OPTIMAL WEIGHT LEARNING FOR COUPLED TENSOR FACTORIZATION WITH MIXED DIVERGENCES

Umut Şimşekli, Beyza Ermiş, A. Taylan Cemgil

Boğaziçi University
Dept. of Computer Engineering
34342, Bebek, İstanbul, Turkey

Evrim Acar

University of Copenhagen
Faculty of Science
DK-1958 Frederiksberg C, Denmark

ABSTRACT

Incorporating domain-specific side information via coupled factorization models is useful in source separation applications. Coupled models can easily incorporate information from source modalities with different statistical properties by estimating shared factors via divergence minimization. Here, it is useful to use mixed divergences, a specific divergence for each modality. However, this extra freedom requires choosing the correct divergence as well as an optimal weighting mechanism to select the relative 'importance'. In this paper, we present an approach for determining the relative weights, framed as dispersion parameter estimation, based on an inference framework introduced previously as Generalized Coupled Tensor Factorization (GCTF). The dispersion parameters play a key role on inference as they form a balance between the information obtained from multimodal observations. We tackle the problem of optimal weighting by maximum likelihood exploiting the relation between β-divergences and the family of Tweedie distributions. We demonstrate the usefulness of our approach on a drum source separation application.

***Index Terms*—** Informed source separation, Coupled Matrix/Tensor Factorization, Tweedie models, β-divergence

1. INTRODUCTION

Blind source separation methods are popular tools and have been shown to be useful in many domains [1]. However, when it comes to separating complex acoustic signals using only a single channel of observations, these methods fall short as they don't exploit the highly structured patterns in such signals. Recent studies show that providing domain-specific information to source separation methods increases the separation performance [2, 3, 4].

One way to incorporate domain-specific knowledge is to jointly analyze side information with the observed audio mixtures using coupled factorization models. For instance, Kim et al. present a coupled matrix factorization model for drum source separation in polyphonic music signals by incorporating side information obtained from a collection of drum recordings [5]. Similarly, coupled factorizations are used for audio source separation in [3], where the audio mixture is coupled with isolated sounds and symbolic music data.

Coupled factorization models jointly factorize data from multiple information sources by extracting common factors. We first consider the following simple coupled matrix factorization model and, in the next section, we construct a drum separation model based on a more complicated coupled factorization problem. We define the toy coupled model as follows:

$$X_1(f,t) \approx \hat{X}_1(f,t) = \sum_i Z_1(f,i) Z_2(i,t)$$
$$X_2(f,n) \approx \hat{X}_2(f,n) = \sum_i Z_1(f,i) Z_3(i,n)$$

where X_1 and X_2 are two observed matrices decomposed by a Nonnegative Matrix Factorization (NMF) model. In an audio processing application, X_1 and X_2 may denote audio spectra (where f denotes frequency and t and n denote time frames), Z_1 may correspond to a spectral dictionary while Z_2 and Z_3 may denote the corresponding excitations. Here, the factor Z_1 exists in both models, which makes these models coupled. If X_1 consists of a mixture of sound sources to be separated, we can incorporate further information to the source separation model using such coupled factorization models.

Given the observed matrices X_1 and X_2, the coupled factorization problem estimates the factors, i.e., $Z_{1:3} \equiv \{Z_1, Z_2, Z_3\}$; in other words, solves the following optimization problem:

$$(Z_{1:3})^* = \arg\min_{Z_{1:3}} \frac{1}{\phi_1} d_1(X_1 || \hat{X}_1) + \frac{1}{\phi_2} d_2(X_2 || \hat{X}_2) \quad (1)$$

where $d_1(\cdot)$ and $d_2(\cdot)$ are (possibly different) divergences, usually chosen as the Euclidean, Kullback-Leibler (KL) or the Itakura-Saito (IS) divergences. In our context, the (inverse)

Funded by the scientific and technological research council of Turkey (TÜBİTAK) grant number 110E292, project Bayesian matrix and tensor factorizations (BAYTEN). Umut Şimşekli is also supported by a Ph.D. scholarship from TÜBİTAK. The authors would like to thank Hakan Güldaş for fruitful discussions.

weights ϕ_1 and ϕ_2 will be called *dispersion parameters* and balance the information obtained from different modalities. We have previously illustrated the effect of correctly choosing a divergence on coupled models [6]. Similarly, dispersion parameters are also expected to play a significant role on the performance of coupled models.

