PHASE SPACE ANALYSIS OF THE HARMONIC OSCILLATOR WITH LÉVY NOISE: SPECTRAL MEASURE AND DEVIATION FROM ELLIPTICITY

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ABSTRACT

We show that the location and velocity of the harmonic oscillator with Lévy-stable noise are mutually Lévy-stable distributed. We give explicitly the associated spectral measure, exhibiting both the non independence and the non ellipticity of the location-velocity couple. We then propose measures of deviation from ellipticity.

1. INTRODUCTION

In a recent paper [1], Sokolov et al. study the behavior of the harmonic oscillator with Lévy-stable noise. They compute the characteristic function in the location-velocity phase space, what allows them to show numerically that, except in the Gaussian case, the location and the velocity of the oscillator are not independent. They also show numerically that the joint distribution is not elliptically distributed either. Finally they provide a measure of dependence between these components. The aim of our study is twofold: (i) we provide a more direct proof that the joint distribution of these components is itself Lévy-stable (this result by itself is not new: see for example [2]) and give explicitly the spectral measure associated with this distribution; (ii) this explicit expression of the spectral measure allows to prove that, as claimed in [1], the components are not independent and also that their joint distribution is not elliptical. Finally, we propose possible measures of deviation from ellipticity and provide some numerical illustrations.

2. BASICS ABOUT LÉVY-STABLE RANDOM VECTORS

2.1 Definition and characteristic function

By definition, a random vector $\mathbf{X}$ of $\mathbb{R}^d$ is Lévy-stable (or α-stable) distributed if, for any positive real numbers a and b, there is a positive number c and a constant vector $\mathbf{d}$ so that $a\mathbf{X}_{(1)} + b\mathbf{X}_{(2)} \stackrel{\mathrm{d}}{=} c\mathbf{X} + \mathbf{d}$ where $\mathbf{X}_{(1)}$ and $\mathbf{X}_{(2)}$ are independent copies of $\mathbf{X}$ and where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution [3, Def. 2.1.1]. Except for the well known Gaussian, Cauchy or Lévy random vectors, explicit forms of the Lévy-stable probability density functions (pdf) involve elaborated combinations of special functions that make them difficult to handle: an extensive review of these explicit forms can be found in [4]. However, their characteristic functions $\Phi_{\mathbf{X}}(\mathbf{u}) = E[e^{\imath \mathbf{u}^t \mathbf{X}}]$ have a closed form expression. In the following, $\imath = \sqrt{-1}$ and $\mathbf{a}^t$ stands for the transpose of vector $\mathbf{a}$.

The classical representation of the characteristic function of a d-dimensional Lévy-stable real random vector $\mathbf{X}$ reads [3, th. 2.3.1]

$$\Phi_{\mathbf{X}}(\mathbf{u}) = \exp\left(-\int_{\mathbb{S}_d} \left|\mathbf{u}^t \mathbf{s}\right|^{\alpha} \left(1 + \imath \eta_\alpha\left(\mathbf{u}^t \mathbf{s}\right)\right) \Lambda(d\mathbf{s}) + \imath \mathbf{u}^t \mu\right) \quad (1)$$

where $\alpha \in [0\,;2]$ is called the stability index, $\mathbb{S}_d = \{\mathbf{u} \in \mathbb{R}^d : \|\mathbf{u}\| = 1\}$ denotes the unit sphere in $\mathbb{R}^d$ and Λ a finite measure on $\mathbb{S}_d$ called the *spectral measure*. Moreover the vector μ in (1) is a location parameter (except for $\alpha \neq 1$ where the term in the asymmetry function η_α contributes to the location parameter [3, ex. 2.3.4]), and the asymmetry function $\eta_\alpha(u)$ reads

$$\eta_\alpha(u) = \begin{cases} -\operatorname{sign}(u) \tan\left(\dfrac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1 \\[2ex] \dfrac{2}{\pi} \operatorname{sign}(u) \log|u| & \text{if } \alpha = 1. \end{cases}$$

The couple (Λ, μ) is unique and the value of the asymmetry function η_α is irrelevant in the Gaussian case $\alpha = 2$.

We note that in the univariate case $d = 1$, the unit sphere $\mathbb{S}_1$ reduces to the discrete set $\{1, -1\}$ so that the characteristic function $\Phi_{\mathbf{X}}$ drastically simplifies [3, Def. 1.1.6 & Ex. 2.3.3] to

$$\Phi_X(u) = \exp\left(-\sigma^\alpha |u|^\alpha \left(1 + \imath \beta \eta_\alpha(u)\right) + \imath u \mu\right) \quad (2)$$

where $\sigma = (\Lambda(\{1\}) + \Lambda(\{-1\}))^{\frac{1}{\alpha}}$ represents the scale parameter and $\beta = \frac{\Lambda(\{1\}) - \Lambda(\{-1\})}{\Lambda(\{1\}) + \Lambda(\{-1\})} \in [-1\,;\,1]$ the skewness of the Lévy-stable random variable X.

