ON SPECTRAL ANALYSIS WITH NONUNIFORM FREQUENCY RESOLUTION OF NONSTATIONARY STOCHASTIC PROCESSES

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ABSTRACT

Spectral analysis with nonuniform frequency resolution of nonstationary stochastic processes is addressed. The frequency-warping operation aimed at increasing the frequency resolution is shown to modify the nonstationarity kind of the analyzed process. Specifically, in several cases of interest, the frequency-warped process is shown to belong to the recently introduced class of the spectrally correlated processes. Spectral correlation density estimation is performed by frequency smoothing the periodogram along curves in the bifrequency plane instead of lines with unit slopes as in the case of wide-sense stationary and almost-cyclostationary processes. Application to cyclic spectral analysis of the GPS-L1 signal is considered.

Index Terms— Spectrally correlated processes, cyclostationarity, spectral analysis.

1. INTRODUCTION

Spectral analysis with nonuniform (or unequal) frequency resolution finds applications in several fields such as frequency and spectral estimation. This problem has been investigated with reference to deterministic signals in [2], [11], [16]. The nonuniform frequency resolution is obtained by frequency-warping techniques. For this purpose, in [2] and [11] the warped discrete Fourier transform is introduced.

In this paper, the problem of spectral analysis with nonuniform frequency resolution is addressed for some classes of discrete-time stochastic processes. Spectral analysis with nonuniform frequency resolution of a given process is equivalent to spectral analysis with uniform frequency resolution of a frequency-warped version of the original process. Since frequency-warping is a linear time-variant transformation, spectral analysis techniques for deterministic signals cannot be easily extended to the case of nonstationary processes. In fact, frequency warping modifies the the nonstationarity properties of the original stochastic process under analysis.

In the paper it is shown that in the case of a wide-sense stationary (WSS) process, the frequency-warped process is still WSS, but jointly spectrally correlated (SC) with the original process. In addition, it is shown that for an almost-cyclostationary (ACS) process, after frequency warping, the resulting process is spectrally correlated (SC) and jointly SC with the original process. SC processes are a recently introduced class of nonstationary processes that have Loève bifrequency spectrum with spectral masses concentrated on a countable set of support curves in the bifrequency plane [12], [13], [14, Chap. 4]. ACS processes are obtained as special case of SC processes when the support curves are lines with unit slope in the principal frequency domain.

In the paper, the periodogram frequency smoothed along curves in the bifrequency plane rather than along lines with unit slope is proposed as effective method for spectral analysis of SC processes obtained by frequency-warping WSS and ACS processes. Performance analysis is carried out via Monte Carlo simulations for a GPS-L1 signal.

The paper is organized as follows. In Section 2, discrete-time SC processes are briefly reviewed. In Section 3 the problem of spectral analysis of nonstationary processes with nonuniform frequency resolution is theoretically addressed. Numerical results are presented in Section 4. Conclusions are drawn in Section 5.

2. SPECTRALLY CORRELATED PROCESSES

The complex-valued discrete-time processes $x_1(n)$ and $x_2(n)$ are said to be *second-order jointly harmonizable* if their cross-correlation function can be expressed by the Fourier-Stieltjes integral

$$\mathrm{E}\left\{x_1(n_1)\, x_2^{(*)}(n_2)\right\} = \int_{I^2} e^{j2\pi[\nu_1 n_1 + (-)\nu_2 n_2]}\, \mathrm{d}\gamma_{\boldsymbol{x}}(\nu_1, \nu_2) \tag{1}$$

where $I \triangleq [-1/2, 1/2]$ and $\gamma_{\boldsymbol{x}}(\nu_1, \nu_2)$ is a (spectral) correlation function of bounded variation [10] . In (1), superscript $(*)$ denotes optional complex conjugation, $(-)$ is an optional minus sign which is linked to $(*)$, and subscript $\boldsymbol{x} = [x_1, x_2^{(*)}]$.

For complex-valued processes, both cross-correlation function $\mathrm{E}\left\{x_1(n_1)\, x_2^*(n_2)\right\}$ and conjugate cross-correlation

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function $\mathrm{E}\{x_1(n_1)\,x_2(n_2)\}$ must be considered for a complete second-order characterization [17]. Notation in (1) allows to treat both second-order cross-moments by considering or not the optional complex conjugation.