While coupled factorization models have been widely studied in many fields [6, 7, 8], determining the right weighting scheme remains to be a major challenge in data fusion studies [9]. Wilderjans et al. [10] proposed a maximum likelihood approach for estimating the weights under Gaussian observation models. In this study, we tackle this problem by using a probabilistic approach, which makes use of the relation between the β-divergence and the family of Tweedie distributions and enables us to learn the dispersion parameters by maximizing the likelihood. We demonstrate that estimating the dispersion is both straightforward and useful in audio source separation.

2. DRUM SEPARATION MODEL

In this section, we present a coupled matrix factorization model for drum separation for professionally recorded audio. This model combines the information that is gathered from the audio mixture, isolated drum sounds and an approximate transcription of drum track of the audio mixture.

Suppose we observe the magnitude spectrum of an audio mixture $X_1(f,t)$, where f and t denote the frequency and time frame indices, respectively. Here, we assume that matrix X_1 is decomposed by using an NMF model:

$$X_1(f,t) \approx \hat{X}_1(f,t) = \sum_i D(f,i)G(i,t) \tag{2}$$

where D is the spectral dictionary and G is the corresponding excitation matrix. Since the aim of this model is drum track separation, we will assume that some of the spectral templates, say the first I_b columns of D, denoted as $D(:,1:I_b)$ model the background sources and the remaining model the drum track. Suppose we observe another magnitude spectrum $X_2(f,n)$, which is obtained from a database of drum sounds. Here f is again the frequency index and n is the time frame index. We can also decompose this matrix using a similar approach:

$$X_2(f,n) \approx \hat{X}_2(f,n) = \sum_i D(f,i)T_1(f,i)E(i,n) \tag{3}$$

where D is the same spectral dictionary in Equation 2 and E is the excitation matrix for the example drum sounds. Here, T_1 is a pre-determined binary matrix that makes sure that the drum sounds use only the related spectral templates: T_1 takes values of 0 for the background part of the dictionary and 1 otherwise: $T_1(:,1:I_b) = 0$ and $T_1(:,I_b+1:I) = 1$, where I is the number of spectral templates. So far, we have the coupled factorization model of [5], which incorporates the spectral information that is obtained from the drum sounds to the drum separation model.

As we are modeling musical signals, we may also assume the excitation matrix G is composed of the superposition of some certain patterns that repeat over time. With this assumption, we can also factorize the matrix G using another NMF model as follows: $G(i,t) = \sum_k B(i,k)F(k,t)$, where B is the dictionary for the excitations and F denotes the excitations that correspond to this dictionary. By making use of the relation between an excitation matrix and a musical score, we can also couple the matrices B and F with an approximate transcription of the drum track as follows:

$$X_3(i,t) \approx \hat{X}_3(i,t) = \sum_k B(i,k)T_2(i,k)F(k,t). \tag{4}$$

Here, $X_3(i,t)$ takes a constant value if a drum event i (e.g. snare hit, hi-hat hit, etc.) is present at time frame t, and becomes 0 otherwise. Furthermore, T_2 is another predetermined binary matrix similar to T_1, where $T_2(1:I_b,:) = 0$ and $T_2(I_b+1:I,:) = 1$. Note that, decomposition of the excitation matrix by this approach is shown to be useful in musical source separation [3].

