

# EFFICIENT TIME RECURSIVE COHERENCE SPECTRUM ESTIMATION

Kostas Angelopoulos\*, George-Othon Glentis\*, and Andreas Jakobsson†

\*University of Peloponnese, Dept. Telecommunications, Greece

†Mathematical Statistics, Lund University, Sweden

## ABSTRACT

The coherence spectrum is of notable interest as a bivariate spectral measure in a variety of application, and the topic has lately attracted notable interest with the recent formulation of several high-resolution data adaptive estimators. In this work, we present computationally efficient *time recursive* implementations of the recent iterative adaptive approach (IAA) estimator, examining both the case of complete data sets and when some observations are missing. The algorithms continues the recent development of exploiting the estimators' inherently low displacement rank of the necessary products of Toeplitz-like matrices, extending these to time-updating formulations for the IAA-based coherence estimation algorithm. Numerical simulations together with theoretical complexity measures illustrate the performance of the proposed algorithm.

**Index Terms**— Coherence spectrum, data adaptive estimators, efficient algorithms

## 1. INTRODUCTION

Finding accurate coherence estimates between various forms of signals is of interest in a wide variety of applications, such as speech processing, time series analysis, geophysics, biomedical engineering, and synthetic aperture radar imaging. The topic has lately attracted renewed interest with the proposal of the non-parametric data-dependent Capon-based magnitude squared coherence (MSC) estimator proposed in [1], and then further explored in [2–4]. The one- and two-dimensional (2-D) Capon and APES-based approaches introduced in [1,2] show that these estimators allow for proper high-resolution MSC estimates, by forming data-adaptive filter banks, with each filter being constrained to pass its center frequency undistorted while suppressing the contribution of all other components. In [4], this work was further extended to allow for non-uniformly sampled data by exploiting a formulation based on the recent Iterative Adaptive Approach

(IAA) [5, 6]. The resulting IAA-based MSC algorithm, as well as a segmented version termed SIAA-MSC, was there shown to yield reliable estimates even if a large proportion of the measurements are missing, albeit with the drawback of being computationally cumbersome. To alleviate this problem, we recently developed efficient implementations for the batch formulations of the IAA-based MSC algorithms, making use of the inherently low displacement rank of the necessary products of Toeplitz-like matrices, thereby allowing for the development of appropriate Gohberg-Semencul (GS) representations of these matrices [7, 8]. In this paper, we further this development by proposing also *time-recursive* formulations of the IAA-based MSC estimator for both the complete and missing data cases.

## 2. TIME-RECURSIVE IAA-MSC ESTIMATION

Let  $x_{1,n}$  and  $x_{2,n}$ , for  $n = 0, 1, \dots, N$ , represent two complex valued data sequences under consideration for which subsets of length  $L$  may be assumed to be reasonably stationary. These subsets may be expressed as

$$\mathbf{x}_L^{(i)}(n) = [x_i(n) \quad x_i(n+1) \quad \dots \quad x_i(n+L-1)]^T \quad (1)$$

for  $i = 1, 2$ . The here proposed time-recursive IAA-based MSC (IAA-MSC) estimate is then formed over a sliding window of the  $L$  most recent samples of each sequence, with the corresponding coherence spectrum being defined as

$$\gamma_{x_1 x_2}^2(\omega) = \frac{|S_{x_1 x_2}(\omega)|^2}{S_{x_1}(\omega) S_{x_2}(\omega)} \quad (2)$$

where  $S_{x_1 x_2}(\omega)$  denotes the cross-spectral density between  $\mathbf{x}_L^{(1)}(n)$  and  $\mathbf{x}_L^{(2)}(n)$ , whereas  $S_{x_1}(\omega)$  and  $S_{x_2}(\omega)$  denote the spectral densities for the respective subsets of signals. The IAA-MSC of the measurement subsets is then formed as [7,8]

$$\gamma_{x_1 x_2, n}^2(\omega) = \frac{|\mathbf{f}_L^H(\omega) \mathbf{P}_L(n) \mathbf{f}_L(\omega)|^2}{\prod_{i=1}^2 \left( \mathbf{f}_L^H(\omega) [\mathbf{R}_L^{(i)}(n)]^{-1} \mathbf{f}_L(\omega) \right)} \quad (3)$$