Let $X_i(\nu)$ be the Fourier transform of the process $x_i(n)$, $(i = 1, 2)$, assumed to exist (at least) in the sense of distributions [4], [5]. It results $\mathrm{d}\chi_i(\nu) = X_i(\nu)\,\mathrm{d}\nu$, where $\chi_i(\nu)$ is the integrated spectrum of $x_i(n)$, provided that $\chi_i(\nu)$ does not contain singular components [6] and derivatives are intended in a generalized sense that accommodates Dirac deltas in correspondence of jumps in $\chi_i(\nu)$. The *Loève bifrequency cross-spectrum* of two discrete-time complex-valued second-order jointly harmonizable stochastic processes $x_1(n)$ and $x_2(n)$ is defined as [10]

$$\mathcal{S}_{\boldsymbol{x}}(\nu_1, \nu_2) \triangleq \mathrm{E}\left\{X_1(\nu_1)\,X_2^{(*)}(\nu_2)\right\} . \quad (2)$$

If $\gamma_{\boldsymbol{x}}(\nu_1, \nu_2)$ and the integrated spectra $\chi_i(n)$ do not contain singular components, $\mathrm{d}\gamma_{\boldsymbol{x}}(\nu_1, \nu_2) = E\{\,\mathrm{d}\chi_1(\nu_1)\ \mathrm{d}\chi_2^*(\nu_2)\}$ $= \mathcal{S}_{\boldsymbol{x}}(\nu_1, \nu_2)\,\mathrm{d}\nu_1\,\mathrm{d}\nu_2$.

Two processes $x_1(n)$ and $x_2(n)$ not containing any additive finite-strength sinewave component are said to be *jointly spectrally correlated* if their Loève bifrequency cross-spectrum can be expressed as [13], [14, Chap. 4]

$$\mathcal{S}_{\boldsymbol{x}}(\nu_1, \nu_2) = \sum_{k\in\mathbb{I}} S_{\boldsymbol{x}}^{(k)}(\nu_1)\,\widetilde{\delta}\left(\nu_2 - \Psi_{\boldsymbol{x}}^{(k)}(\nu_1)\right) . \quad (3)$$

In (3), $\mathbb{I}$ is a countable set and $\widetilde{\delta}(\nu) \triangleq \sum_{p\in\mathbb{Z}} \delta(\nu - p)$ is the periodic Dirac delta train with period 1. The complex valued functions $S_{\boldsymbol{x}}^{(k)}(\nu)$, referred to as *spectral cross-correlation densities*, and the real-valued functions $\Psi_{\boldsymbol{x}}^{(k)}(\nu)$ referred to as *spectral support functions*, are periodic functions of ν with period 1. Each $\Psi_{\boldsymbol{x}}^{(k)}(\cdot)$ is assumed to be differentiable and locally invertible in every interval of width 1.

From (3) it follows that discrete-time jointly SC processes have spectral masses concentrated on the countable set of support curves $\nu_2 = \Psi_{\boldsymbol{x}}^{(k)}(\nu_1) \bmod 1,\ k \in \mathbb{I}$, where $\bmod 1$ is the modulo 1 operation with values in $[-1/2, 1/2)$. Moreover, the spectral mass distribution is periodic with period 1 in both frequency variables ν_1 and ν_2. Without lack of generality, it can be assumed that two support curves intersect at most in a finite or countable set of points (ν_1, ν_2).

The density of Loève bifrequency cross-spectrum is obtained by replacing in (3) the periodic Dirac delta train with the periodic Kronecker delta train and is denoted by $S_{\boldsymbol{x}}(\nu_1, \nu_2)$

