2000.

[3] P. Balazs, *Basic definition and properties of Bessel multipliers*, J. Math. Anal. Appl. **325** (2007), no. 1, 571–585.

[4] ———, *Hilbert-Schmidt operators and frames - classification, best approximation by multipliers and algorithms*, Int. J. Wavelets Multiresolut. Inf. Process. **6** (2008), no. 2, 315 – 330.

[5] ———, *Matrix-representation of operators using frames*, Sampl. Theory Signal Image Process. **7** (2008), no. 1, 39–54.

[6] P. Balazs, J. P. Antoine, and A. Grybos, *Weighted and controlled frames: Mutual relationship and first numerical properties*, Int. J. Wavelets Multiresolut. Inf. Process. **8** (2010), no. 1, 109–132.

[7] P. Balazs, D. Bayer, and A. Rahimi, *Multipliers for continuous frames in Hilbert spaces*, J. Phys. A: Mathematical and Theoretical **45** (2012), 244023.

[8] P. Balazs, B. Laback, G. Eckel, and W. A. Deutsch, *Time-frequency sparsity by removing perceptually irrelevant components using a simple model of simultaneous masking*, IEEE Trans Audio Speech Lang Processing **18** (2010), no. 1, 34–49.

[9] J. Benedetto and G. Pfander, *Frame expansions for Gabor multipliers*, Appl. Comput. Harmon. Anal. **20** (2006), no. 1, 26–40.

[10] P. Casazza, *The art of frame theory*, Taiwanese J. Math. **4** (2000), no. 2, 129–202.

[11] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[12] M. Dörfler and B. Torrésani, *Representation of operators in the time-frequency domain and generalized Gabor multipliers*, J. Fourier Anal. Appl. **16** (2010), no. 2, 261–293.

[13] H. G. Feichtinger and K. Nowak, *A first survey of Gabor multipliers*, ch. 5, pp. 99–128, Birkhäuser, Boston, 2003.

[14] C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

[15] P. Majdak, P. Balazs, W. Kreuzer, and M. Dörfler, *A time-frequency method for increasing the signal-to-noise ratio in system identification with exponential sweeps*, Proceedings of the 36th International Conference on Acoustics, Speech and Signal Processing, ICASSP 2011, Prag, 2011.

[16] G. Matz and F. Hlawatsch, *Linear time-frequency filters: On-line algorithms and applications*, ch. 6 in 'Application in Time-Frequency Signal Processing', pp. 205–271, eds. A. Papandreou-Suppappola, Boca Raton (FL): CRC Press, 2002.

[17] A. Rahimi and P. Balazs, *Multipliers for p-Bessel sequences in Banach spaces*, Integral Equations Oper. Theory **68** (2010), no. 2, 193–205.

[18] P. Soendergaard, B. Torrésani, and P. Balazs, *The linear time frequency analysis toolbox*, Int. J. Wavelets Multiresolut. Inf. Process. **10** (2012), no. 4, 1250032.

[19] D. T. Stoeva and P. Balazs, *Weighted frames and frame multipliers*, Annual of the University of Architecture, Civil Engineering and Geodesy XLIII-XLIV 2004-2009 (Fasc. II Mathematics Mechanics), 2012, 33–42.

[20] ———, *Invertibility of multipliers*, Appl. Comput. Harmon. Anal. **33** (2012), no. 2, 292–299.

[21] ———, *Canonical forms of unconditionally convergent multipliers*, J. Math. Anal. Appl. **399** (2013), 252–259.

[22] ———, *Detailed characterization of unconditional convergence and invertibility of multipliers*, Sampl. Theory Signal Image Process **12** (2013), no. 2, to appear.

[23] ———, *Riesz bases multipliers*, Proceedings IWOTA 2011 (Manuel Cepedello Boiso, Håkan Hedenmalm, Marinus A. Kaashoek, Alfonso Montes-Rodríguez, and Sergei R. Treil, eds.), Operator Theory: Advances and Applications **236** (2013), 477–484, Springer Basel AG.

[24] D. Wang and G. J. Brown, *Computational Auditory Scene Analysis: Principles, Algorithms, and Applications*, Wiley-IEEE Press, 2006.