The research work of K. Angelopoulos and G. O. Glentis was funded through a grant (THALES PROTOMI) in the framework of the O.P. Education and Lifelong Learning from Community (ESF) and national funds. The research work of A. Jakobsson was supported by the Swedish Research Council and Carl Trygger's foundation.

where  $\mathbf{R}_L^{(i)}(n)$  and  $\mathbf{R}_L^{(12)}(n)$  denote the covariance and cross-covariance matrices of  $\mathbf{x}_L^{(i)}(n)$ , for  $i = 1, 2$ , defined as

$$\mathbf{R}_L^{(i)}(n) = \mathcal{E} \left[ \mathbf{x}_L^{(i)}(n) \mathbf{x}_L^{(i)H}(n) \right] \quad (4)$$

$$\mathbf{R}_L^{(12)}(n) = \mathcal{E} \left[ \mathbf{x}_L^{(1)}(n) \mathbf{x}_L^{(2)H}(n) \right] \quad (5)$$

with  $\mathcal{E}[\cdot]$  denoting expectation, and

$$\mathbf{f}_L(\omega) = [1 \quad e^{j\omega} \quad \dots \quad e^{j(L-1)\omega}]^T \quad (6)$$

$$\mathbf{P}_L(n) \triangleq \left[ \mathbf{R}_L^{(1)}(n) \right]^{-1} \mathbf{R}_L^{(12)}(n) \left[ \mathbf{R}_L^{(2)}(n) \right]^{-1} \quad (7)$$

Here,  $(\cdot)^T$  and  $(\cdot)^H$  denote the transpose and the conjugate transpose, respectively. To estimate the covariance and cross-covariance matrices, the IAA-based covariance estimate is formed by iteratively computing [7, 8]

$$\alpha_n^{(i)}(\omega_k) = \frac{\mathbf{f}_L^H(\omega_k) [\mathbf{R}_L^{(i)}(n)]^{-1} \mathbf{x}_L^{(i)}(n)}{\mathbf{f}_L^H(\omega_k) [\mathbf{R}_L^{(i)}(n)]^{-1} \mathbf{f}_L(\omega_k)} \triangleq \frac{\psi_n^{(i)}(\omega_k)}{\varphi_n^{(i)}(\omega_k)} \quad (8)$$

$$\mathbf{R}_L^{(i)}(n) = \sum_{k=0}^{K-1} |\alpha_n^{(i)}(\omega_k)|^2 \mathbf{f}_L(\omega_k) \mathbf{f}_L^H(\omega_k) \quad (9)$$

until practical convergence, for  $\omega_k = 2\pi(k/K)$ ,  $k = 0, 1, \dots, K-1$ , where  $K > L$ , with  $\mathbf{R}_L^{(i)}(n)$  initialized to the identity matrix,  $\mathbf{I}_L$ . Upon convergence, the cross-covariance matrix is then estimated using

$$\mathbf{R}_L^{(12)}(n) = \sum_{k=0}^{K-1} \left( \alpha_n^{(1)}(\omega_k) \right)^* \alpha_n^{(2)}(\omega_k) \mathbf{f}_L(\omega_k) \mathbf{f}_L^H(\omega_k) \quad (10)$$

where  $(\cdot)^*$  denotes the complex conjugate. Finally, the IAA-MSD for the current data subsets is estimated via (3). Thus, to form the brute-force time-updating, the subsets are then updated, and the entire estimation process repeated, for each time update, clearly being a quite inefficient implementation, with large amounts of redundant calculations. To alleviate this, we now proceed with examining how the estimates may be formed in a time-recursive manner, exploiting the inherent structure of the estimates.

## 2.1. Efficient Time-Recursive Implementation

To form the time-recursive IAA-based MSD estimation scheme for the complete data case, one may instead express (3) through the use of trigonometric polynomials as

$$\gamma_{x_1 x_2, n}^2(\omega_k) = \frac{|\varphi_n^{(12)}(\omega_k)|^2}{\varphi_n^{(1)}(\omega_k) \varphi_n^{(2)}(\omega_k)} \quad (11)$$

where  $\varphi_n^{(1)}(\omega_k)$  and  $\varphi_n^{(2)}(\omega_k)$  are defined as in (8), and with

$$\varphi_n^{(12)}(\omega_k) \triangleq \mathbf{f}_L^H(\omega_k) \mathbf{P}_L(n) \mathbf{f}_L(\omega_k) \quad (12)$$

