

STATISTICAL MECHANICS APPROACH TO SPARSE NOISE DENOISING

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ABSTRACT

Reconstruction fidelity of sparse signals contaminated by sparse noise is considered. Statistical mechanics inspired tools are used to show that the ℓ_1 -norm based convex optimization algorithm exhibits a phase transition between the possibility of perfect and imperfect reconstruction. Conditions characterizing this threshold are derived and the mean square error of the estimate is obtained for the case when perfect reconstruction is not possible. Detailed calculations are provided to expose the mathematical tools to a wide audience.

Index Terms— sparse signals and noise, replica method, statistical mechanical analysis

1. INTRODUCTION

Sparse signal estimation for linear underdetermined systems has attracted wide interest in signal processing community during the recent years. This is not surprising since the general class of sparse problems is encountered in many applications, such as, linear regression [1], multimedia [2, 3], and compressive sampling (CS) [4, 5], to name just a few.

The present paper considers a CS setup where the sparse vector $\mathbf{x} \in \mathbb{R}^N$ is observed via noisy linear measurements

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w}, \quad (1)$$

where $\mathbf{A} \in \mathbb{R}^{M \times N}$ represents the compressive ($M < N$) sampling system and $\mathbf{y} \in \mathbb{R}^M$ is the observed vector. The measurement errors are captured by the additive noise vector $\mathbf{w} \in \mathbb{R}^M$. The task is to reconstruct \mathbf{x} from \mathbf{y} , given \mathbf{A} , but without detailed information about the statistics of \mathbf{x} and \mathbf{w} .

A prominent approach for finding a sparse solution to (1) is by solving a (convex) optimization problem of the form

$$\hat{\mathbf{x}}_\lambda = \arg \min_{\mathbf{x} \in \mathbb{R}^N} \{C_{\mathbf{y}, \mathbf{A}}(\mathbf{x}) + \lambda \|\mathbf{x}\|_1\}, \quad (2)$$

where $\|\mathbf{x}\|_1 = \sum_n |x_n|$. The cost function $C_{\mathbf{y}, \mathbf{A}}(\mathbf{x}) \geq 0$, that may depend on the realizations of \mathbf{y} and \mathbf{A} , is typically chosen so that (2) can be obtained using convex optimization tools like CVX [6]. In addition to the choice of $C_{\mathbf{y}, \mathbf{A}}(\mathbf{x})$, the solution also depends on the regularization parameter λ . In general, finding the optimal value of λ is not a trivial task.

For the case of (dense) Gaussian noise, the standard approach is to set $C_{\mathbf{y}, \mathbf{A}}(\mathbf{x}) = \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2$, reducing (2) to the

so-called LASSO estimator [7]. For non-zero noise variance, the solution obtained through LASSO is not exact, but if the noise has some structure, like sparsity, perfect reconstruction may again be feasible [8]. Some applications where sparse noise can be encountered are: impulsive noise [9], salt-and-pepper noise in an image, a sensor scenario where few measurements are corrupted but the other ones are good [10], and dictionary learning with sparse noise [11].

Let us consider a setup similar to [8], where both \mathbf{x} and \mathbf{w} are sparse, and the cost function is chosen as

$$C_{\mathbf{y}, \mathbf{A}}(\mathbf{x}) = \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_1, \quad (3)$$

to guarantee that (2) is a convex optimization problem. Then, we ask the following questions:

1. Given that the signal and noise are sparse and convex optimization based on (2) and (3) is used for reconstruction, what compression ratios $\alpha = M/N$ allow a perfect reconstruction of \mathbf{x} ?
2. What is the mean square error (MSE) of the sparse estimate of \mathbf{x} outside of this region?

We answer these questions in the large system limit (LSL) and report the sharp threshold for α that separates the two phases of reconstruction fidelity. The key technique is the replica method¹ developed in equilibrium statistical mechanics, where it is used to study large-scale behavior of disordered physical systems, such as, spin glasses. It has also been used in information theory [12–15] and CS [16–19], where quantities like mutual information and MSE play the role of thermodynamic variables.