Finally, we can define the combined model as follows:

$$\hat{X}_1(f,t) = \sum_{i,k} D(f,i)B(i,k)F(k,t) \qquad \text{Mixture} \tag{5}$$

$$\hat{X}_2(f,n) = \sum_i D(f,i)T_1(f,i)E(i,n) \qquad \text{Drum} \tag{6}$$

$$\hat{X}_3(i,t) = \sum_k B(i,k)T_2(i,k)F(k,t) \qquad \text{MIDI} \tag{7}$$

Figure 1 illustrates this model. The goal is to estimate the latent factors D, B, F, and E as the sources can be separated by Wiener filtering after the factors D, B, and F are obtained. This problem can be formulated as an optimization problem with the following objective function with mixed divergence:

$$d(X_{1:3}||\hat{X}_{1:3}) = \sum_{\nu=1}^{3} \frac{1}{\phi_\nu} d_\nu(X_\nu||\hat{X}_\nu), \tag{8}$$

where $d_\nu(\cdot)$ are divergence functions and ϕ_ν are dispersion parameters as defined in Section 1. Since we are dealing with audio signals, selecting $d_1(\cdot)$ and $d_2(\cdot)$ as Itakura-Saito divergence would be appropriate as suggested in [11]. Besides, we may want to select $d_3(\cdot)$ as the KL divergence. However, we may wish to give different weights to each observed matrix. In particular, we know that the transcription matrix X_3 is not very accurate, we don't need to fit this matrix precisely. The dispersion parameters play a key role here, as they determine the noise variance of each observed matrix. For this particular example, selecting a large ϕ_3 seems to be an accurate modeling strategy.

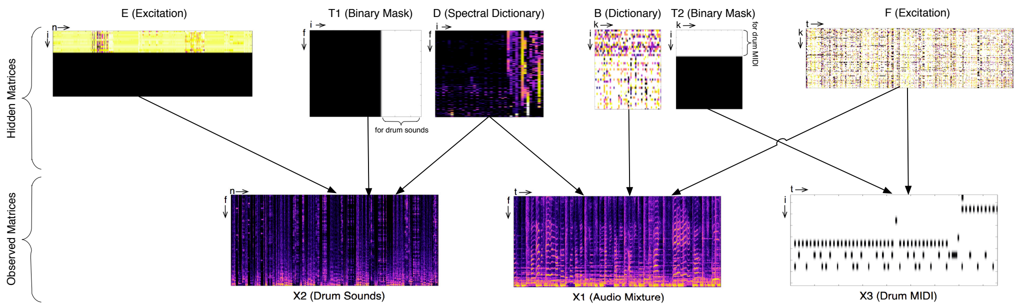


Fig. 1: Illustration of the drum separation model. The blocks visualize the matrices and the relation between them. The lower-case letters and arrows near the blocks represent the indices of a particular matrix.

The estimation of factor matrices under mixed divergences can be achieved using the Generalized Coupled Tensor Factorization framework, which we will explain in the next section. In this study, we propose a method for automatically determining the dispersion parameters ϕ_ν under mixed cost functions.

3. GENERALIZED COUPLED TENSOR FACTORIZATION

The Generalized Coupled Tensor Factorization (GCTF) framework [12] is a generalization of matrix and tensor factorization models to jointly factorize more than one multiway array (tensor or matrix). The formal definition of the GCTF framework is as follows:

$$X_\nu(u_\nu) \approx \hat{X}_\nu(u_\nu) = \sum_{\bar{u}_\nu} \prod_\alpha Z_\alpha(v_\alpha)^{R^{\nu,\alpha}} \qquad (9)$$

where $\nu = 1, ...|\nu|$ is the observed tensor index and $\alpha = 1, ...|\alpha|$ is the factor index. In this framework, the goal is computing an approximate factorization of given observed tensors X_ν in terms of a product of individual factors Z_α, some of which are possibly shared. Here, we define V as the set of all indices in a model, U_ν as the set of visible indices of the tensor X_ν, V_α as the set of indices in Z_α, and $\bar{U}_\nu = V - U_\nu$ as the set of invisible indices that is not present in X_ν. We use small letters as v_α to refer to a particular setting of indices in V_α. Furthermore, R is a *coupling matrix* that is defined as follows: $R^{\nu,\alpha} = 1$ (0) if X_ν and Z_α connected (otherwise). In other words, the coupling matrix $R^{\nu,\alpha}$ specifies the factors Z_α that effect the observed tensor X_ν.