Here, and in the following,  $n$  indicates matrices and products estimated for the current subset of data, as opposed to previous  $(n-1)$  and next  $(n+1)$  subsets or windows of data. It is further noted that the (Hermitian Toeplitz) covariance matrices and the (Toeplitz) cross-covariance matrix allow for a low rank displacement representation via the GS factorization of the  $[\mathbf{R}_L^{(i)}(n)]^{-1}$  and  $\mathbf{P}_L(n)$  matrices [7, 8]. Indeed, the well known GS representation of the inverse covariance matrix results in [9]

$$[\mathbf{R}_L^{(i)}(n)]^{-1} = \sum_{\ell=1}^2 \sigma_\ell \mathcal{L} \left( \mathbf{t}_L^{\ell, i}(n) \right) \mathcal{L}^H \left( \mathbf{t}_L^{\ell, i}(n) \right) \quad (13)$$

where  $\sigma_1 = 1$  and  $\sigma_2 = -1$ , with  $\mathcal{L}(\boldsymbol{\xi}_L)$  denoting a  $L \times L$  lower triangular Toeplitz matrix with  $\boldsymbol{\xi}_L$  along its first column, whereas  $\mathbf{t}_L^{1, i}(n) \triangleq \hat{\mathbf{a}}_L^{(i)}(n)$  and  $\mathbf{t}_L^{2, i}(n) \triangleq \mathbf{Z}_L \mathbf{J}_L \hat{\mathbf{a}}_L^{(i)*}(n)$ , with  $\mathbf{Z}_L$  and  $\mathbf{J}_L$  denoting the down-shift and the exchange matrix respectively, and with  $\hat{\mathbf{a}}_L^{(i)}(n)$  denoting the power normalized forward predictor defined as

$$\hat{\mathbf{a}}_L^{(i)}(n) = \mathbf{a}_L^{(i)}(n) \sqrt{\mathbf{a}_L^{(i)T}(n) \mathbf{e}_L} \quad (14)$$

$$\mathbf{a}_L^{(i)}(n) \triangleq [\mathbf{R}_L^{(i)}(n)]^{-1} \mathbf{e}_L \quad (15)$$

with  $\mathbf{e}_L$  denoting the unity vector  $\mathbf{e}_L \triangleq [1 \ 0 \dots 0]^T$ . Due to the Toeplitz structure of  $\mathbf{R}_L^{(i)}(n)$ , (15) may be solved efficiently using the celebrated Levinson-Durbin (LD) algorithm, whereas the first column of each matrix can be computed from the relevant spectra as is dictated by (9) at low cost using a Toeplitz to circulant embedding approach and the Fast Fourier Transform (FFT), as detailed in [10]. Given the GS representation in (13), the coefficients of the associated trigonometric polynomials  $\varphi_n^{(i)}(\omega)$ , appearing in the denominator of (8) as well as in (11), may be efficiently computed and evaluated on the frequencies of interest,  $\omega_k$ , using operations that can be implemented using the FFT [10–12], resulting in an efficient implementation of (8) and (9), requiring about  $L^2 + 12\phi(2L) + 3\phi(K)$  operations, where  $\phi(N)$  denotes the complexity of forming the FFT of length  $N$ . Furthermore, the trigonometric polynomial  $\varphi_n^{(12)}(\omega_k)$ , defined in (12), can be computed efficiently. Given (13) and the fact that  $\mathbf{R}_L^{(12)}(n)$  is Toeplitz, a GS representation of (7) may be formed as [7]

$$\mathbf{P}_L(n) = \sum_{\ell=1}^4 \sigma_\ell^P \mathcal{L} \left( \mathbf{v}_L^{\ell, i}(n) \right) \mathcal{L}^H \left( \mathbf{z}_L^{\ell, i}(n) \right) \quad (16)$$