2. PROBLEM FORMULATION AND METHODS

Consider the set of noisy measurements (1) and assume that both the signal and the noise are sparse random vectors (RVs). Let us define a parametrized mixture distribution

$$p(z; \rho, \sigma^2) = (1 - \rho)\delta(z) + \rho g_z(0, \sigma^2), \quad (4)$$

where $g_z(\mu, \sigma^2) = e^{-(z-\mu)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$, and $\delta(z)$ is the Dirac delta function. Let the elements of \mathbf{x} (resp. \mathbf{w}) be

¹Drawback of the replica method is that some of its steps are still lacking formal proof. Hence, it can be considered to be at most a “semi-rigorous” analytical tool. It is, however, routinely used in equilibrium statistical mechanics and its predictions are often verified by experiments.

independently and identically distributed (IID) according to $p(\mathbf{x}; \rho_x, \sigma_x^2)$ (resp. $p(w; \rho_w, \sigma_w^2)$). The $\rho \in [0, 1]$ in (4) is the fraction of non-zero elements in the vector and σ^2 is their variance. The measurement process is taken to be random so that the elements of \mathbf{A} are IID with density $g_A(0, 1/N)$. The ratio between the number of observables and the unknown parameters is denoted $\alpha = M/N$. To use statistical mechanics tools, we next write the problem in a probabilistic framework.

Let us consider the optimization problem (2) with the ℓ_1 -cost (3). Assume the system is in the LSL $M, N \rightarrow \infty$, where the compression ratio $\alpha = M/N$ and the density of the signal and noise ρ_x, ρ_w remain as finite constants. Let the postulated prior of \mathbf{x} be proportional to the Laplace distribution, namely, $q_{\beta, \lambda}(\mathbf{x}) \propto e^{-\beta \lambda \|\mathbf{x}\|_1}$, where $\beta \geq 0$. The postulated distribution of the measurement process has the same form, that is, $q_{\beta}(\mathbf{y} | \mathbf{A}, \mathbf{x}) \propto e^{-\beta \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_1}$, and the (mismatched) conditional mean estimator of \mathbf{x} reads by definition

$$\langle \mathbf{x}; \lambda \rangle_{\beta} = \frac{1}{Z_{\beta}(\mathbf{y}, \mathbf{A}; \lambda)} \int \mathbf{x} q_{\beta}(\mathbf{y} | \mathbf{A}, \mathbf{x}) q_{\beta, \lambda}(\mathbf{x}) d\mathbf{x}, \quad (5)$$

where $Z_{\beta}(\mathbf{y}, \mathbf{A}; \lambda) = \int e^{-\beta(\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_1 + \lambda \|\mathbf{x}\|_1)} d\mathbf{x}$. Then, the zero temperature estimate $\hat{\mathbf{x}}_{\lambda} = \langle \mathbf{x}; \lambda \rangle_{\beta \rightarrow \infty}$, is the solution to the original optimization problem defined by (2) and (3).

2.1. Replica Method

The key for finding the statistical properties of the reconstruction (5) is the normalization factor or *partition function* $Z_{\beta}(\mathbf{y}, \mathbf{A}; \lambda)$. Based on the statistical mechanics approach, our goal is to assess the *free energy* $f_{\beta}(\mathbf{y}, \mathbf{A}; \lambda) = -\frac{1}{\beta N} \ln Z_{\beta}(\mathbf{y}, \mathbf{A}; \lambda)$, when $N \rightarrow \infty$ and obtain the desired statistical properties from it. This is, however, difficult since f_{β} depends on the observations and the measurement process. If the averaged quantity $f_{\beta}(\lambda) = \mathbb{E} f_{\beta}(\mathbf{y}, \mathbf{A}; \lambda)$ is considered instead, a new problem arises in assessing the expectation over logarithm. We may reformulate the problem by writing

$$f(\lambda) = - \lim_{\beta, N \rightarrow \infty} \frac{1}{\beta N} \lim_{u \rightarrow 0^+} \frac{\partial}{\partial u} \ln \mathbb{E} \{ [Z_{\beta}(\mathbf{y}, \mathbf{A}; \lambda)]^u \}, \quad (6)$$

and remark that so-far the development has been rigorous. Unfortunately, obtaining an expression for (6) is still difficult so we resort to the replica trick in order to proceed.

Replica trick. Consider the free energy in (6). Assume that the limits commute, which in conjunction with the expression

$$[Z_{\beta}(\mathbf{y}, \mathbf{A}; \lambda)]^u = \int \prod_{a=1}^u e^{-\beta(\|\mathbf{y} - \mathbf{A}\mathbf{x}^a\|_1 + \lambda \|\mathbf{x}^a\|_1)} d\mathbf{x}^a \quad (7)$$

for $u = 1, 2, \dots$ allows the evaluation of the expectation in (6) as a function of $u \in \mathbb{R}$. The functional expression is utilized in taking the limit of $u \rightarrow 0^+$.

