TIME-FREQUENCY REPRESENTATIONS BASED ON COMPRESSIVE SAMPLES

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ABSTRACT

A recent rise of compressive sensing (CS) algorithms has prompted many questions about the analysis of such sensed signals. Specifically, calculating a time-frequency representation (TFR) of these signals is an open question. In this paper, we propose an approach for calculating TFRs of compressed sensed signals based on recently proposed CS algorithm using modulated discrete prolate spheroidal sequences (MDPSS). The results of our numerical analysis show that a visually reliable TFR of compressed sensed signals can be obtained using the proposed approach. Furthermore, these compressed sensed signals can also be used for accurate estimation of signal parameters such as the instantaneous frequency.

Index Terms— Sparse signals, compressive sensing, time-frequency analysis.

1. INTRODUCTION

A widespread use of monitoring devices in recent years has prompted severe constraints on data acquisition and processing systems. To alleviate this problem, CS was proposed as a potential approach [1]. CS enables one to acquire the data at sub-Nyquist rates, and recover it accurately from such sparse samples [1], [2], [3]. However, an immediate question is how can we process such data. Specifically, how can we obtain a TFR of a signal? One way would be to recover a signal from its samples and calculate TFR based on the reconstructed signal, but this would defeat the purpose of CS. In their seminal work which married the idea of TFRs and compressive sensing [4], Flandrin and Borgnat proposed to acquire compressed samples in the ambiguity domain which requires calculating the ambiguity function of the signal. To avoid these additional computational steps, one should consider calculating TFRs based on the actual compressed samples.

In this paper, we propose to calculate TFR of a signal based on compressed samples obtained through a recently proposed CS approach [5]. The CS approach is based on a dictionary of MDPSS [6], since these bases are more suitable for CS than discrete prolate sequences (DPSS) [5]. MDPSS are obtained by modulation and variation of the bandwidth of DPSS to reflect the varying time-frequency nature of many real-life signals (e.g., dual-axis swallowing accelerometry signals considered in [5]). The results of the numerical analysis show that the proposed algorithm can be used to obtain a reliable TFR of compressed sensed signals.

2. PROPOSED SCHEME

CS, as a transform coding method, converts input signals from a high-dimensional space into signals that lie in a space of significantly smaller dimensions (e.g., wavelet transforms) [1], [7]. These CS approaches are well suited for signals that can be represented with significant K coefficients and N-dimensional basis. This is accomplished by computing a measurement vector y that consists of $M << N$ linear projections of the vector x:

$$y = \Phi x \tag{1}$$

where Φ represents an $M \times N$ matrix and is often refer to as the sensing matrix [1].

Given the CS framework, the immediate question is how to define the sensing matrix Φ. In [5], [8], MDPSS were proposed as suitable bases for CS. In particular, given N such that $n = 0, 1, ..., N-1$, MDPSS are defined as

$$\Psi_k(N, W, \omega_m; n) = \exp(j\omega_m n) v_k(N, W; n) \tag{2}$$

where $\omega_m = 2\pi f_m$ is a modulating frequency, W is the normalized half-bandwidth ($0 < W < 0.5$) and $v_k(n, N, W)$, is defined as the real solution to the system of equations [9]:

$$\sum_{m=0}^{N-1} \frac{\sin[2\pi W(n-m)]}{\pi(n-m)} v_k(m, N, W) = \lambda_k(N, W) v_k(n, N, W) \tag{3}$$

with $k = 0, 1, ..., N-1$ and $\lambda_k(N, W)$ being the ordered non-zero eigenvalues of (3):

$$\lambda_0(N, W) > \lambda_1(N, W), ..., \lambda_{N-1}(N, W) > 0. \tag{4}$$

MDPSS are doubly orthogonal and are bandlimited to the frequency band $[-W + \omega_m : W + \omega_m]$ [6]. To choose the modulating frequency, it is assumed that the spectrum $S(\omega)$ of the signal is confined to a known band $[\omega_1; \omega_2]$ [6], [5]. Then the modulating frequency is given by

$$\omega_m = (\omega_1 + \omega_2)/2 \tag{5}$$

and the bandwidth of the DPSSs is given by

$$W = |(\omega_2 - \omega_1)/2| \tag{6}$$

as long as both satisfy:

