

THE INVERSE QRD SET-MEMBERSHIP RLS ALGORITHM

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ABSTRACT

This paper introduces a robust QR-decomposition version of the Set Membership RLS (SM-RLS) algorithm. The SM-RLS algorithm was recently proposed to improve the numerical stability of the so-called BEACON algorithm, while keeping the same performance. Our algorithm takes the numerical stability a step further, by considering the Cholesky factor of the input autocorrelation matrix and using it in the Givens rotation based update equations. The performance of the proposed algorithm in comparison to the SM-RLS algorithm is verified through computer simulation. The results show a superior performance of the proposed algorithm as compared to the original SM-RLS in finite precision environment, without any compromise on the convergence and tracking performance.

Index Terms— Inverse QRD-RLS, RLS, Set-Membership, BEACON

1. INTRODUCTION

Set-membership filtering (SMF), while keeping the same peak complexity of competing algorithms, reduces the average computational complexity of adaptive filters [1]. These algorithms are derived from a deterministic objective function with bounded error constraint, enforcing updates only from a feasible solution set. This results in data-selective updates of the adaptive filtering algorithm and therefore a computational complexity advantage over their conventional counterparts. A lower computational cost directly impacts the power consumption and therefore leads to “green” and environmentally friendly solutions. The sparse update also allows better tracking capability and enables efficient usage of shared resources in case of a multi-channel problem.

The SMF concept has been successfully employed to a number of traditional algorithms including those minimizing least-squares (LS) errors [2]. It is well-known that LS based algorithms (e.g. the recursive least-squares (RLS) algorithm) outperform MSE based algorithms (e.g. the least-mean

squares (LMS) algorithm) in terms of convergence speed and misadjustment. Therefore, applications requiring fast convergence would favor the use of RLS-type algorithms. However, the computational cost of the conventional RLS algorithm as well as its instability prohibit its utilization in a wide variety of applications. In [2], the combination of SMF and LS objective functions results in the BEACON (Bounding Ellipsoidal Adaptive CONstrained least-squares) algorithm; a computationally efficient version of the RLS algorithm that combines fast tracking and data selectiveness capabilities of SMF with fast convergence and low misadjustment of RLS. In some application scenarios, the BEACON (also referred to as *quasi*-optimal bounding ellipsoid or QOBE in [3]) algorithm utilizes only 5% updates to match the performance of a conventional RLS-type algorithm. These features make the BEACON algorithm an attractive choice for modern day wireless communication applications requiring high convergence speed, low cost, and good tracking capabilities.

Despite all the advantages of the BEACON algorithm, it turns out to be numerically unstable [4, 5]. The instability of RLS-type algorithms strongly depends on two factors: the condition number of the auto-correlation matrix and the tracking parameter (forgetting factor) of the algorithm [6]. The SM-RLS solution proposed in [4] and later renamed to modified *quasi*-optimal bounding ellipsoid or MQOBE¹ in [5], unlike the BEACON algorithm, bounds the time-varying forgetting factor of the algorithm which results in improved numerical stability. However, as will be seen in a later section, this algorithm is unable to address arithmetic stability issues when the auto-correlation matrix becomes ill-conditioned.

We solve this problem by proposing an algorithm which, using QR-decomposition [10], works on the square-root factor of the auto-correlation matrix. In this way, the arithmetic stability is ensured as the condition number of the squared-root is an order of magnitude lower than the actual auto-correlation matrix, while keeping the time-varying forgetting factor within a tight bound. An equalization scenario

¹Although keeping the original name, SM-RLS, the (time-varying forgetting factor exponentially weighted) MQOBE algorithm is not to be confused with the Set-Membership Weighted RLS (SM-WRLS) algorithm detailed in [7, 8]. Also note that a QR version of the SM-WRLS algorithm can be found in [9].

*J. A. Apolinário Jr. thanks the Brazilian agencies CNPq and CAPES for partial funding of this work.

for a wireless channel is used to compare our solution with the BEACON and the SM-RLS algorithms. The simulation results verify the improved stability of our method when compared to these algorithms.

The paper is organized as follows: in the next section, we review the basic concepts of Set-Membership filtering algorithms. The BEACON and the SM-RLS algorithms are detailed in Section 3. Section 4 shows the proposed algorithm while simulation results are presented in Section 5. Finally, conclusions are drawn in Section 6.