The assumption that the variable u (number of replicas) can be first treated as a non-negative integer and then extended to the set of real numbers has no rigorous mathematical proof in general. The predictions of the replica method, however, tend to be accurate when compared to experiments.

The general scheme of the following analysis consists of first assessing (6) using the replica trick and then identify the parameters that describe the MSE of the reconstruction. Finally, requiring that the MSE vanishes provides the threshold for perfect recovery. The next section reports the outcomes of the analysis and Section 4 contains the derivations.

3. RESULTS AND DISCUSSION

Let Q denote the standard Q-function and define

$$s(x) = \frac{1}{x^2} [1 - 2Q(x)] - \sqrt{\frac{2}{\pi x^2}} e^{-\frac{x^2}{2}}, \quad (8)$$

$$r_{\lambda}(h) = \lambda \sqrt{\frac{h}{2\pi}} e^{-\frac{\lambda^2}{2h}} - (\lambda^2 + h) Q\left(\frac{\lambda}{\sqrt{h}}\right). \quad (9)$$

Then, under the (technical) assumption of *replica symmetric ansatz* (see Section 4 for definition and [12–14] for further discussion), the following results are obtained.

Proposition 1. Fix $\lambda, \alpha, \rho_x, \rho_w$ and let the variances σ_x^2 and σ_w^2 be finite and non-zero. Then, the critical threshold for the perfect reconstruction, $\text{mse} \rightarrow 0$, is given by the solution of

$$\kappa = \frac{\rho_x(\lambda^2 + \hat{\chi}) - 2(1 - \rho_x)r_{\lambda}(\hat{\chi})}{[2(1 - \rho_x)Q(\lambda/\sqrt{\hat{\chi}}) + \rho_x]^2}, \quad (10)$$

$$\hat{\chi} = \alpha(1 - \rho_w) \left\{ \kappa \left[1 - 2Q\left(\frac{1}{\sqrt{\kappa}}\right) \right] - \sqrt{\frac{2\kappa}{\pi}} e^{-1/(2\kappa)} + 2Q\left(\frac{1}{\sqrt{\kappa}}\right) \right\} + \alpha\rho_w, \quad (11)$$

that satisfies the condition

$$\alpha(1 - \rho_w) \left[1 - 2Q\left(\frac{1}{\sqrt{\kappa}}\right) \right] = (1 - \rho_x)2Q\left(\frac{\lambda}{\sqrt{\hat{\chi}}}\right) + \rho_x. \quad (12)$$

The solution can be found by numerically iterating (10) and (11) until convergence and then checking if (12) holds.

The above result gives the *critical threshold for the compression ratio* $\alpha_c(\lambda, \rho_x, \rho_w)$ that guarantees vanishing MSE of reconstruction. More precisely, if $\alpha_c(\lambda, \rho_x, \rho_w)$ is a solution to Proposition 1, then for all $\alpha < \alpha_c(\lambda, \rho_x, \rho_w)$ we have perfect reconstruction in the MSE sense, while $\alpha > \alpha_c(\lambda, \rho_x, \rho_w)$ leads to non-vanishing MSE. Note that the threshold depends on the regularization parameter λ and densities of the source and noise $\{\rho_x, \rho_w\}$, but is independent of the variances of the non-zero elements of signal σ_x^2 and noise σ_w^2 . With Proposition 1, we have thus answered the first question laid out in Section 1.

Proposition 2. Let the system be outside of the perfect reconstruction phase given by Proposition 1, i.e., the compression ratio is above the threshold $\alpha > \alpha_c(\lambda, \rho_x, \rho_w)$. The MSE of the sparse signal estimate obtained with (2) and (3) is then

$$\text{mse} = \rho_x \sigma_x^2 - 4\sigma_x^2 \rho_x Q\left(\frac{\lambda}{\sqrt{\hat{\chi} + \sigma_x^2 \hat{m}^2}}\right) - 2\hat{m}^{-2} [(1 - \rho_x)r_{\lambda}(\hat{\chi}) + \rho_x r_{\lambda}(\hat{\chi} + \sigma_x^2 \hat{m}^2)], \quad (13)$$

where the required parameters can be obtained by solving the following set of coupled equations