Almost all modulated signals encountered in communications, radar, sonar, and telemetry can be modeled as almost-cyclostationary. That is, their statistical functions such as distribution functions, moments, and cumulants are almost-periodic functions of time [3]. Second-order jointly ACS signals in the wide-sense are characterized by an almost-periodic (conjugate) cross-correlation function

$$\mathrm{E}\left\{x_1(n+m)\,x_2^{(*)}(n)\right\} = \sum_{\alpha\in\mathcal{A}} R_{\boldsymbol{x}}^{\alpha}(m)\,e^{j2\pi\alpha n} \quad (4)$$

where the Fourier coefficients

$$R_{\boldsymbol{x}}^{\alpha}(m) \triangleq \left\langle \mathrm{E}\left\{x_1(n+m)\,x_2^{(*)}(n)\right\}\ e^{-j2\pi\alpha n}\right\rangle_n \quad (5)$$

with $\langle\cdot\rangle_n$ denoting infinite-time average, are referred to as cyclic cross-correlation functions and

$$\mathcal{A} \triangleq \{\alpha \in [-1/2, 1/2)\ :\ R_{\boldsymbol{x}}^{\alpha}(m) \not\equiv 0\} \quad (6)$$

is the countable set of cycle frequencies α in the principal domain $[-1/2, 1/2)$. By double Fourier transforming both sides of (4), the following expression for the Loève bifrequency cross-spectrum is obtained

$$\mathcal{S}_{\boldsymbol{x}}(\nu_1, \nu_2) = \sum_{\alpha\in\mathcal{A}} S_{\boldsymbol{x}}^{\alpha}(\nu_1)\,\widetilde{\delta}(\nu_2 - {\scriptstyle(-)}(\alpha - \nu_1)) \quad (7)$$

where $S_{\boldsymbol{x}}^{\alpha}(\nu)$ are the Fourier transforms of $R_{\boldsymbol{x}}^{\alpha}(m)$ and are referred to as cyclic spectra. From (7) it follows that discrete-time jointly ACS processes are obtained as a special case of jointly SC processes when the spectral support curves are lines with unit slope in the principal frequency domain I^2. In such a case, correlation exists only between spectral components that are separated by quantities equal to the cycle frequencies. WSS processes have distinct spectral components that are uncorrelated and the support of the Loève bifrequency spectrum in I^2 is contained in the main diagonal.

The case of support lines with not necessarily unit slope is addressed in [8]. Continuous-time fractional Brownian motion is shown to have spectral masses concentrated on the main diagonal and the frequency axes of the bifrequency plane [15].

3. SPECTRAL ANALYSIS WITH NONUNIFORM FREQUENCY RESOLUTION

(Cross-)spectral analysis techniques of a discrete-time process $x(n)$ based on the Fourier transform $X(\nu)$ and its inverse

$$x(n) = \int_{-1/2}^{1/2} X(\nu)\,e^{j2\pi\nu n}\,\mathrm{d}\nu \quad (8)$$

have uniform frequency resolution. That is, the discrete Fourier transform (DFT) $X(k/N)$, $k = -N/2, \ldots, N/2-1$ (N even) has frequency bins uniformly spaced in the main frequency interval $[-1/2, 1/2]$ with spacing $1/N$.

Let $\psi(\nu)$ a real-valued strictly-increasing differentiable possibly nonlinear function defined in $[-1/2, 1/2]$ and with values contained in $[-1/2, 1/2]$. The process

$$y(n) = \int_{-1/2}^{1/2} X(\psi(\nu))\,e^{j2\pi\nu n}\,\mathrm{d}\nu \quad (9)$$

is a *frequency-warped* version of $x(n)$. Due to the nonlinear behavior of $\psi(\cdot)$, $X(\psi(k/N))$ is a DFT of $x(n)$ with nonuniform (or unequal) frequency resolution [2], [11], [16]. Therefore, spectral analysis with uniform frequency resolution of a frequency-warped version of $x(n)$ is equivalent to spectral analysis with nonuniform frequency resolution of $x(n)$.

Relationship (9) describes a linear time-variant (LTV) transformation of $x(n)$ into its frequency-warped version $y(n)$. The corresponding LTV system belongs to the class of the deterministic linear systems in the fraction-of-time probability framework [14, Chap. 6]. In fact, it transforms an almost-periodic input signal into an almost-periodic output signal with different frequencies of the (generalized) Fourier series expansion. Since the system is LTV, it modifies the nonstationarity properties of the input process.

In the following, input ACS and WSS processes are considered. It is shown that in both cases the statistical characterization of the output process an the joint characterization of the input and output processes involves (jointly) SC processes.