with the auxiliary variables defined in Table 1, where  $\mathbf{v}_L^1(n) = \hat{\mathbf{a}}_L^{(1)}(n)$ ,  $\mathbf{z}_L^1(n) = \mathbf{c}_L^{(12)}(n)$ ,  $\mathbf{v}_L^2(n) = \mathbf{d}_L^{(2)}(n)$ ,  $\mathbf{z}_L^2(n) = \hat{\mathbf{a}}_L^{(2)}(n)$ ,  $\mathbf{v}_L^3(n) = \mathbf{Z}_L \mathbf{J}_L \hat{\mathbf{a}}_L^{(1)*}(n)$ ,  $\mathbf{z}_L^3(n) = \mathbf{Z}_N \mathbf{c}_L^{(12)}(n)$ ,  $\mathbf{v}_L^4(n) = \mathbf{Z}_L \mathbf{c}_L^{(2)}(n)$ ,  $\mathbf{z}_L^4(n) = \mathbf{Z}_L \mathbf{J}_L \hat{\mathbf{a}}_L^{(2)*}(n)$ , and  $\sigma_1^P = \sigma_2^P = 1$  and  $\sigma_3^P = \sigma_4^P = -1$ . Given this GS factorization, the coefficients of the associated polynomial in (12) may be efficiently computed using the FFT as detailed in [13], at a

complexity of no more than  $18\phi(2L)$  operations, whereas the computational cost for computing the variables that appear in Table 1 using fast Toeplitz vector multiplication methods requires no more than  $40\phi(2L)$  operations. The time-recursive scheme developed thus far comprises two distinct steps, namely the estimation of the covariance and cross covariance sequences of the two signals of interest using a fast implementation of the IAA algorithm by iterating until convergence (8) and (9), with convergence usually being reached after 10-15 steps [5], followed by the computation of (12) and subsequently of (11) using fast trigonometric polynomial evaluation techniques based on the GS representation of the Toeplitz-like matrices associated with the pertinent polynomials. Interestingly, one may reduce the complexity even further by allowing for an approximate solution, noting that due to the sliding time windowing of the data formulation in (1), upon convergence,  $\mathbf{R}_L^{(i)}(n) \approx \mathbf{R}_L^{(i)}(n-1)$ , suggesting that an approximate solution may be found by applying a single IAA iteration, simply using the estimate of  $\mathbf{R}_L^{(i)}(n-1)$  at time index  $(n-1)$  for the initialization of the IAA algorithm in (8) and (9) at the successive time index  $n$ , as opposed to the initialization of  $\mathbf{R}_L^{(i)}(n)$  by the identity matrix  $\mathbf{I}_L$  suggested by the original IAA algorithm. Finally, in an attempt to reduce the variance of the IAA-MS estimates a reduced size IAA-MS estimator may be formed by using a smaller fraction,  $L_R$ , of the full sized cross-correlation sequence for the computation of (7) and (12), such that [4, 7]

$$\varphi_n^{(12)}(\omega) \triangleq \mathbf{f}_{L_R}^T(\omega) \mathbf{P}_{L_R}(n) \mathbf{f}_{L_R}^*(\omega), \quad (17)$$

$$\mathbf{P}_{L_R}(n) \triangleq [\mathbf{R}_{L_R}^{(1)}(n)]^{-1} \mathbf{R}_{L_R}^{(12)}(n) [\mathbf{R}_{L_R}^{(2)}(n)]^{-1} \quad (18)$$

with  $L_R \leq L$ , while keeping  $\varphi_n^{(1)}(\omega)$  and  $\varphi_n^{(2)}(\omega)$  as originally defined in (8). Due to the order recursive structure of the LD algorithm, lower order GS factorizations are produced at no extra cost. Moreover, the computation of the displacement of  $\mathbf{P}_{N_R}$  required for the efficient computation of (18) is in this case lower than that of original full order approach. It is worth noticing that the auto- and cross-correlation sequences of the input signals are still estimated using the full order IAA algorithms. We term the resulting approximate scheme the time-recursive IAA-MS (TR-IAA MS) algorithm. The computational complexity of the proposed implementation is approximately  $\mathcal{C}_{MSC}^{TR-IAA} \approx 2L^2 + 12\phi(2L) + 58\phi(2L_R) + 7\phi(K)$  operations, which is a major improvement over the  $\mathcal{O}(L^3 + L^2K)$  operations required by the direct implementation of (3)-(10).