As the product $\prod_\alpha Z_\alpha(v_\alpha)$ is collapsed over a set of indices, the factorization is latent. The optimization problem is to minimize the total discrepancy between the observations X_ν and the model output $\hat{X}_\nu$, as given by a divergence function $d_\nu(X_\nu||\hat{X}_\nu)$. This divergence is a quasi-squared distance typically taken as Euclidean, KL or IS. The objective function to be minimized has a similar form as in Equation 8:

$$Z^\star_{1:|\alpha|} = \underset{Z_{1:|\alpha|}}{\arg\max} \sum_\nu \frac{1}{\phi_\nu} d_\nu(X_\nu||\hat{X}_\nu). \qquad (10)$$

In order to illustrate our notation, we define the drum separation model in the GCTF notation as follows. We can define $Z_{1:6} \equiv \{D, B, F, T_1, E, T_2\}$, the observed index sets: $U_1 = \{f, t\}$, $U_2 = \{f, n\}$, and $U_3 = \{i, t\}$, the index sets of the factors: $V_1 = \{f, i\}$, $V_2 = \{i, k\}$, $V_3 = \{k, t\}$, $V_4 = \{f, i\}$, $V_5 = \{i, n\}$, and $V_6 = \{i, k\}$.

3.1. Estimation

Estimation of the latent factors Z_α can be achieved via iterative methods, by fixing all factors $Z_{\alpha'}$ for $\alpha' \neq \alpha$ but one Z_α and updating in an alternating fashion(see [12]). For nonnegative data and factors, the update has a simple form:

$$Z_\alpha \leftarrow Z_\alpha \circ \frac{\sum_\nu R^{\nu,\alpha} \phi_\nu^{-1} \Delta_{\alpha,\nu}(\hat{X}_\nu^{-p_\nu} \circ X_\nu)}{\sum_\nu R^{\nu,\alpha} \phi_\nu^{-1} \Delta_{\alpha,\nu}(\hat{X}_\nu^{1-p_\nu})}, \qquad (11)$$

where $\circ$ is the element-wise product (Hadamard product) and the parameter p_ν determines the cost function to be used for X_ν: for $p_\nu = \{0, 1, 2\}$ correspond to the β-divergence [13] that unifies Euclidean, KL, and IS cost functions, respectively. The key quantity in the above update equation is the $\Delta_{\alpha,\nu}$ function that is defined as follows:

$$\Delta_{\alpha,\nu}(A) = \left[\sum_{v_{0,\nu} \cap \bar{v}_\alpha} A(v_{0,\nu}) \sum_{\bar{v}_0 \cap \bar{v}_\alpha} \prod_{\alpha' \neq \alpha} Z_{\alpha'}(v_{\alpha'})^{R^{\nu,\alpha'}} \right] \qquad (12)$$

For updating Z_α, we need to compute this function twice for arguments $A = \hat{X}_\nu^{-p_\nu} \circ X_\nu$ and $A = \hat{X}_\nu^{1-p_\nu}$.