Let $x(n)$ be an ACS process with Loève bifrequency spectrum (7) with $x_1 \equiv x_2 \equiv x$ and let us denote by $\widetilde{\psi}(\nu)$ the periodic replication with period 1 of the frequency-warping function $\psi(\nu)$, that is, $\widetilde{\psi}(\nu) \triangleq \sum_{m\in\mathbb{Z}} \psi(\nu - m)$. From (9) and (7), and using the variable change property in the argument of Dirac deltas we have

$$\begin{aligned} \mathrm{E}\Big\{Y(\nu_1)\, Y^*(\nu_2)\Big\} &= \mathrm{E}\Big\{X\Big(\widetilde{\psi}(\nu_1)\Big)\, X^*\Big(\widetilde{\psi}(\nu_2)\Big)\Big\} \\ &= \sum_{\alpha\in\mathcal{A}} S_{\boldsymbol{x}}^{\alpha}\Big(\widetilde{\psi}(\nu_1)\Big)\, \widetilde{\delta}\Big(\widetilde{\psi}(\nu_2) - \widetilde{\psi}(\nu_1) + \alpha\Big) \\ &= \sum_{\alpha\in\mathcal{A}} S_{\boldsymbol{x}}^{\alpha}\Big(\widetilde{\psi}(\nu_1)\Big)\, \Big|\widetilde{\phi}'(\widetilde{\psi}(\nu_1) - \alpha)\Big| \\ &\quad\ \widetilde{\delta}\Big(\nu_2 - \widetilde{\phi}(\widetilde{\psi}(\nu_1) - \alpha)\Big) \end{aligned} \tag{10}$$

where $\widetilde{\phi}(\cdot)$ is the periodic replication with period 1 of $\phi(\cdot)$, the inverse function of $\psi(\cdot)$. From (10) it follows that the Loève bifrequency spectrum of $y(n)$ has spectral masses concentrated on a countable set of support curves in the bifrequency plane. Specifically, $y(n)$ is a SC process characterized by spectral correlation densities and support curves

$$S_{\boldsymbol{y}}^{(k)}(\nu_1) = S_{\boldsymbol{x}}^{\alpha}\Big(\widetilde{\psi}(\nu_1)\Big)\, \Big|\widetilde{\phi}'(\widetilde{\psi}(\nu_1) - \alpha)\Big| \tag{11}$$

$$\Psi_{\boldsymbol{y}}^{(k)}(\nu_1) = \widetilde{\phi}(\widetilde{\psi}(\nu_1) - \alpha) \tag{12}$$

respectively, where k is an integer index in one-to-one correspondence with the cycle frequencies α in the countable set $\mathcal{A}$. In addition, the Loève bifrequency cross-spectrum of $y(n)$ and $x(n)$ is given by

$$\begin{aligned} \mathrm{E}\Big\{Y(\nu_1)\, X^*(\nu_2)\Big\} &= \mathrm{E}\Big\{X\Big(\widetilde{\psi}(\nu_1)\Big)\, X^*(\nu_2)\Big\} \\ &= \sum_{\alpha\in\mathcal{A}} S_{\boldsymbol{x}}^{\alpha}\Big(\widetilde{\psi}(\nu_1)\Big)\, \widetilde{\delta}\Big(\nu_2 - \widetilde{\psi}(\nu_1) + \alpha\Big) \end{aligned} \tag{13}$$

that is, $y(n)$ and $x(n)$ are jointly SC processes.

In the special case where $x(n)$ is WSS, the set $\mathcal{A}$ contains the only element $\alpha = 0$. Thus,

$$\mathrm{E}\Big\{Y(\nu_1)\, Y^*(\nu_2)\Big\} = S_{\boldsymbol{x}}^{0}\Big(\widetilde{\psi}(\nu_1)\Big)\, \Big|\widetilde{\phi}'(\widetilde{\psi}(\nu_1))\Big|\, \widetilde{\delta}(\nu_2 - \nu_1) \tag{14}$$

that is, $y(n)$ is in turn WSS accordingly with the results of [9]. However, $x(n)$ and $y(n)$ are jointly SC with Loève bifrequency cross-spectrum

$$\mathrm{E}\Big\{Y(\nu_1)\, X^*(\nu_2)\Big\} = S_{\boldsymbol{x}}^{0}\Big(\widetilde{\psi}(\nu_1)\Big)\, \widetilde{\delta}\Big(\nu_2 - \widetilde{\psi}(\nu_1)\Big)\,. \tag{15}$$