## 2.2. Time Recursive QN-IAA MSC estimation

Exploiting the ideas in [14], further substantial computational savings can be achieved by instead using the there proposed approximative IAA algorithm for the estimation of the covariance and the cross covariance sequences required for the computation of the MSC, where the inverse of Toeplitz-like

**Table 1.** Auxiliary variables required for the displacement representation of  $\mathbf{P}_L(n)$ .

$$\begin{aligned} \beta(n) &= \hat{\mathbf{a}}_L^{(1)T}(n) \mathbf{J}_L \mathbf{R}_L^{(12)}(n) \mathbf{J}_L \hat{\mathbf{a}}_L^{(2)*}(n) \\ \mathbf{c}_L^{(1)}(n) &= \begin{bmatrix} [\mathbf{R}_{L-1}^{(2)}(n)]^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} \mathbf{R}_L^{(12)H}(n) \mathbf{J}_L \hat{\mathbf{a}}_L^{(1)*}(n) \\ \mathbf{c}_L^{(2)}(n) &= \begin{bmatrix} [\mathbf{R}_{L-1}^{(1)}(n)]^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} \mathbf{R}_L^{(12)}(n) \mathbf{J}_L \hat{\mathbf{a}}_L^{(2)*}(n) \\ \mathbf{c}_L^{(12)}(n) &= \mathbf{c}_L^{(1)}(n) + \mathbf{J}_L \hat{\mathbf{a}}_L^{(2)*}(n) \beta^*(n) \\ \delta(n) &= \hat{\mathbf{a}}_L^{(1)H}(n) \mathbf{R}_L^{(12)}(n) \hat{\mathbf{a}}_L^{(2)}(n) \\ \mathbf{d}_L^{(1)}(n) &= \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & [\mathbf{R}_{L-1}^{(2)}(n)]^{-1} \end{bmatrix} \mathbf{R}_L^{(12)H}(n) \hat{\mathbf{a}}_L^{(1)}(n) \\ \mathbf{d}_L^{(2)}(n) &= \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & [\mathbf{R}_{L-1}^{(1)}(n)]^{-1} \end{bmatrix} \mathbf{R}_L^{(12)}(n) \hat{\mathbf{a}}_L^{(2)}(n) \\ \mathbf{d}_L^{(12)}(n) &= \mathbf{d}_L^{(1)} + \hat{\mathbf{a}}_L^{(2)}(n) \delta^*(n) \end{aligned}$$

matrices are approximated by extrapolating the inverse of a lower sized matrix, treated as it have been associated with an autoregressive (AR) model of lower order  $M \leq L$  (see also [15]). Thus, instead of computing  $[\mathbf{R}_L^{(i)}(n)]^{-1}$ , a low order extrapolated estimate is adopted by iteratively estimating  $\alpha_n^{(i)}(\omega_k)$  and  $\mathbf{Q}_L^{(i)}(n)$  in place of  $[\mathbf{R}_L^{(i)}(n)]^{-1}$  as

$$\alpha_n^{(i)}(\omega_k) = \frac{\mathbf{f}_L^H(\omega_k) \mathbf{Q}_L^{(i)}(n) \mathbf{x}_L^{(i)}(n)}{\mathbf{f}_L^H(\omega_k) \mathbf{Q}_L^{(i)}(n) \mathbf{f}_L(\omega_k)}, \quad (19)$$

$$\mathbf{R}_M^{(i)}(n) = \sum_{k=0}^{K-1} |\alpha_n^{(i)}(\omega_k)|^2 \mathbf{f}_M(\omega_k) \mathbf{f}_M^H(\omega_k) \quad (20)$$

until practical convergence, where

$$\mathbf{Q}_L^{(i)}(n) = \begin{bmatrix} \mathbf{0} & \mathbf{0}^T \\ \mathbf{0} & [\mathbf{R}_M^{(i)}]^{-1} \end{bmatrix} + \mathbf{A}_{L,L-M}^{(i)}(n) \mathbf{A}_{L,L-M}^{(i)H}(n)$$

with

$$\mathbf{A}_{L,L-M}^{(i)}(n) \triangleq [\mathbf{Z}_L \bar{\mathbf{a}}_L^{(i)}(n) \dots \mathbf{Z}_L^{L-M} \bar{\mathbf{a}}_L^{(i)}(n)],$$

whereas

$$\bar{\mathbf{a}}_L^{(i)}(n) = [\hat{\mathbf{a}}_M^{(i)T}(n) \mathbf{0}_{L-M}^T]^T$$

and where  $\mathbf{R}_M^{(i)}(n)$  is the auto correlation matrix of order  $M$ . Using  $M \ll L$ , a significant computation reduction can then be achieved, at the expense of a minor degradation in the quality of the resulting spectrum estimate. The LD algorithm is subsequently employed for the computation of the generators of  $[\mathbf{R}_M^{(i)}]^{-1}$ . Due to the special structure of  $\mathbf{Q}_L^{(i)}$ , the required matrix vector products can be computed using FFT based schemes. We denote the resulting time recursive approximative algorithm the TR-QN-IAA MSC algorithm. The overall complexity of this approach is roughly  $\mathcal{C}_{MSC}^{TR-QN-IAA} \approx$