$$\chi = \frac{2}{\hat{m}} \left[(1 - \rho_x) Q \left(\frac{\lambda}{\sqrt{\hat{\chi}}} \right) + \rho_x Q \left(\frac{\lambda}{\sqrt{\hat{\chi} + \sigma_x^2 \hat{m}^2}} \right) \right], \quad (14)$$

$$\hat{m} = \frac{\alpha(1 - \rho_w)}{\chi} \left[1 - 2Q \left(\frac{\chi}{\sqrt{\text{mse}}} \right) \right] + \frac{\alpha\rho_w}{\chi} \left[1 - 2Q \left(\frac{\chi}{\sqrt{\text{mse} + \sigma_w^2}} \right) \right], \quad (15)$$

$$\hat{\chi} = \alpha(1 - \rho_w) \left[s \left(\frac{\chi}{\sqrt{\text{mse}}} \right) + 2Q \left(\frac{\chi}{\sqrt{\text{mse}}} \right) \right] + \alpha\rho_w \left[s \left(\frac{\chi}{\sqrt{\text{mse} + \sigma_w^2}} \right) + 2Q \left(\frac{\chi}{\sqrt{\text{mse} + \sigma_w^2}} \right) \right]. \quad (16)$$

The solution can be found by numerically iterating the equations until convergence is reached.

With Proposition 2 we have answered the second question in Section 1, namely, how does the MSE behave when perfect reconstruction is not possible. It is important to note that Proposition 2 reduces to Proposition 1 when we enforce the condition $\text{mse} \rightarrow 0$. Taking the limit is, however, somewhat subtle as explained in Section 4. Note that in principle, one could observe the vanishing MSE also by setting $\alpha < \alpha_c(\lambda, \rho_x, \rho_w)$ and numerically evaluating (13) – (16). Some numerical difficulties, however, arise in this case since $\hat{m} \rightarrow \infty$ and $\chi \rightarrow 0$ holds for perfect reconstruction.

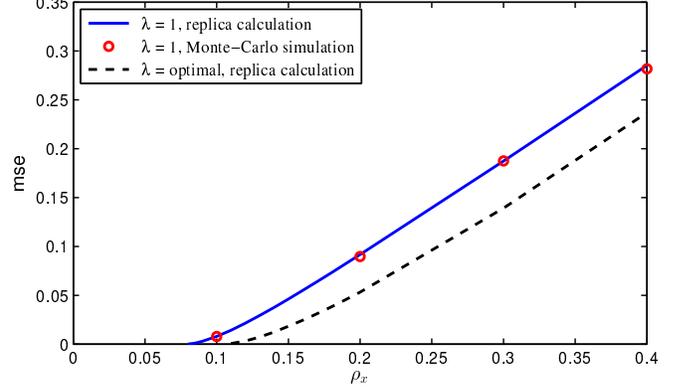
Mean square error predicted by Proposition 2 is shown in Fig. 1a. Numerical experiments obtained with `cvx` [6] are also given. Below the thresholds $\rho_x = 0.0770$ and $\rho_x = 0.1030$ for $\lambda = 1$ and $\lambda = \text{optimal}$, respectively, the MSE of the reconstruction vanishes. Figure 1b shows the effect of λ on the perfect recovery threshold given in Proposition 1. Here $\rho_w = \delta\rho_x$, where $\delta = 1/5, 1/10, 1/50$. For given ρ_x we find the critical threshold $\alpha_c(\lambda, \rho_x, \rho_w)$ that admits perfect reconstruction, so that the MSE vanishes for the set of parameters that lie above the selected curve. The results demonstrate that the choice of the regularization parameter λ has a significant impact on the performance. Note that optimization of λ with simulations is very time consuming, while it is easy to do even with brute-force search using Proposition 1.

4. REPLICANALYSIS

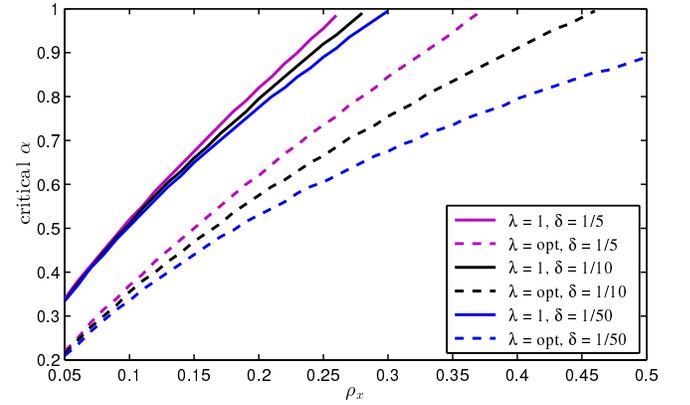
In this section a sketch of derivation is given for Propositions 1 and 2. Throughout the rest of the paper, the replica trick given in Section 2.1 is assumed to be valid. With this in mind, recall (7) and denote $\mathbf{v}^a = \mathbf{A}(\mathbf{x}^0 - \mathbf{x}^a)$. The term inside \ln in (6) can then be written as