4. LEARNING THE DISPERSION PARAMETERS

Exponential dispersion models (EDM's) are a well-studied family of distributions and have found place in various fields. The Tweedie distribution is a special case of the EDM [14], itself a generalization of the more familiar natural exponential family. There is a close connection between β-divergences and the family of Tweedie distributions where the relation was stated by Cichocki *et al.* [13] and elaborated in [15]. A

Table 1: Tweedie distributions with corresponding normalizing constants, divergence forms and maximum a-posteriori estimation of the dispersion parameters. Here N_ν denotes the number of elements in X_ν.

p	Divergence	Distribution	Normalizing Constant	Divergence Form	$\phi_\nu^\star$ (MAP)
$2-\beta$	β-divergence	Tweedie	$z_{p_\nu}(X_\nu, \phi_\nu)$	$d_{p_\nu}(X_\nu, \hat{X}_\nu)$	–
0	EUC	Gaussian	$(2\pi\phi_\nu)^{-1/2}$	$\frac{1}{2}\sum_{u_\nu}(X_\nu - \hat{X}_\nu)^2$	$\frac{d_0(X_\nu, \hat{X}_\nu) + \beta_\nu}{N_\nu/2 + \alpha_\nu + 1}$
1	KL	Poisson	$\frac{(X_\nu/\phi_\nu)^{X_\nu/\phi_\nu}}{e^{X_\nu/\phi_\nu}\Gamma(X_\nu/\phi_\nu + 1)}$	$\sum_{u_\nu} X_\nu \log\big(\frac{X_\nu}{\hat{X}_\nu}\big) - X_\nu + \hat{X}_\nu$	$\frac{d_1(X_\nu, \hat{X}_\nu) + \beta_\nu}{3N_\nu/2 + \alpha_\nu + 1}$
2	IS	Gamma	$\big(\Gamma(1/\phi_\nu)(e\phi_\nu)^{1/\phi_\nu} X_\nu\big)^{-1}$	$\sum_{u_\nu} \frac{X_\nu}{\hat{X}_\nu} - \log\big(\frac{X_\nu}{\hat{X}_\nu}\big) - 1$	$\frac{d_2(X_\nu, \hat{X}_\nu) + \beta_\nu}{N_\nu/2 + \alpha_\nu + 1}$
3	–	Inverse Gaussian	$(2\pi X_\nu^3 \phi_\nu)^{-1/2}$	$\frac{1}{2}\sum_{u_\nu} \frac{(X_\nu - \hat{X}_\nu)^2}{X_\nu \hat{X}_\nu^2}$	$\frac{d_3(X_\nu, \hat{X}_\nu) + \beta_\nu}{N_\nu/2 + \alpha_\nu + 1}$

Tweedie distribution is specified by a simple power relation between its mean and variance as: $Var\{x\} = \phi\hat{x}^p$, where $\hat{x}$ is the mean (also called expectation parameter), $p = 2 - \beta$ is the index parameter of the β-divergence and ϕ is the dispersion parameter. They also fully characterize the dispersion model and can be written in the following moment form

$$\mathbb{P}(x; \hat{x}, \phi, p) = z_p(x, \phi) \exp\big(-\frac{1}{\phi} d_p(x; \hat{x})\big) \qquad (13)$$

where $z_p(\cdot)$ is a suitable normalization constant that depends on the index parameter p. Since z_p does not depend on $\hat{x}$, for fixed p and ϕ, solving a maximum likelihood problem for $\hat{x}$ is indeed equivalent to minimization of the β-divergence. However, for fixed $\hat{x}$, solving a maximum likelihood problem for ϕ depends on $z_p(\cdot)$ that has a closed form characteristic function:

$$\phi^\star = \arg\max_\phi \ \log z_p(x, \phi) - \frac{1}{\phi} d_p(x; \hat{x}). \qquad (14)$$

In this study, we focus on learning the dispersion parameters ϕ_ν for integer values of $p_\nu = \{0, 1, 2, 3\}$. Other values of p_ν are possible but technically more difficult as the normalizing constants z_p are given as power series without an explicit analytical form [16]. From the probabilistic interpretation, maximization of likelihood with respect to the dispersion parameters ϕ_ν when the other parameters are given of a Tweedie model, provides a data driven formulation for choosing relative weights. In addition, it is easy to verify that when mean is fixed, the conjugate prior of the dispersion parameter is the inverse Gamma distribution. Hence, we assume an inverse Gamma prior on ϕ_ν: $\phi_\nu \sim \mathcal{IG}(\phi_\nu; \alpha_\nu, \beta_\nu)$. Surprisingly, none of the references we are aware of used this conjugate prior for the dispersion parameter.