2. SET-MEMBERSHIP FILTERING

The set-membership approach solves for a weight vector \mathbf{w} (order N , having $N + 1$ coefficients) that belongs to a set of vectors satisfying an output error constraint defined as

$$|d - \mathbf{w}^T \mathbf{x}| = |e| \leq \bar{\gamma}, \quad (1)$$

where e is the error, d is the desired signal, \mathbf{x} is the input signal vector, and $\bar{\gamma}$ is the constraint. The set of all possible solutions within the set-membership approach is termed the *feasibility set*, defined as

$$\Theta = \bigcap_{(\mathbf{x}, d) \in \mathcal{S}} \{\mathbf{w} \in \mathbb{R}^{N+1} : |d - \mathbf{w}^T \mathbf{x}| \leq \bar{\gamma}\}, \quad (2)$$

where $\bar{\mathcal{S}}$, a data-space, is the set of all possible pairs (\mathbf{x}, d) . An iterative solution to the set-membership approach defines a *constraint set*, $\mathcal{H}(k)$, based on the k -th observed pairs, $(\mathbf{x}(k), d(k))$,

$$\mathcal{H}(k) = \{\mathbf{w} \in \mathbb{R}^{N+1} : |d(k) - \mathbf{w}^T \mathbf{x}(k)| \leq \bar{\gamma}\}. \quad (3)$$

The intersection of the constraint sets over all observations $0, \dots, k$, leads to the *exact membership set* defined as

$$\psi(k) = \bigcap_{i=0}^k \mathcal{H}(i). \quad (4)$$

It is important to note that, as k grows, $\psi(k)$ approaches the subset Θ [1]. The update of $\psi(k)$ is illustrated in Figure 1 which shows the reduction, typical in early iterations, due to the updating process.

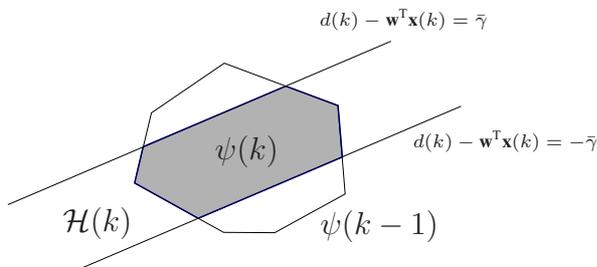


Fig. 1. Updating the exact membership set $\psi(k)$.

The approach in the adaptive algorithm is to find an analytically tractable outer bound for the set $\psi(k)$. The object bounded ellipsoid (OBE) algorithms use ellipsoids as outer bound [4]. In the next section we summarize two of such algorithms.

3. BEACON AND SM-RLS ALGORITHMS

The basic idea of OBE algorithms is to outer bound the membership set at each instant by a mathematically tractable ellipsoid $\varepsilon(k)$ defined as

$$\varepsilon(k) = \{\mathbf{w} \in \mathbb{R}^{N+1} : [\mathbf{w} - \mathbf{w}(k)]^T \mathbf{R}(k) [\mathbf{w} - \mathbf{w}(k)] \leq \sigma(k)\}, \quad (5)$$

where $\sigma(k) > 0$ and $\mathbf{R}(k)$ is the deterministic weighted auto-correlation matrix of the input signal.

3.1. The BEACON Algorithm

The BEACON [2], an OBE algorithm, minimizes the cost function $[\mathbf{w} - \mathbf{w}(k-1)]^T \mathbf{R}(k-1) [\mathbf{w} - \mathbf{w}(k-1)] - \sigma(k-1)$ subject to $|d(k) - \mathbf{w}^T \mathbf{x}(k)|^2 \leq \bar{\gamma}^2$, which, as pointed out in [4] and [11], is equivalent to minimizing the following objective function:

$$\xi(k) = \sum_{i=0}^k l(i) [d(i) - \mathbf{x}^T(i) \mathbf{w}]^2. \quad (6)$$

The resulting algorithm can be considered as a form of RLS with a time-varying forgetting factor $l(k)$ and with the auto-correlation matrix update equation given by

$$\mathbf{R}(k) = \mathbf{R}(k-1) + l(k) \mathbf{x}(k) \mathbf{x}^T(k). \quad (7)$$