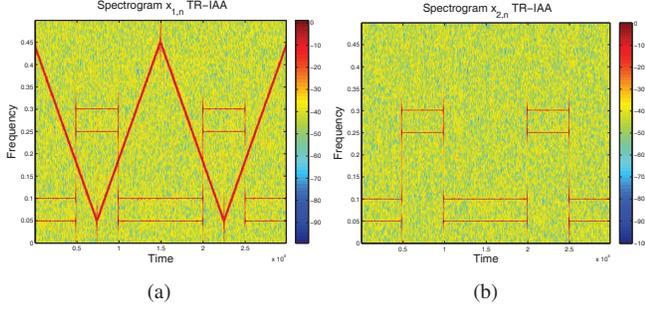


Fig. 1. Spectrogram estimates of the two examined signal.

$2M^2 + 10\phi(N) + 12\phi(2M) + 58\phi(2L_R) + 7\phi(K)$ , being substantially lower than what is required by the TR-IAA MSC, especially when  $M \ll L$ .

### 3. MSC ESTIMATION IN THE MISSING DATA CASE

The time-recursive estimators presented so far have been designed for evenly sampled data. However, in a wide range of applications the measured data could be unevenly sampled, or might suffer from lost samples, here referred to as the missing samples MSC estimation case. By extending the algorithms presented above and utilizing the work presented in [10], where efficient batch processing implementation of IAA spectral estimation in the case when some data are not available have been proposed, we proceed to formulate a fast time-recursive IAA-MS C method for case when input data sets contain missing data samples at arbitrarily (but known) selected positions. Denote the vectors of the available data at time instant  $n$

$$\mathbf{x}_{L_g^i(n)}^{(i)}(n) = \mathbf{S}_{L_g^i(n),L} \mathbf{x}_L^{(i)}(n) \quad (21)$$

for  $i = 1, 2$ , where  $\mathbf{S}_{L_g^i(n),L}$  are time varying selection matrices with zeros and ones in proper places, of time varying dimensions  $L_g^i(n) \times L$ , where  $\mathbf{S}_{L_g^i(n),L} \mathbf{S}_{L_g^i(n),L}^T = \mathbf{I}_{L_g^i(n)}$ , with  $L_g^i(n) \leq L$  denoting the number of the available data samples in  $\mathbf{x}_L^{(i)}(n)$ . Defining the frequency vector accordingly as

$$\mathbf{f}_{L_g^i(n)}(\omega_k) = \mathbf{S}_{L_g^i(n),L} \mathbf{f}_L(\omega_k),$$

the missing data IAA (MIAA) spectral estimation algorithm may then form by iterating

$$\alpha_n^{(i)}(\omega_k) = \frac{\mathbf{f}_{L_g^i(n)}^H(\omega_k) [\mathbf{R}_{L_g^i(n)}^{(i)}(n)]^{-1} \mathbf{x}_{L_g^i(n)}^{(i)}(n)}{\mathbf{f}_{L_g^i(n)}^H(\omega_k) [\mathbf{R}_{L_g^i(n)}^{(i)}(n)]^{-1} \mathbf{f}_{L_g^i(n)}(\omega_k)}$$

$$\mathbf{R}_{L_g^i(n)}^{(i)}(n) = \sum_{k=0}^{K-1} |\alpha_n^{(i)}(\omega_k)|^2 \mathbf{f}_{L_g^i(n)}(\omega_k) \mathbf{f}_{L_g^i(n)}^H(\omega_k)$$

until practical convergence, with  $\mathbf{R}_{L_g^i(n)}^{(i)}(n)$  initialized by the