Tweedie distribution generalizes the well-known distributions where the choices of $p_\nu = \{0, 1, 2, 3\}$ correspond to the Gaussian, Poisson, Gamma, and inverse Gaussian distributions, respectively. We estimate the optimal dispersion for these distributions by setting the derivative of the log-likelihood to zero and then solving it for ϕ_ν. Note that, since the normalizing constants of the Poisson and the Gamma distributions are problematic (see Table 1), we replace the terms involving $\log\Gamma(\cdot)$ functions with the so called Stirling's approximation: $\log\Gamma(n) \approx -\frac{1}{2}\log n + n\log n - n$. Finally, we obtain the optimal dispersion parameters in closed form as presented in Table 1.

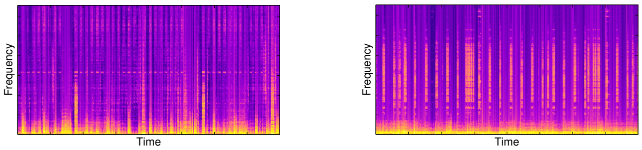
As a summary, at each iteration of the estimation algorithm, we first update all factors Z_α via Equation 11 and compute the mean parameters $\hat{X}_\nu$ via Equation 9. Then, for each observed tensor we compute the maximum a-posteriori (MAP) estimation of the dispersion parameters $\phi_\nu^\star$ as in Table 1. This coordinate ascent procedure is iterated until convergence.

Note that, the Poisson distribution in its well-known form, is an exponential dispersion model with $\phi_\nu = 1$. When we introduce a dispersion parameter, we re-define domain of the probability distribution on integer multiples of ϕ_ν: on a grid $\{0, \phi_\nu, 2\phi_\nu, 3\phi_\nu, ...\}$. Therefore, for the observations with $p_\nu = 1$, we should set $X_\nu \leftarrow X_\nu^{\text{org}}/\phi_\nu$ at the beginning of each iteration, where X_ν^{org} denotes the original observation.

5. EXPERIMENTS AND CONCLUSION

In this section, we illustrate our method on the drum source separation model defined in Section 2. Here, we conduct our experiments on a famous pop song 'Chasing Pavements' by Adele. Firstly, we estimate the factors without using the dispersion parameters (i.e. setting $\phi_{1:3} = \phi$) and then we also estimate the dispersion parameters by using the proposed method while making inference.

We compute the magnitude spectrum of a 20 second excerpt of the piece and obtain X_1. For X_2, we compute the spectra of the drum sounds that are obtained from the RWC Musical Instrument Sound Database. Finally, we compute X_3



(a) (b)

Fig. 2: Example results of the experiment. The figures illustrate the reconstructed audio spectra of the drum track X_d, computed via Wiener filtering: $X_d = X_1 \circ \frac{D(:,I_b+1:I)B(I_b+1:I,:)F}{\hat{X}_1}$. In (a) the dispersion parameters are set to the same value; they do not contribute to the estimation process. In (b) the dispersion parameters are also estimated by the proposed approach.

by using an approximate transcription of the drum track of the piece, which can be obtained from online MIDI databases. In our experiments we used $|i| = 813$ (the number of spectral templates): 13 templates for the drum part and 800 templates for the harmonic part, $|k| = 100$, and we set the divergence parameters as $p_1 = 2$, $p_2 = 2$, and $p_3 = 1$. The hyperparameters are selected as $\alpha_{1:3} = 100$, $\beta_1 = 9$, $\beta_2 = \beta_3 = 9900$. The sources are separated via Wiener filtering.