$$\mathbb{E}_{\mathbf{x}^0} \left\{ \int \prod_{a=1}^u \left[e^{-\beta\lambda\|\mathbf{x}^a\|_1} d\mathbf{x}^a \right] \mathbb{E}_{\mathbf{A}, \mathbf{w}} \prod_{a=1}^u e^{-\beta\|\mathbf{v}^a + \mathbf{w}\|_1} \right\}, \quad (17)$$

where \mathbf{x}^0 has IID elements drawn according to $p(x; \rho_x, \sigma_x^2)$. We first concentrate on evaluating the latter term $\mathcal{I}_{u, \beta}(\mathcal{X}) = \mathbb{E}_{\mathbf{A}, \mathbf{w}} \prod_{a=1}^u e^{-\beta\|\mathbf{v}^a + \mathbf{w}\|_1}$, for a fixed set $\mathcal{X} = \{\mathbf{x}^a\}_{a=0}^u$.



(a) MSE of reconstruction for $\alpha = 1/2$, $\rho_w = 0.1$ and $\sigma_x^2 = \sigma_w^2$. Lines for replica analysis based results and markers for simulations.



(b) Replica method based critical condition for perfect reconstruction. Solid lines for $\lambda = 1$ and dashed lines for optimal λ .

Fig. 1: Reconstruction performance vs. signal density ρ_x .

Since \mathbf{A} has IID elements with density $g_A(0, 1/N)$, conditioned on \mathcal{X} the vectors $\{\mathbf{v}^a\}$ tend to jointly Gaussian RVs by the central limit theorem as $N \rightarrow \infty$. More precisely, if $\mathbf{v} \in \mathbb{R}^{uN}$ is formed by stacking $\{\mathbf{v}^a\}_{a=1}^u$ then \mathbf{v} is a zero-mean Gaussian RV with covariance matrix $\mathbf{R} = \mathbb{E}_{\mathbf{A}} \mathbf{v} \mathbf{v}^T$. We write this as $\mathbf{v} \sim g_{\mathbf{v}}(\mathbf{0}, \mathbf{R})$ and remark that the (a, b) th ($a, b = 1, \dots, u$) block of \mathbf{R} is given by

$$\mathbf{R}_{a,b} = [Q_{00} - (Q_{a0} + Q_{0b}) + Q_{ab}] \mathbf{I}_M, \quad (18)$$

where $Q_{ab} = N^{-1}(\mathbf{x}^a \cdot \mathbf{x}^b)$. For later use, let the matrix $\mathbf{Q} \in \mathbb{R}^{(u+1) \times (u+1)}$ be composed of the elements $\{Q_{ab}\}$. Thus, for large N ,

$$\mathcal{I}_{u, \beta}(\mathcal{X}) = \mathbb{E}_{\mathbf{w}} \int g_{\mathbf{v}}(\mathbf{0}, \mathbf{R}) \exp \left[-\beta \sum_{a=1}^u \|\mathbf{v}^a + \mathbf{w}\|_1 \right] d\mathbf{v}, \quad (19)$$

where we have omitted terms that vanish as $N \rightarrow \infty$ [12, 13].

In the large system limit of $N \rightarrow \infty$, Laplace's (the saddle point) method with respect to \mathbf{R} yields the exact assessment of $N^{-1} \ln E\{[Z_{\beta}(\mathbf{y}, \mathbf{A}; \lambda)]^u\}$ for $\forall n \in \mathbb{N}$ and $\forall \beta > 0$. We here assume that the dominant saddle point in the assessment is invariant under any permutation of the replica indexes