$$|\omega_m| + W < 0.5. \tag{7}$$

The next step is to formalize how to recover the samples using the sensing matrix. In other words, we seek a solution to the problem

$$\min ||x||_0 \text{ subject to } \|y - \Phi x\|_2 < \eta \tag{8}$$

where η is the expected noise of measurements and $\|\bullet\|_2$ is the Euclidian norm. Unfortunately, this minimization is NP-hard and not suitable for many applications. To avoid the computational burden, matching pursuit (MP) (e.g., [10]) can be used to avoid some of the computational burden associated with the CS. MP is a greedy, sparse function approximation scheme with the squared error loss, which iteratively adds new functions (i.e. basis functions) to the linear expansion. In other words, it decomposes a signal into a linear expansion of waveforms that are selected from a redundant dictionary of functions [10]. If the dictionary is orthogonal, MP achieves perfect reconstruction. Otherwise, if it is desired to achieve compact representation of the signal with MP, the members of the dictionary should mimic the signal structure and behavior. MP is computationally more efficient than a basis pursuit, since the basis pursuit minimizes a global cost function over all members of the dictionary [10].

To obtain TFR of a compressed sensed signal, the proposed algorithm starts with initial approximation for the signal, $\widehat{x}$, and the residual, R:

$$\widehat{x}^{(0)}(m) = 0 \tag{9}$$

$$R^{(0)}(m) = x(m) \tag{10}$$

where m represent the M uniformly or non-uniformly distributed time indices (i.e., the compressed samples). Then, MP calculates the sparse approximation of a signal by reducing the norm of the residue, $R = \widehat{x} - x$. In other words, at step k, MP identifies the member of the dictionary that best approximates the residual. Secondly, MP adds a scalar multiple of the member of the dictionary to the current approximation:

$$\widehat{x}^{(k)}(m) = \widehat{x}^{(k-1)}(m) + \alpha_k \phi_k(m) \tag{11}$$

$$R^{(k)}(m) = x(m) - \widehat{x}^{(k)}(m) \tag{12}$$

where $\alpha_k = \langle R^{(k-1)}(m), \phi_k(m)\rangle / \|\phi_k(m)\|^2$. The process continues until we reach a stoping criterion. The stoping criterion can be based on the idea that the normalized mean square error should be below a certain threshold value or it can be based on the number of bases used for approximation should be below a certain number.

Using the signal approximation with L bases obtained by MP, we can obtain a TFR of the signal as follows:

$$\mathcal{TF}\{x(n)\} = \sum_{l=1}^{L} \langle x(m), \phi_l(m)\rangle \mathcal{TF}\{\phi_l(n)\} \tag{13}$$

where ϕ_l are L bases from the dictionary with the strongest contributions, and $\mathcal{TF}\{\}$ is a time-frequency operator (e.g., short-time Fourier transform) [11], [12].

3. RESULTS AND DISCUSSION

To calculate TFRs of sample signals, we use spectrogram with the Gaussian window with $\sigma = 0.02$. For compressive sensing, we used a 25-band MDPSS-based dictionary with the normalized half-bandwidth equal to $W = 0.495$ and $N = 256$. The stopping criterion used in all four examples was that the normalized mean square error has to be less than 10^{-20}.

Let's begin the analysis of the proposed algorithm by examining the spectrogram of the following signal:

$$x_1(t) = [e^{-60(t-0.25)^2} + e^{-60(t-0.75)^2}] \cos(110\pi t) + e^{-60(t-0.5)^2}[\cos(180\pi t) + \cos(40\pi t)] \tag{14}$$

where $0 < t < 1$ and the assumed sampling rate is $T_s = 1/256$ seconds. The signal consists of four short transients in different frequency and time bands. Figure 1 (a) depicts the spectrogram of the original signal, while Figures 1 (b)-(c) depict the signal acquired compressively with only 60% of samples used. These samples were acquired either using uniform sampled times or nonuniform sampled times. Nonuniform sampled times were obtained using a standard uniform distribution on the open interval $(0, 1)$. When comparing these TFRs to the TFR of the original signal shown in Figure 1(a) (i.e., the signal sampled using traditional schemes), no differences can be observed. Next, we calculated the mean square error denoting the normalized difference between the TFR based on compressive samples and the TFR based on the original signal. The presented results were obtained using 1000 realizations. Clearly, as we increased the number of samples, the error became smaller regardless whether uniform or nonuniform sampling was used.