Figure 2 visualizes the drum source separation results for the particular experiment. It can be visually observed that estimating the dispersion parameters while estimating the other factors yields better results. We can see that, the high frequency components of the drum sounds cannot be recovered if the dispersion parameters are fixed to the same number. The resulting audio files and further experimental results on a tensor completion problem can be found in `http://www.cmpe.boun.edu.tr/~umut/eusipco2013`.

Conclusion: The dispersion parameters play a significant role on the performance of coupled matrix/tensor factorization models. In this study, we presented a method for estimating the dispersion parameters in coupled models. Our method follows a systematic approach, where we formulated this problem as a maximum likelihood estimation by making use of the relation between the β-divergence and the family of Tweedie distributions.

We applied our method on a coupled drum source separation model. We observed a certain improvement on results when the dispersion parameters are estimated while making inference on the coupled factorization model.

6. REFERENCES

[1] P. Comon and C. Jutten, *Handbook of Blind Source Separation: Independent Component Analysis and Applications*, Academic Press, 2010.

[2] N. Sturmel and L. Daudet, "Informed source separation using iterative reconstruction," *IEEE TASLP*, vol. 21, no. 1, pp. 176–183, 2013.

[3] U. Simsekli and A. T. Cemgil, "Score guided musical source separation using generalized coupled tensor factorization," in *EUSIPCO*, 2012.

[4] A. Ozerov, A. Liutkus, R. Badeau, and G. Richard, "Coding-based informed source separation: Nonnegative tensor factorization approach," Rapport de recherche, Sept. 2012.

[5] M. Kim, J. Yoo, K. Kang, and S. Choi, "Nonnegative matrix partial co-factorization for spectral and temporal drum source separation," *J. Sel. Topics Signal Processing*, vol. 5, no. 6, pp. 1192–1204, 2011.

[6] B. Ermis, E. Acar, and A. T. Cemgil, "Link prediction via generalized coupled tensor factorisation," in *ECML/PKDD Workshop on Collective Learning and Inference on Structured Data*, 2012 (arXiv:1208.6231).

[7] A. Banerjee, S. Basu, and S. Merugu, "Multi-way clustering on relation graphs," in *SDM*, 2007, pp. 145–156.

[8] E. Acar, T. G. Kolda, and D. M. Dunlavy, "All-at-once optimization for coupled matrix and tensor factorizations," in *KDD MLG*, 2011 (arXiv:1105.3422).

[9] T. Wilderjans, E. Ceulemans, and I. Van Mechelen, "Simultaneous analysis of coupled data blocks differing in size: A comparison of two weighting schemes," *Comput. Stat. Data. An.*, 2009.

[10] T.F. Wilderjans, E. Ceulemans, I. Van Mechelen, and R.A. van den Berg, "Simultaneous analysis of coupled data matrices subject to different amounts of noise.," *Br J Math Stat Psychol*, vol. 64, pp. 277–90, 2011.

[11] C. Févotte, N. Bertin, and J. L. Durrieu, "Nonnegative matrix factorization with the Itakura-Saito divergence. with application to music analysis," *Neural Computation*, vol. 21, pp. 793–830, 2009.

[12] Y. K. Yilmaz, *Generalized Tensor Factorization*, Ph.D. thesis, Bogazici University, 2012.

[13] A. Cichocki, R. Zdunek, A. H. Phan, and S. Amari, *Nonnegative Matrix and Tensor Factorization*, Wiley, 2009.

[14] B. Jørgensen, *The Theory of Dispersion Models*, Chapman & Hall/CRC Monographs on Statistics & Applied Probability, 1997.

[15] Y. K. Yılmaz and A. T. Cemgil, "Alpha/beta divergences and tweedie models," *arXiv:1209.4280 v1*, 2012.

[16] P. K. Dunn and G. S. Smyth, "Series evaluation of tweedie exponential dispersion model densities," *Stats. & Comp.*, vol. 15, pp. 267–280, 2005.