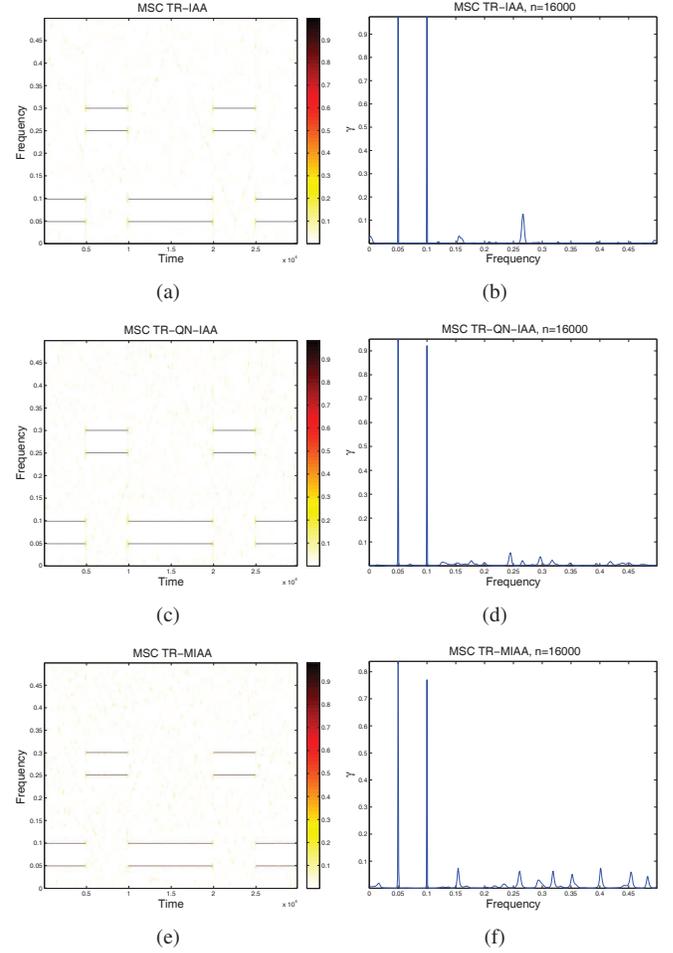


Fig. 2. The evolution of the MSC estimates as well as snapshots of the MSC estimates at time instant  $n = 16000$  for the (a,b) TR-IAA MSC, (c,d) the TR-QN-IAA MSC, and (e,f) the TR-MIAA MSC estimates.

identity matrix. Fast implementations of the MIAA algorithm have been proposed for the batch case in [10, 16], exploiting that the relevant auto-correlation matrices may be extracted from the corresponding full size counterparts, which for the case of time varying processing considered in this work, takes the form

$$\mathbf{R}_{L_g^i(n)}^{(i)} = \mathbf{S}_{L_g^i(n),L} \mathbf{R}_L^{(i)}(n) \mathbf{S}_{L_g^i(n),L}^T, \quad (22)$$

whereas  $\mathbf{R}_L^{(i)}(n)$  is estimated using (9) as if it had been associated with the full data case. A time recursive scheme can thus be constructed using the estimate of  $\mathbf{R}_L^{(i)}(n-1)$  at time index  $(n-1)$  for the initialization of the MIAA algorithm at the successive time index  $n$ . As a by product, the MIAA algorithm is capable of producing, at each time instant, estimates for the full sized correlation matrices, which in conjunction with (10) and (7) are utilized further for the MSC estimation, eventually implemented efficiently as discussed above, result-

ing in a time recursive algorithm, hereafter termed the *time recursive MIAA based MSC estimator* (TR-MIAA MSC), which can be implemented at a cost of approximately  $C_{MSC}^{TR-MIAA} \approx (L_g^1(n))^3 + (L_g^2(n))^3 + 58\phi(2L_R) + 7\phi(K)$  operations.

#### 4. NUMERICAL EXAMPLES

In order to evaluate the performance of the proposed algorithms, we examine two time-varying signals consisting of mixtures of complex sinusoids corrupted by additive zero-mean complex Gaussian noise. The first of these signals is composed by two complex sinusoids of abruptly changing frequencies and a complex valued linear chirp with descending/ascending linear frequency variation, whereas the second signal is composed by complex sinusoids of abruptly changing frequencies (see [11] for a more detailed description of these signals). The amplitude of each signal component is set equal to one, and the phase is selected randomly. The signal-to-noise ratio (SNR) is set to 20 dB. The spectrogram of each of signals, estimated by means of the TR-IAA algorithm is depicted in Fig. 1 (a) and (b). The MSC estimate obtained using the proposed TR-IAA MSC is depicted in Fig. 2 (a), with parameters set at values  $L = 128$ ,  $L_R = 32$  and  $K = 1000$ . The TR-QN-IAA MSC estimate is shown in Fig. 2 (c), where the additional parameter is set equal to  $M = 32$ . We proceed to examine the performance of the proposed TR-MIAA MSC estimate in the missing data case, by randomly (with a uniform distribution) omitting 70% and 75% samples for each input signal,  $x_{1,n}$  and  $x_{2,n}$ , respectively, where, as shown in Fig. 2 (e), despite the heavy data loss, the proposed method succeed in obtaining MSC estimates similar to those obtained in the full data case. Finally, snapshots of the MSC estimates at time instant  $n = 16000$  are shown in Fig. 2 (b), (d) and (f) for the TR-IAA MSC, the TR-QN-IAA MSC and the TR-MIAA MSC method, respectively, indicating the difference in performance between the different approximations.