An illustrative example is presented to show the effects of spectral analysis with nonuniform frequency resolution on an ACS process. A discrete-time pulse-amplitude modulated (PAM) signal $x(n)$ is obtained by uniformly sampling with period T_s a continuous-time PAM signal with raised cosine pulse with excess bandwith $\eta = 0.85$ and symbol period $T_p = 4T_s$. It is a discrete-time ACS process with three cycle frequencies $\alpha \in \{0, \pm T_s/T_p\}$. Its frequency-warped version $y(n)$ is also considered, with frequency warping function $\psi(\nu) = B_m \tan^{-1}(\nu/B_s)$ which is typical in spectral analysis with non uniform frequency resolution [16]. In the example, $B_m = B_s = 0.2$.

In Figure 1, the magnitude of the bifrequency spectral correlation density for (a) $x(n)$ and (b) $y(n)$ is reported as function of ν_1 and ν_2. The frequency-warping operation transforms the ACS process into a SC process. The support of the power spectral density (PSD) of the process, which is contained in the main diagonal, after frequency warping remains contained in the main diagonal even if the shape of the PSD is modified. This result is in accordance with the fact that frequency warping transforms WSS processes into WSS processes [1], [9].

Cyclic spectra of $x(n)$ estimated with nonuniform frequency resolution are obtained by estimating with uniform frequency resolution the spectral correlation densities (11). Since the support curves (12) of the SC process $y(n)$ are not lines with unit slope, the classical frequency-smoothed cyclic periodogram method that is adopted for cyclic spectral analysis of (jointly) ACS processes [3] cannot be adopted in this case.

The frequency-smoothed cyclic periodogram at cycle frequency α of the ACS process $x(n)$ is obtained by frequency smoothing the bifrequency cross-periodogram

$$I_{\boldsymbol{x}}(\nu_1, \nu_2) \triangleq \frac{1}{N} X_N(\nu_1)\, X_N^*(\nu_2) \tag{16}$$

along the support line $\nu_2 = \nu_1 - \alpha$, where $X_N(\nu)$ is the short-time Fourier transform of $x(n)$ for $n = 0, 1, \ldots, N-1$. Motivated by this, an estimator of the spectral correlation density (11) is obtained by considering the bifrequency cross-periodogram frequency smoothed along the support curve (12). That is,

$$S_{\boldsymbol{y}}^{(k)}(\nu_1)_{N,\Delta\nu} = I_{\boldsymbol{x}}(\nu_1, \nu_2)\big|_{\nu_2 = \Psi_{\boldsymbol{y}}^{(k)}(\nu_1)} \otimes A_{\Delta\nu}(\nu_1) \tag{17}$$

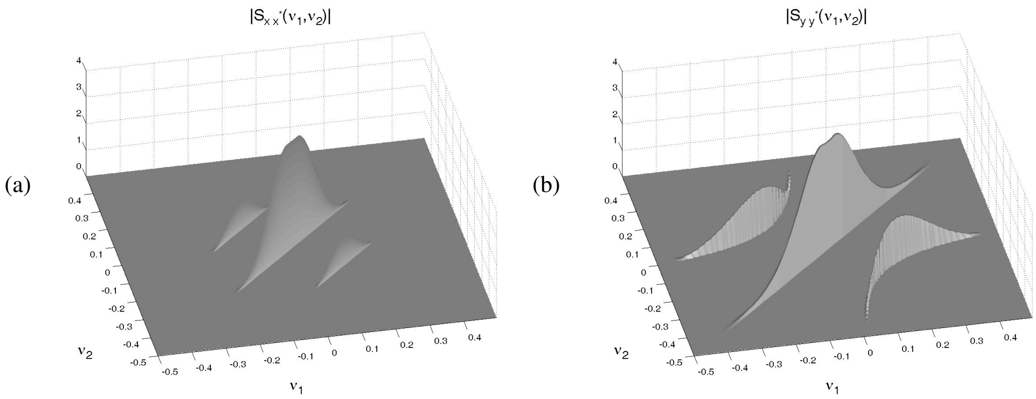

Fig. 1. Magnitude of the bifrequency spectral cross-correlation density of (a) $x(n)$ and (b) $y(n)$ as a function of ν_1 and ν_2.