$a = 1, 2, \dots, u$, which is often termed the replica symmetric (RS) ansatz and is characterized as $Q_{a0} = Q_{0b} = m, Q_{aa} = Q, a = 1, \dots, u$ and $Q_{ab} = q$ for $a \neq b \in \{1, \dots, u\}$ in the current case. This allows us to express v^a in (19) as $v^a = z_a \sqrt{Q - q} + t \sqrt{p - 2m + q} \in \mathbb{R}^M$, which means that $\mathcal{I}_{u,\beta}(\mathcal{X})$ is proportional to

$$\left[\mathbb{E} \left\{ \left(\int e^{-\beta |z \sqrt{Q - q} + t \sqrt{p - 2m + q} + w| - \frac{1}{2} z^2} dz \right)^u \right\} \right]^M, \quad (20)$$

where $\mathbb{E}\{\dots\} = \int (\dots) p(w; \rho_w, \sigma_w^2) dw Dt$ and $Dt = dt e^{-t^2/2} / \sqrt{2\pi}$. Since we are interested in the zero temperature solution $\beta \rightarrow \infty$, Laplace's method for the integral w.r.t. z implies

$$\mathcal{I}_{u,\beta}(\mathbf{Q}) \propto \left[\mathbb{E} e^{-u\beta\psi(t,w;\mathbf{Q})} \right]^{\alpha N}. \quad (21)$$

We write next the exponential term in (20) in a slightly different form by denoting $\chi = \beta(Q - q) \geq 0$ and $\varphi(t, w; \mathbf{Q}) = t \sqrt{p - 2m + Q} + w$. We also use the fact that $p - 2m + q = p - 2m + Q - \chi/\beta \rightarrow p - 2m + Q$ for any finite χ . The Laplace's method requires then that

$$\psi(t, w; \mathbf{Q}) = \min_z \left\{ |z \sqrt{\chi} + \varphi(t, w; \mathbf{Q})| + \frac{z^2}{2} \right\}. \quad (22)$$

Examining the critical points of ψ for a fixed set $\{t, w, \chi, \mathbf{Q}\}$ shows that the minimizing z gives

$$\psi(t, w; \mathbf{Q}) = \begin{cases} \varphi(t, w; \mathbf{Q})^2 / (2\chi), & |\varphi(t, w; \mathbf{Q})| < \chi; \\ |\varphi(t, w; \mathbf{Q})| - \chi/2, & |\varphi(t, w; \mathbf{Q})| > \chi. \end{cases} \quad (23)$$

The next task is to average $\mathcal{I}_{u,\beta}(w; \mathbf{Q})$ over the set \mathcal{X} . The expectation w.r.t. \mathbf{Q} can be carried out under the RS ansatz by defining first the probability weight

$$\begin{aligned} \mu(\mathbf{Q}) &= \int d\mathbf{x}^0 p(\mathbf{x}^0) \prod_{a=1}^u \left(d\mathbf{x}^a e^{-\beta \lambda \|\mathbf{x}^a\|_1} \right) \\ &\times \prod_{0 \leq a \leq b \leq u} \delta(\mathbf{x}^a \cdot \mathbf{x}^b - N Q_{ab}), \end{aligned} \quad (24)$$

and integrating then w.r.t. the measure $\mu(\mathbf{Q})$. Under the RS ansatz, measure (24) has the same form as in [16, 17] so we skip the derivation here due to space constraints and arrive straight at the expression

$$\begin{aligned} \mu(\mathbf{Q}) &\propto \int d\hat{\mathbf{Q}} \exp \left[\beta N \left(u \frac{\hat{\mathbf{Q}}\mathbf{Q} - \hat{\chi}\chi}{2} - um\hat{m} \right. \right. \\ &\left. \left. + \frac{u^2}{2} (\hat{\chi}\chi - \beta\hat{\chi}Q) + \frac{1}{\beta} \log \mathcal{M}_u(\hat{\mathbf{Q}}; \beta, \lambda) \right) \right], \end{aligned} \quad (25)$$

where $\hat{\mathbf{Q}}$ is a short-hand for $\{\hat{\chi}, \hat{\mathbf{Q}}, \hat{m}\}$. We also have the moment generating function for the elements of $\{\mathbf{x}^a\}_{a=0}^u$

$$\begin{aligned} \mathcal{M}_u(\hat{\mathbf{Q}}; \beta, \lambda) &= (1 - \rho_x) \mathbb{E}_z e^{-\beta u \phi_\lambda(z \sqrt{\hat{\chi}}; \hat{\mathbf{Q}})} \\ &+ \rho_x \mathbb{E}_z e^{-\beta u \phi_\lambda(z \sqrt{\hat{\chi} + \sigma_x^2 \hat{m}^2}; \hat{\mathbf{Q}})}, \end{aligned} \quad (26)$$

where $\mathbb{E}_z(\dots)$ denotes $\int (\dots) Dz$ and ϕ satisfies

$$\phi_\lambda(h; \hat{\mathbf{Q}}) = \begin{cases} -(|h| - \lambda)^2 / (2\hat{\mathbf{Q}}), & \text{if } |h| > \lambda \\ 0, & \text{if } |h| \leq \lambda. \end{cases} \quad (27)$$