#### 5. REFERENCES

- [1] J. Benesty, J. Chen, and Y. Huang, "A Generalized MVDR Spectrum," *IEEE Signal Processing Letters*, vol. 12, no. 12, pp. 827–830, December 2005.
- [2] A. Jakobsson, S. R. Alty, and J. Benesty, "Estimating and Time-Updating the 2-D Coherence Spectrum," *IEEE Transactions on Signal Processing*, vol. 55, no. 5, pp. 2350–2354, May 2007.
- [3] M. Zhou, C. Zheng, and X. Li, "On the relationship of non-parametric methods for coherence function estimation," *Signal Processing*, vol. 88, pp. 2863–2867, 2008.
- [4] N. R. Butt and A. Jakobsson, "Coherence Spectrum Estimation From Nonuniformly Sampled Sequences," *IEEE Signal Processing Letters*, vol. 17, no. 4, pp. 339–342, April 2010.
- [5] T. Yardibi, J. Li, P. Stoica, M. Xue, and A. B. Baggeroer, "Source Localization and Sensing: A Nonparametric Iterative Approach Based on Weighted Least Squares," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 46, no. 1, pp. 425–443, January 2010.
- [6] P. Stoica, J. Li, and J. Ling, "Missing Data Recovery via a Nonparametric Iterative Adaptive Approach," *IEEE Signal Processing Letters*, vol. 16, no. 4, pp. 241–244, April 2009.
- [7] K. Angelopoulos, G. O. Glentis, and A. Jakobsson, "Efficient Implementation of the IAA-based Magnitude Squared Coherence Estimator," in *International Conference on Digital Signal Processing*, Corfu, July 6-8 2011.
- [8] K. Angelopoulos, G. O. Glentis, and A. Jakobsson, "Computationally Efficient Coherence Spectrum Estimation," submitted.
- [9] T. Kailath and A. H. Sayed, *Fast Reliable Algorithms for Matrices with Structure*, SIAM, Philadelphia, USA, 1999.
- [10] G.-O. Glentis and A. Jakobsson, "Efficient Implementation of Iterative Adaptive Approach Spectral Estimation Techniques," *IEEE Trans. Signal Processing*, vol. 59, no. 9, pp. 4154–4167, Sept. 2011.
- [11] G.-O. Glentis and A. Jakobsson, "Time-Recursive IAA Spectral Estimation," *IEEE Signal Processing Letters*, vol. 18, no. 2, pp. 111–114, Feb. 2011.
- [12] M. Xue, L. Xu, and J. Li, "IAA Spectral Estimation: Fast Implementation using the Gohberg-Semencul Factorization," *IEEE Trans. Signal Processing*, vol. 59, no. 7, pp. 3251 – 3261, July 2011.
- [13] G.-O. Glentis, "A Fast Algorithm for APES and Capon Spectral Estimation," *IEEE Transactions on Signal Processing*, vol. 56, no. 9, pp. 4207–4220, Sept. 2008.
- [14] G.-O. Glentis and A. Jakobsson, "Superfast Approximative Implementation of the IAA Spectral Estimate," *IEEE Trans. Signal Processing*, vol. 60, no. 1, pp. 472–478, Jan. 2012.
- [15] G. V. Moustakides and S. Theodoridis, "Fast Newton Transversal Filters - A New Class of Adaptive Estimation Algorithms," *IEEE Trans. Signal Processing*, vol. 39, no. 10, pp. 2184–2193, Oct. 1991.
- [16] D. Vu, L. Xu, M. Xue, and J. Li, "Nonparametric Missing Sample Spectral Analysis and Its Applications to Interrupted SAR," *IEEE Journal of Selected Topics in Signal Processing*, vol. 6, no. 1, pp. 1–14, Feb. 2012.