We set $\mathbf{S}(k) = \mathbf{R}^{-1}(k)$ and summarize the rest of BEACON update equations in Algorithm 1.

```

S(0) =  $\delta \mathbf{I}$ ;    % small constant multiplying an identity matrix
w(0) = 0;
for  $k = 1, 2, \dots$  do
     $e(k) = d(k) - \mathbf{w}^T(k-1) \mathbf{x}(k)$ ;
    if  $|e(k)| > \bar{\gamma}$  then
         $\bar{\mathbf{x}}(k) = \mathbf{S}(k-1) \mathbf{x}(k)$ ;
         $\mathcal{G}(k) = \mathbf{x}^T(k) \bar{\mathbf{x}}(k)$ ;
         $l(k) = \frac{1}{\mathcal{G}(k)} \left( \frac{|e(k)|}{\bar{\gamma}} - 1 \right)$ ;
         $\mathbf{S}(k) = \mathbf{S}(k-1) - \frac{l(k) \bar{\mathbf{x}}(k) \bar{\mathbf{x}}^T(k)}{1 + l(k) \bar{\mathbf{x}}^T(k) \bar{\mathbf{x}}(k)}$ ;
         $\mathbf{w}(k) = \mathbf{w}(k-1) + l(k) e(k) \mathbf{S}(k) \mathbf{x}(k)$ ;
    else
         $\mathbf{S}(k) = \mathbf{S}(k-1)$ ;
         $\mathbf{w}(k) = \mathbf{w}(k-1)$ ;
    end
end

```

Algorithm 1: The BEACON algorithm [2].

The forgetting factor $l(k)$ and the norm of the input auto-correlation matrix $\mathbf{R}(k)$ in BEACON are tangled in a positive feedback, resulting in ever increasing values of both variables, causing overflow in finite-precision. Consequently, the continuous decrease in $\mathbf{S}(k)$ triggers underflow.

3.2. The SM-RLS Algorithm

In an attempt to solve this problem, [4] proposed the SM-RLS algorithm, i.e., the use of an exponentially weighted RLS with a time-varying forgetting factor (more similar to the RLS algorithm) instead of a variable weight as in the BEACON algorithm. The cost function of the SM-RLS algorithm is expressed as

$$\xi(k) = \sum_{i=0}^k [\lambda(i)]^{k-i} [d(i) - \mathbf{x}^T(i)\mathbf{w}(k)]^2. \quad (8)$$

The BEACON and the SM-RLS algorithms yield the same coefficients; yet, the SM-RLS algorithm has a negative feedback between the autocorrelation matrix and the forgetting factor which preserved its numerical (infinite precision) stability. The SM-RLS algorithm is shown in Algorithm 2.

```

S(0) =  $\delta\mathbf{I}$ ;
w(0) =  $\mathbf{0}$ ;
for  $k = 1, 2, \dots$  do
     $e(k) = d(k) - \mathbf{w}^T(k-1)\mathbf{x}(k)$ ;
    if  $|e(k)| > \bar{\gamma}$  then
         $\bar{\mathbf{x}}(k) = \mathbf{S}(k-1)\mathbf{x}(k)$ ;
         $\mathcal{G}(k) = \mathbf{x}^T(k)\bar{\mathbf{x}}(k)$ ;
         $\lambda(k) = \frac{\mathcal{G}(k)}{(\frac{|e(k)|}{\bar{\gamma}} - 1)}$ ;
         $\mathbf{S}(k) = \frac{1}{\lambda(k)} \left( \mathbf{S}(k-1) - \frac{\bar{\mathbf{x}}(k)\bar{\mathbf{x}}^T(k)}{\lambda(k) + \mathbf{x}^T(k)\bar{\mathbf{x}}(k)} \right)$ ;
         $\mathbf{w}(k) = \mathbf{w}(k-1) + e(k)\mathbf{S}(k)\mathbf{x}(k)$ ;
    else
         $\mathbf{S}(k) = \mathbf{S}(k-1)$ ;
         $\mathbf{w}(k) = \mathbf{w}(k-1)$ ;
    end
end

```

Algorithm 2: The SM-RLS algorithm [4].