The final form of $\mu(\mathbf{Q})$ seems undoubtedly cryptic for a casual reader, so let us sketch the derivation briefly (more details in [16, 17]). The first task in obtaining (25) is to write the Dirac's delta functions using (inverse) Fourier transform and integrating over \mathbf{x}^0 with the help of the Gaussian integral

$$\sqrt{\frac{1}{2\pi}} \int e^{-ax^2/2 + bx} dx = \frac{1}{\sqrt{a}} \exp\left(\frac{b^2}{2a}\right). \quad (28)$$

Then (28) is used right-to-left to decouple the replicated terms $\{\mathbf{x}^a\}$ and the average over them is obtained using the saddle point method as $\beta \rightarrow \infty$. These last two steps give arise to (27) and the integrals in (26). Rest of the terms in (25) come essentially from the (inverse) Fourier transform of the Dirac's delta functions where the hatted variables represent scaled transform domain variables.

Combining (21) and (25) yields an expression for (17) as

$$\begin{aligned} \mathbb{E}\{[Z_\beta(\mathbf{y}, \mathbf{A}; \lambda)]^u\} &\propto \int d\mathbf{Q} \mathcal{I}_{u,\beta}(\mathbf{Q}) \mu(\mathbf{Q}) \\ &= \int d\mathbf{Q} d\hat{\mathbf{Q}} \exp \left\{ \beta N \left(\frac{\alpha}{\beta} \log \mathbb{E}_{t,w} e^{-u\beta\psi(t,w;\mathbf{Q})} - um\hat{m} \right. \right. \\ &\left. \left. + u \frac{\hat{\mathbf{Q}}\mathbf{Q} - \hat{\chi}\chi}{2} + \frac{u^2}{2} (\hat{\chi}\chi - \beta\hat{\chi}Q) + \frac{1}{\beta} \log \mathcal{M}_u(\hat{\mathbf{Q}}; \beta, \lambda) \right) \right\}. \end{aligned} \quad (29)$$

For the integration w.r.t. \mathbf{Q} and $\hat{\mathbf{Q}}$ we use again the saddle point method as $N \rightarrow \infty$. Note that we have then by the law of large numbers $p \rightarrow \sigma_x^2 \rho_x$ as well (see (20)). Thus, the replica symmetric expression for (6) reads

$$\begin{aligned} f_{rs}(\lambda) &= -\text{extr} \left\{ \frac{\hat{\mathbf{Q}}\mathbf{Q}}{2} - \frac{\hat{\chi}\chi}{2} - m\hat{m} \right. \\ &\left. + \lim_{u \rightarrow 0^+} \frac{\partial}{\partial u} \left[\frac{\alpha}{\beta} \log \mathbb{E} e^{-u\beta\psi(t,w;\mathbf{Q})} + \frac{1}{\beta} \log \mathcal{M}_u(\hat{\mathbf{Q}}; \beta, \lambda) \right] \right\}, \end{aligned} \quad (30)$$

where we used the fact that the order of extremization $\text{extr}\{\dots\}$ w.r.t. $\{\chi, m, Q, \hat{\chi}, \hat{m}, \hat{\mathbf{Q}}\}$ and the partial derivative w.r.t. u can be exchanged [12]. Solving the remaining derivatives finally gives the form

$$\begin{aligned} f_{rs}(\lambda) &= \text{extr} \left\{ \frac{\hat{\chi}\chi - \hat{\mathbf{Q}}\mathbf{Q}}{2} + m\hat{m} + \alpha \mathbb{E}_{t,w} \psi(t, w; \chi, m, Q) \right. \\ &\left. + (1 - \rho_x) \mathbb{E}_z \phi_\lambda(z \sqrt{\hat{\chi}}; \hat{\mathbf{Q}}) + \rho_x \mathbb{E}_z \phi_\lambda(z \sqrt{\hat{\chi} + \sigma_x^2 \hat{m}^2}; \hat{\mathbf{Q}}) \right\} \end{aligned} \quad (31)$$

in the limit $u \rightarrow 0^+$.