where $\otimes$ denotes periodic convolution with period 1 with respect to ν_1 and $A_{\Delta\nu}(\nu_1)$ is a frequency-smoothing window with bandwidth $\Delta\nu$ such that it approaches $\widetilde{\delta}(\nu)$, in the sense of distributions, as $\Delta\nu \to 0$. By following the guidelines in [14, Chap. 4] where the estimator for the continuous-time case is considered, the frequency-smoothed periodogram (17) can be shown to be a mean-square consistent and asymptotically complex Normal (as $N \to \infty$ and $\Delta\nu \to 0$ with $N\Delta\nu \to \infty$) estimator of $S_{\boldsymbol{y}}^{(k)}(\nu_1)\,\mathcal{E}^{(k)}(\nu_1)$, where $\mathcal{E}^{(k)}(\nu)$ is a known multiplicative bias factor depending on $A_{\Delta\nu}(\nu)$ and the first-order derivative of $\Psi_{\boldsymbol{y}}^{(k)}(\nu)$. When the process is ACS, $\Psi_{\boldsymbol{y}}^{(k)}(\nu) = \nu_1 - \alpha$, and estimator (17) reduces to the well-known frequency-smoothed cyclic periodogram [3].

4. NUMERICAL RESULTS

In this section, a numerical experiment is conducted aimed at corroborating the theoretical results of Section 3.

Spectral analysis with nonuniform frequency resolution is carried out for the discrete-time signal $x(n)$ obtaining by uniformly sampling the complex envelope of a continuous-time GPS-L1 signal [7]

$$\begin{aligned} x_a(t) &= \sqrt{2}\,P\,d(t)\,c(t)\,\cos(2\pi f_{\mathrm{L1}}t + \phi_0) \\ &\quad + P\,d(t)\,p(t)\,\sin(2\pi f_{\mathrm{L1}}t + \phi_0) \end{aligned} \tag{18}$$

where f_{L1} is the $L1$ carrier frequency, $d(t)$ is the navigation message, $c(t)$ is the course acquisition (C/A) code, and $p(t)$ is the precision P(Y) code. The signal $d(t)$ is obtained by interleaving two periodic components with periods T_{frame} (frame period) and $T_{\mathrm{page}} = 5T_{\mathrm{frame}}$ (page period), respectively, and a binary PAM signal with bit period T_{b} such that $T_{\mathrm{frame}} = 300T_{\mathrm{b}}$; the PAM signal is in turn multiplied by a periodic signal with period T_{page}. The signal $c(t)$ is the periodic replication of a fixed sequence of $N_{\mathrm{c}} = 1023$ chips that identifies the satellite, it is periodic with period $N_{\mathrm{c}}T_{\mathrm{c}}$ where T_{c} is the chip period such that $T_{\mathrm{b}} = 20N_{\mathrm{c}}T_{\mathrm{c}}$. The signal $p(t)$ is a periodic signal with period equal to 1 week obtained by periodic replication of a fixed pseudo-noise sequence, it is modeled as a binary PAM signal with bit period $T_{\mathrm{p}} = T_{\mathrm{c}}/10$ within realistic observation intervals. All periods of periodic signals and bit periods of PAM signals are multiple of T_{p}. Therefore, the complex envelope of $x_a(t)$ is a cyclostationary process with period T_{p}. An accurate cyclic spectral analysis shows that the complex envelope of $x_a(t)$ exhibits strong cyclic features at cycle frequencies $\alpha \in \{k_1/(N_{\mathrm{c}}T_{\mathrm{c}}) \pm k_2/T_{\mathrm{b}} \pm k_3/T_{\mathrm{page}},\ k_1/(N_{\mathrm{c}}T_{\mathrm{c}}) \pm k_4/T_{\mathrm{frame}} \pm k_5/T_{\mathrm{page}},\ k_6/T_{\mathrm{p}} \pm k_2/T_{\mathrm{b}} \pm k_3/T_{\mathrm{page}},\ k_6/T_{\mathrm{p}} \pm k_4/T_{\mathrm{frame}} \pm k_5/T_{\mathrm{page}}\}$ where $k_1, \ldots, k_6$ are small integers.