4. THE INVERSE QRD SM-RLS ALGORITHM

Despite having a bounded forgetting factor, the SM-RLS algorithm was unable to solve the stability issue of the BEACON algorithm in the presence of an ill-conditioned auto-correlation matrix. This problem is solved here by factorizing the auto-correlation matrix, using QR decomposition, and updating the Cholesky factor (or its inverse) [10]. The key idea in the family of QRD-RLS adaptive algorithms is to use unitary rotation matrices (Givens matrices in this case) to

update the factorized part of auto-correlation matrix, i.e., the Cholesky factor matrix. These update equations are known to be numerically stable [10].

The QR-decomposition RLS (QRD-RLS) algorithm is the simplest form of QR-decomposition based algorithms that updates the Cholesky factor using Givens rotation matrix. However, in order to compute the weight vector, an extra backward substitution step is required. The inverse QRD-RLS (IQRD-RLS) algorithm avoids this step by directly updating the inverse of the Cholesky factor. Here we present the IQRD version of SM-RLS algorithm. The Cholesky factor matrix $\mathbf{U}(k)$ is defined in,

$$\mathbf{U}^T(k)\mathbf{U}(k) = \mathbf{X}^T(k)\mathbf{Q}^T(k)\mathbf{Q}(k)\mathbf{X}(k) = \mathbf{R}(k), \quad (9)$$

where $\mathbf{U}(k) \in \mathbb{R}^{(N+1) \times (N+1)}$ is the Cholesky factor matrix, $\mathbf{X}(k) \in \mathbb{R}^{(k+1) \times (N+1)}$ is the input data matrix, $\mathbf{Q}(k) \in \mathbb{R}^{(k+1) \times (k+1)}$ is the orthogonal Givens rotation matrix, and $\mathbf{R}(k) \in \mathbb{R}^{(N+1) \times (N+1)}$ is the deterministic auto-correlation matrix. Using (9), the variable $\mathcal{G}(k)$ in Algorithm 2 is redefined as [11]:

$$\mathcal{G}(k) = \mathbf{x}^T(k) \underbrace{\mathbf{U}^{-1}(k-1)\mathbf{U}^{-T}(k-1)}_{\mathbf{S}(k-1)} \mathbf{x}(k) = \bar{\mathbf{a}}^T(k)\bar{\mathbf{a}}(k). \quad (10)$$

The proposed algorithm gets rid of the numerical stability issue by replacing the update equation for $\mathbf{S}(k)$ with that of $\mathbf{U}^{-1}(k)$. The performance of the resulting algorithm is exactly the same as SM-RLS with improved arithmetic stability. All equations are summarized in Algorithm 3.

```

U(0) = lower inferior triangular matrix;
w(0) =  $\mathbf{0}$ ;
for  $k = 1, 2, \dots$  do
     $e(k) = d(k) - \mathbf{w}^T(k-1)\mathbf{x}(k)$ ;
    if  $|e(k)| > \bar{\gamma}$  then
         $\bar{\mathbf{a}}(k) = \mathbf{U}^{-T}(k-1)\mathbf{x}(k)$ ;
         $\mathcal{G}(k) = \bar{\mathbf{a}}^T(k)\bar{\mathbf{a}}(k)$ ;
         $\lambda(k) = \frac{\mathcal{G}(k)}{(\frac{|e(k)|}{\bar{\gamma}} - 1)}$ ;
         $\mathbf{a}(k) = \frac{\bar{\mathbf{a}}(k)}{\sqrt{\lambda(k)}}$ ;
         $\begin{bmatrix} \gamma^{-1}(k) \\ \mathbf{0} \end{bmatrix} = \mathbf{Q}_\theta(k) \begin{bmatrix} 1 \\ -\mathbf{a}(k) \end{bmatrix}$ ;
         $\begin{bmatrix} \mathbf{u}^T(k) \\ \mathbf{U}^{-T}(k) \end{bmatrix} = \mathbf{Q}_\theta(k) \begin{bmatrix} \mathbf{0}^T \\ \mathbf{U}^{-T}(k-1) \\ \sqrt{\lambda(k)} \end{bmatrix}$ ;
         $\mathbf{w}(k) = \mathbf{w}(k-1) - e(k)\gamma(k)\mathbf{u}(k)$ ;
    else
         $\mathbf{U}^{-T}(k) = \mathbf{U}^{-T}(k-1)$ ;
         $\mathbf{w}(k) = \mathbf{w}(k-1)$ ;
    end
end

```

Algorithm 3: The proposed IQRD-SM-RLS algorithm.