We have now managed to write the normalized free energy under RS ansatz as the solution of an extremization problem that has a couple of expectations inside. Let us first consider the derivatives w.r.t. the variables $\{\hat{\chi}, \hat{m}, \hat{\mathbf{Q}}\}$. Since ψ

does not depend on them, we only need to solve the expectations and partial derivatives on the second line in (31).

Lemma 1. *Let h be a real positive (function) independent of $z \in \mathbb{R}$. Then, for positive real parameters \hat{Q} and λ we have*

$$\mathbb{E}_z \phi_\lambda(z\sqrt{h}; \hat{Q}) = \hat{Q}^{-1} r_\lambda(h), \quad (32)$$

$$\frac{\partial}{\partial x} r_\lambda(h) = - \left(\frac{\partial h}{\partial x} \right) Q \left(\frac{\lambda}{\sqrt{h}} \right), \quad (33)$$

where $r_\lambda(h)$ is given in (9).

Using the above results, the normalized free energy reads

$$f_{\text{rs}}(\lambda) = \text{extr} \left\{ \frac{\hat{Q}Q}{2} - \frac{\hat{\chi}\chi}{2} - m\hat{m} - \alpha \mathbb{E}_{t,w} \psi(t, w; \chi, m, Q) \right. \\ \left. + \hat{Q}^{-1} [(1 - \rho_x) r_\lambda(\hat{\chi}) + \rho_x r_\lambda(\hat{\chi} + \sigma_x^2 \hat{m}^2)] \right\}, \quad (34)$$

where χ is given in (14) and

$$m = 2\sigma_x^2 \rho_x \left(\frac{\hat{m}}{\hat{Q}} \right) Q \left(\frac{\lambda}{\sqrt{\hat{\chi} + \sigma_x^2 \hat{m}^2}} \right), \quad (35)$$

$$Q = -2\hat{Q}^{-2} [(1 - \rho_x) r_\lambda(\hat{\chi}) + \rho_x r_\lambda(\hat{\chi} + \sigma_x^2 \hat{m}^2)]. \quad (36)$$

To obtain rest of the parameters, we need the following result.

Lemma 2. *Let $\omega(t\sqrt{a}, x_1, \dots, x_k)$ be a real-valued function, where $a \geq 0$ and $\{t, x_1, \dots, x_k\}$ are independent random variables that do not depend on a . Then,*

$$\frac{\partial}{\partial a} \int \omega(t\sqrt{a}, x_1, \dots, x_k) Dt = \frac{1}{2} \int \omega''(t\sqrt{a}, x_1, \dots, x_k) Dt, \quad (37)$$

where $\omega''(\dots)$ is the 2nd order partial derivative w.r.t. first argument. Also, denoting the indicator function $\mathbb{1}\{\dots\}$,

$$\int \mathbb{1}\{|t| > a\} Dt = 2Q(a), \quad (38)$$

$$\int t^2 \mathbb{1}\{|t| < a\} Dt = 1 - 2Q(a) - \frac{2a}{\sqrt{2\pi}} e^{-a^2/2}, \quad (39)$$

where the integrals are over the set of real numbers.

Using (37) for the partial derivatives w.r.t. m and Q , and then (38) – (39) for the remaining integrals shows that $\hat{Q} = \hat{m}$ as given in (15). Furthermore, $\text{mse} = \sigma_x^2 \rho_x - 2m + Q$ reduces to (13) and gives the MSE of the reconstruction [16, 17]. Similarly, from the derivative of χ and (38) – (39) one gets (16). Thus, we have obtained a full description of the free energy under the RS ansatz in terms of six parameters. More importantly, we obtained as a by product the MSE behavior of the convex optimization problem based on (2) and (3), finishing the proof of Proposition 2.

To obtain Proposition 1, we require that $\text{mse} \rightarrow 0$. This implies $\rho_x \sigma_x^2 = m = Q$ and $\hat{m} = \hat{Q} \rightarrow \infty \implies \chi \rightarrow 0$. For a non-trivial solution we also need $\hat{\chi} \in O(1)$ and $0 < \lambda < \infty$. However, the condition for critical threshold α cannot be directly obtained by plugging this to (13) – (16). Instead, we expand the Q-function and exponential function near zero with the Taylor series, define $\kappa = \text{mse}/\chi^2$ and examine the limits for κ and $\hat{\chi}$. Some algebra provides then Proposition 1.

5. REFERENCES

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