The second investigation is carried for frequency-modulated (FM) signals. In particular, we consider a sinusoidally-modulated signal defined as:

$$x_2(t) = \sin(110\pi t + 2\pi \cos(6\pi t)) \tag{15}$$

where $0 < t < 1$ and the assumed sampling rate is $T_s = 1/256$ seconds. In this case, 80 % of samples are used for

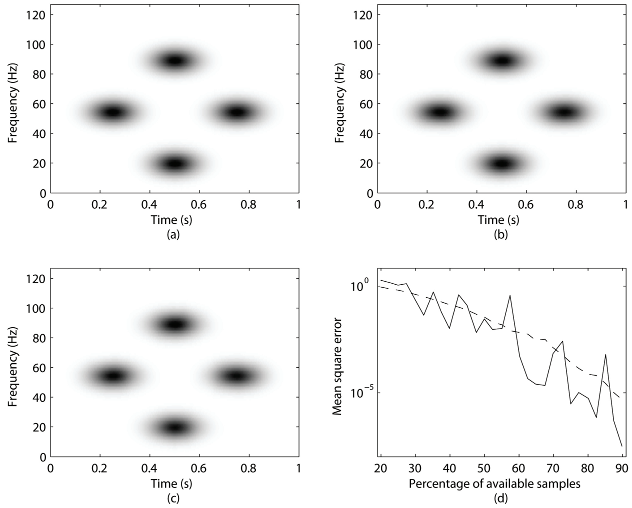

Fig. 1. Spectrograms of: (a) the original signal; (b) the signal based on uniform samples; (c) the signal based on nonuniform samples. Mean square error obtained when using various percentage of samples is shown in (d), with the solid line indicating the error obtained with uniform samples, and the dashed line indicating the error obtained with nonuniform samples.

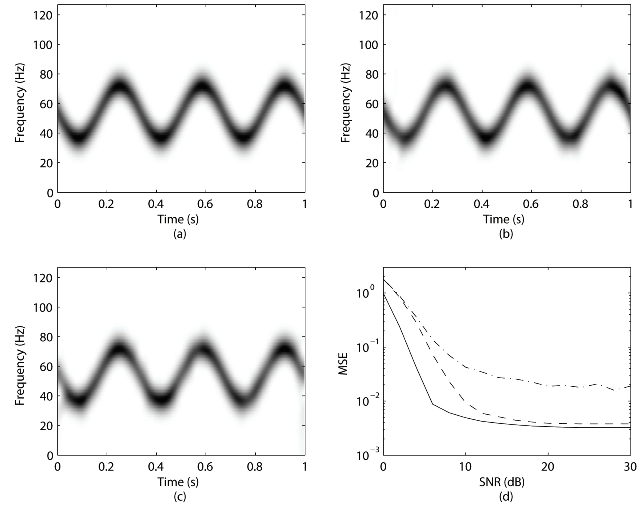

Fig. 2. Spectrograms of: (a) the original signal; (b) the signal based on equal distance samples; (c) the signal based on irregular samples. The performance of the IF estimator is shown in (d) based on the original signal (solid line), equal-distance compressed samples (dashed line) and nonuniform compressed samples (dash-dotted line).

compressive sensing, and the results of the analysis are shown in Figures 2(a)-(c). As in the previous case, there were no differences in TFRs of traditionally-sampled and compressively-sampled versions of the signal. However, here we had to use a greater number of samples in comparison to the first example, due to a wideband nature of the signal. This issue can be alleviated by using modulation techniques that are more suitable for these wideband signals.

The results in Figure 2(d) depict the behavior of the instantaneous frequency (IF) estimator based on the traditional TFR and TFRs obtained from compressed samples. The signal was contaminated with an additive white Gaussian noise, and its variance was proportional to the considered signal-to-noise ratios (SNR). The results shown in Figure 2(d) were obtained using 1000 realizations. As expected, the traditional TFR obtained the lowest values of mean square errors (MSE), but TFRs based on compressive samples closely follow the trend of the traditional TFR, especially the TFR based on uniformly compressed samples. These results clearly demonstrate that even based on compressive samples, we can achieve a reliable estimate of IF. Further studies are needed to understand the effects of the number of acquired samples on MSE.