5. SIMULATION

A finite precision simulation of an adaptive equalizer is carried out to compare the arithmetic stability of the proposed method with that of SM-RLS, BEACON and RLS algorithms at different word lengths, i.e., 64 (assumed *infinite precision*), 16, and 8 bits.

5.1. Setup

An adaptive equalizer scenario as illustrated in Figure 2 is considered for the simulations. The signal-to-noise ratio of the received signal is $40dB$. Abrupt changes in the channel taps are introduced at specified time intervals to verify tracking properties and numerical stability of all algorithms under investigation. The input training sequence is a discrete random signal constructed from the set $(-1, 1)$.

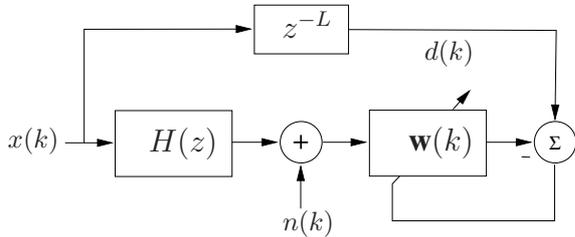


Fig. 2. Simulation scenario (adaptive equalizer with $L = 5$).

5.2. Simulation with 64-bit word length

Learning-curves of all the algorithms are compared in Figure 3. The result shows that the learning curves of algorithms BEACON, SM-RLS and IQRD-SM-RLS match each other, which leads to the conclusion that in infinite precision these algorithms are equivalent in the MSE sense. The learning curve for the RLS algorithm shows slight deviation from those of others.

The weighting factor $l(k)$ for BEACON and the forgetting factor $\lambda(k)$ for SM-RLS and IQRD-SM-RLS are compared in Figures 4. It shows the equivalence of the forgetting factor for both SM-RLS and IQRD-SM-RLS algorithms. On the other hand, the value of $l(k)$ in the BEACON algorithm is ever increasing. In the following, Figure 5 shows the condition number (eigenvalue spread) of the auto-correlation matrices at each iteration. For the IQRD-SM-RLS algorithm, the condition number is computed by reconstructing the inverse auto-correlation matrix ($\mathbf{S} = \mathbf{U}^{-1}\mathbf{U}^{-T}$) and then taking its condition number, as the auto-correlation matrix is a Hermitian matrix. Figure 5 shows that within infinite precision, all three algorithms yield the same condition number for the auto-correlation matrix.

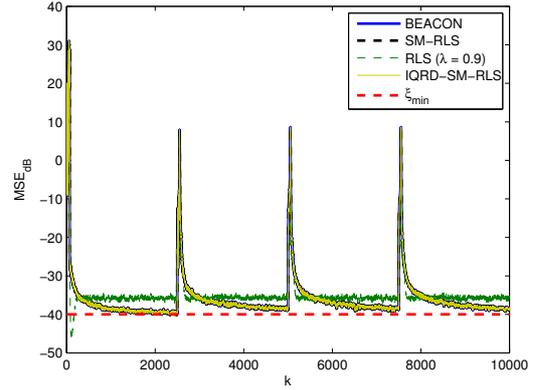


Fig. 3. Learning curves in infinite precision environment.

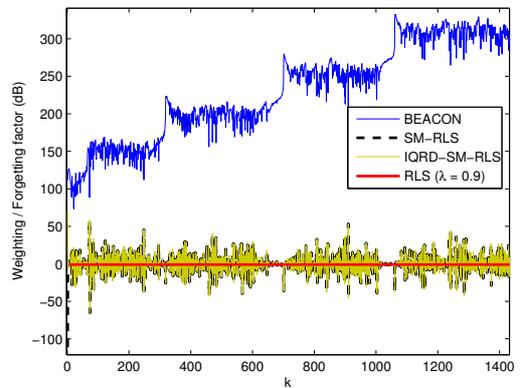


Fig. 4. Weighting / forgetting factor in infinite precision.