Since the base pulse of the PAM and periodic signals is rectangular, the bandwidth of the continuous-time signal is ideally infinite. Therefore, a sampling frequency relatively large compared with the (approximate) bandwidth of the complex envelope of $x_a(t)$ is necessary to contain aliasing. Consequently, cyclic spectra of $x(n)$ are significantly different from zero in small bands in the main frequency interval $[-1/2, 1/2]$.

The frequency-smoothed cyclic periodogram of $x(n)$ at cycle-frequency $\alpha = 1/T_{\mathrm{c}}$ is estimated by $N = 2^{10}$ samples and $\Delta\nu = 1/16$ when $T_{\mathrm{p}} = 4T_s$, with T_s denoting the sampling period. In order to increase the frequency resolution, a frequency-warped version of $x(n)$, say $y(n)$, is constructed with frequency warping function $\psi(\nu) = B_m \tan^{-1}(\nu/B_s)$ with $B_m = 0.5$, $B_s = 1$. Thus, the frequency smoothed periodogram (17) of $y(n)$ with $N = 2^{10}$ and $\Delta\nu = 1/16$ is evaluated by frequency smoothing the bifrequency periodogram (16) along the support curve $\nu_2 = \widetilde{\phi}(\widetilde{\psi}(\nu_1) - \alpha)$ with $\alpha = 1/T_{\mathrm{c}}$. Sample mean (solid line) and standard deviation (shaded area) evaluated by 100 Monte Carlo trials are reported in Fig. 2 for (a) the magnitude of the frequency-smoothed cyclic periodogram of the ACS signal $x(n)$ and (b) the magnitude of the periodogram of the SC signal $y(n)$ frequency smoothed along the support curve. The main three lobes of the cyclic spectrum of $x(n)$, due to the frequency warping, are spread over a larger bandwidth with no significant increase of the estimate variance.

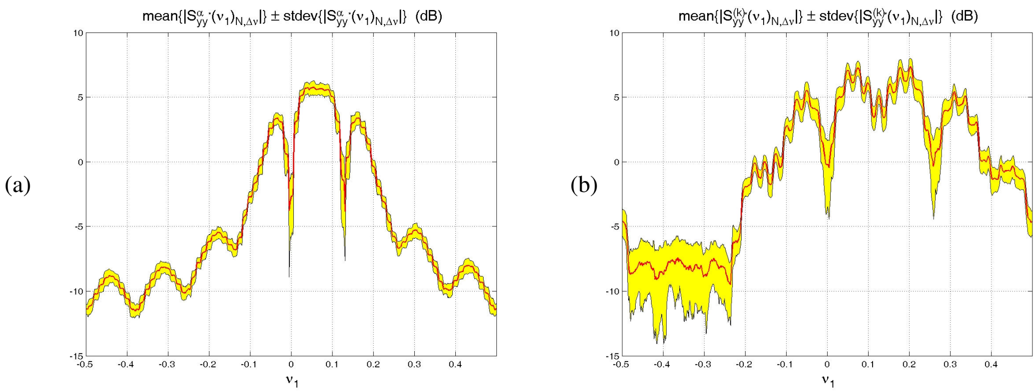

Fig. 2. (a) GPS-L1 signal $x(n)$ with rectangular pulse. Magnitude of the frequency-smoothed cyclic periodogram at cycle frequency $\alpha = 1/T_c$ as a function of ν_1. (b) Frequency warped signal $y(n)$. Magnitude of the bifrequency periodogram frequency smoothed along the support curve $\nu_2 = \widetilde{\phi}(\widetilde{\psi}(\nu_1) - \alpha)$ as a function of ν_1. Solid line: Sample mean; Shaded area: standard deviation.

5. CONCLUSION

Spectral analysis with nonuniform frequency resolution is performed by frequency warping the original process and then using a uniform frequency resolution analysis on the frequency-warped process. Frequency-warping operation modifies the nonstationarity kind of the process. In particular, almost-cyclostationary processes are shown to be transformed into spectrally correlated processes. Therefore, spectral analysis needs to be performed by frequency smoothing the periodogram along curves in the bifrequency plane instead of lines as in the case of almost-cyclostationary and wide-sense stationary processes. The shape of the curves depends on the frequency-warping function and the signal cycle frequencies. Performance analysis of the proposed technique evaluated by Monte Carlo simulations for a GPS-L1 signal shows no significant increase in the estimate variance of the spectral correlation density of the frequency-warped process.

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