To obtain further compression, the bandwidth of MDPSS sequences and partitioning approaches need to be optimized. Here, we consider two ideal cases. First, we consider a signal consisting of a single basis function from the current dictionary. The signal is depicted in Figure 3(a). The reconstructed spectrograms using uniform sample spacing and non-uniform sample spacing are shown in Figures 3(c) and (d), respectively. In both cases, only 5 samples were need to exactly recover the signal, which represents less than 2% of the total number of samples.

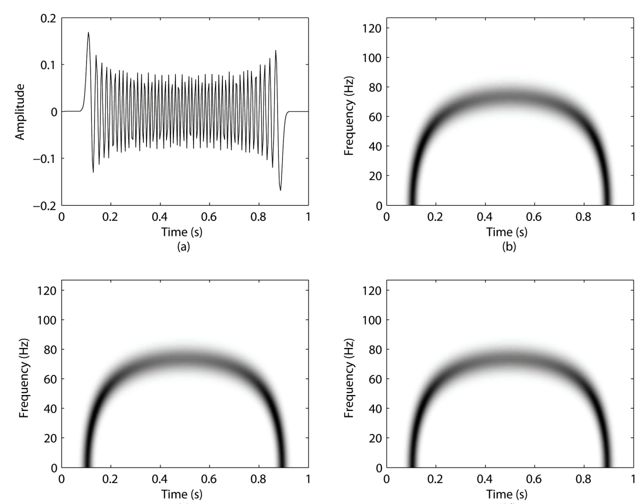

Fig. 3. The time domain representation of the consider signal is shown in (a). Spectrograms of: (b) the original signal; (c) the signal based on equal distance samples; (d) the signal based on irregular samples.

Similar results are obtained when considering a signal consisting of three basis functions from the dictionary as shown in Figure 4(a). For both uniform and non-uniform sampling, only 46 samples were needed to recover the signal exactly. This represents less than 18% of the total number of samples. The obtained spectrograms were recovered exactly as shown Figures 4(c) and (d). In comparison to the signal shown in Figure 3(a), the current signal has more complex time-frequency patterns involved. Hence, a greater number of samples is needed, even though each signal component is a member of the dictionary.

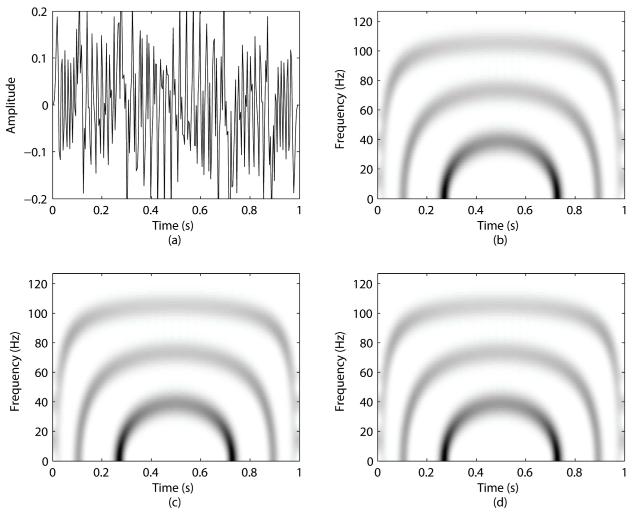

Fig. 4. The time domain representation of the consider signal is shown in (a). Spectrograms of: (b) the original signal; (c) the signal based on equal distance samples; (d) the signal based on irregular samples.

These two ideal cases show that higher accuracies can be obtained for $x_1(t)$ and $x_2(t)$, as long as the dictionaries used for compressive sampling are well suited for the considered signals. This is well known shortcoming of any dictionary-based approaches, and various schemes can be devised to achieve well-suited partitioning of the time-frequency plane. However, such investigations are beyond the scope of the current manuscript.

4. CONCLUSION

A novel approach for obtaining TFRs of compressively sampled signals was proposed in this paper. In particular, we proposed a scheme that used MDPSS. We compared TFRs obtained from traditionally sampled signals and compressively sampled signal. There were no visible differences between these representations. Furthermore, we demonstrated that an IF estimator based on the TFRs from the compressive samples produces reliable results.

5. REFERENCES

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