5.3. Finite Precision Simulation

Figures 6 and 7 show implementations with 16 and 8 bits of floating-point precision, respectively. We observe divergence of the BEACON and the SM-RLS algorithms in both implementations. However, the IQRD-SM-RLS algorithm remains stable in these scenarios. Figure 8 shows the condition number evolution in the 8-bit implementation. We can observe that the IQRD-SM-RLS algorithm, by updating the inverse Cholesky factor, resulted in having a condition number orders of magnitude smaller than those of the other algorithms, retaining arithmetic stability even with an ill-conditioned auto-correlation matrix, confirming the well behavior of the IQRD-SM-RLS algorithm in finite precision environment.

6. CONCLUSION

This paper proposed an inverse QRD-based version of the SM-RLS algorithm to overcome the numerical stability issues in both BEACON and SM-RLS algorithms. The performance and stability of the proposed algorithm was compared with algorithms BEACON and SM-RLS by simulating

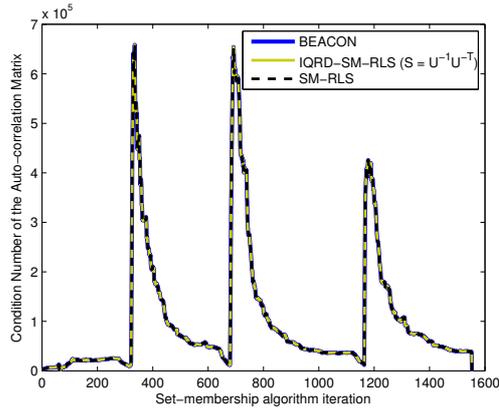


Fig. 5. Condition number in infinite precision.

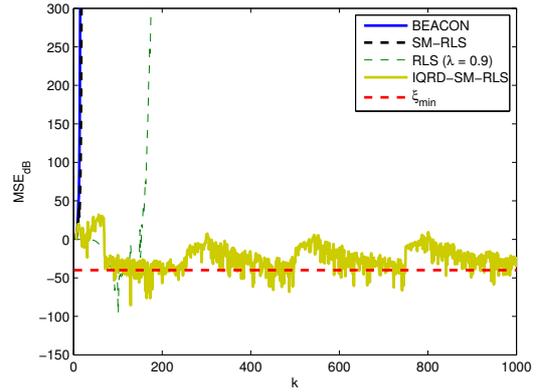


Fig. 7. Learning curves for the 8-bit implementation.

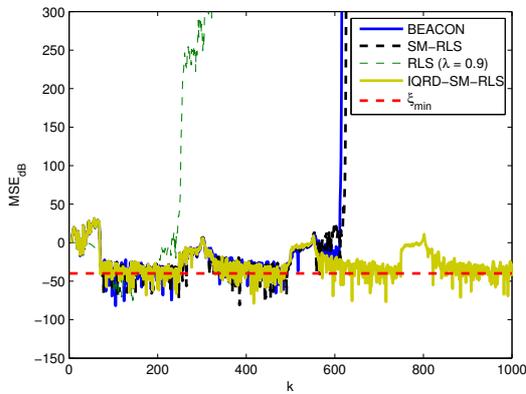


Fig. 6. Learning curves for the 16-bit implementation.

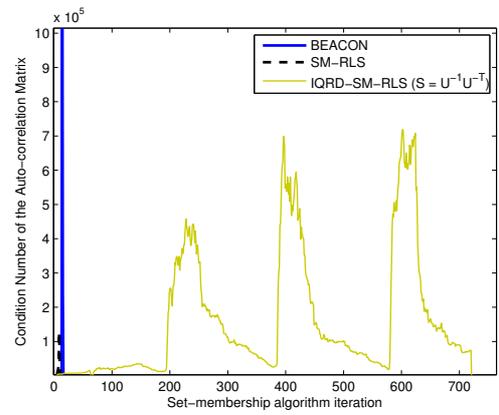


Fig. 8. Condition number for the 8-bit implementation.

an adaptive equalizer in finite precision environment. The results show the superiority of the proposed algorithm over algorithms BEACON and SM-RLS in terms of arithmetic stability. Furthermore, the proposed method does not demonstrate any performance loss in terms of misadjustment and convergence speed.

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