An equivalent representation for the characteristic function of a Lévy-stable random vector is as follows

$$\Phi_{\mathbf{X}}(\mathbf{u}) = \exp\left(-\int_{\mathscr{D}} \left|\mathbf{u}^t \mathbf{G}(\mathbf{y})\right|^{\alpha} \left(1 + \imath \beta(\mathbf{y}) \eta_\alpha\left(\mathbf{u}^t \mathbf{G}(\mathbf{y})\right)\right) \times M(d\mathbf{y}) + \imath \mathbf{u}^t \mu_0\right) \quad (3)$$

where M is a finite measure, $\mathscr{D}$ is an M-measurable space, $\beta : \mathscr{D} \mapsto [-1\,;1]$ a measurable function and $\mathbf{G} : \mathscr{D} \mapsto \mathbb{R}^d$ a measurable function such that $\int_{\mathscr{D}} \|\mathbf{G}(\mathbf{y})\|^\alpha M(d\mathbf{y}) < +\infty$ for any α and moreover $\int_{\mathscr{D}} \mathbf{G}(\mathbf{y}) \log \|\mathbf{G}(\mathbf{y})\| M(d\mathbf{y}) < +\infty$ if $\alpha = 1$ (see [3, §3.2]).

A third useful representation for the characteristic function of a Lévy-stable vector is the hyperspherical representation. We first remark that when the spectral measure admits

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a density with respect to the Haar measure on the sphere, it can be denoted as λ so that $\Lambda(d\mathbf{s}) = \lambda(\mathbf{s})\mathscr{A}(d\mathbf{s})$ where $\mathscr{A}(\cdot)$ denotes the surface element of the sphere. Vector $\mathbf{s}$ in (1) can be represented by its hyperspherical coordinates, namely $s_j(\theta) = \left(\prod_{k=1}^{j-1}\sin\theta_k\right)\cos\theta_j$ where $\theta = [\theta_1 \ \dots \ \theta_{d-1}]^t \in \mathscr{D}_\theta = [0;\pi)^{d-2}\times[0;2\pi)$, where the non-existing angle θ_d is set to zero by convention and where for $j=1$ the empty product is unity by convention. With a slight abuse of notation, we will denote $\lambda(\theta) = \lambda(s(\theta))$ while changing to hyperspherical coordinates makes appear the Jacobian of the transformation $J(\theta) = \prod_{k=1}^{d-2}\sin^{d-k-1}\theta_k$ in the expression (for $d=2$ the Jacobian is unity).

2.2 Some properties

We first remark that Lévy-stable random vectors admit no covariance matrix when $\alpha < 2$. This, together with the stability property, explains the frequent use of such vectors for impulsive noise modeling. Trivially, when $\alpha = 2$, the characteristic function has the form $\Phi_{\mathbf{X}}(\mathbf{u}) = \exp\left(-\frac{1}{2}\mathbf{u}^t\mathbf{R}\mathbf{u}\right)$, where $\mathbf{R} = \int_{\mathbb{S}_d}\mathbf{s}\mathbf{s}^t\Lambda(d\mathbf{s})$.

A Lévy-stable random vector has independent components if and only if its spectral measure is discrete and concentrated on the intersection of the axes with the sphere $\mathbb{S}_d$ [3, Ex. 2.3.5].

When the location parameter $\mu = \mathbf{0}$ and the measure Λ is symmetric, the distribution is said symmetric Lévy-stable and the asymmetry function η_α vanishes in (1) [3, Th. 2.4.3]. This symmetry should be named centro-symmetry, i.e. symmetry with respect to the origin. In the hyperspherical representation, centro-symmetry is expressed by the invariance of density λ under the symmetry $\theta_1 \to \pi - \theta_1$. Note that the probability density is then also centro-symmetric.

When $\alpha < 2$, a Lévy-stable random vector $\mathbf{X}$ is spherically symmetric (or rotationally invariant) if and only if $\mu = \mathbf{0}$ and its spectral measure is uniform, that is $\Lambda(d\mathbf{s}) \propto \mathscr{A}(d\mathbf{s})$. Thus, the characteristic function is a function of the norm $\|\mathbf{u}\|$ only, of the form[1] $\Phi_{\mathbf{X}}(\mathbf{u}) = \exp(-\sigma^\alpha\|\mathbf{u}\|^\alpha)$ [3, §2.6 & Prop. 2.5.5].

Finally, a vector is said elliptically distributed if its iso-probability contours are ellipsoids. Equivalently, the iso-characteristic function contours are also ellipsoids, and one can show that the characteristic function is [3, Chap. 2]

$$\Phi_{\mathbf{X}}(\mathbf{u}) = \exp\left(-\left(\mathbf{u}^t\mathbf{R}\mathbf{u}\right)^{\frac{\alpha}{2}}\right) \tag{4}$$

where $\mathbf{R}$ is a symmetric definite positive matrix. The spectral measure of a Lévy-stable elliptical distribution is studied in [3, Prop. 2.5.8] in a very formal way as a composition of measures. The existence of each measure is proved but the global measure or its density is not expressed explicitly. However, we prove here that a more detailed result can be obtained, namely: when $\alpha < 2$, the Lévy-stable vector $\mathbf{X}$ is elliptical if and only if there exists a symmetric definite positive matrix $\mathbf{R}$ such that the spectral density of $\mathbf{X}$ has the form

$$\lambda(\mathbf{s}) \propto \frac{1}{|\mathbf{R}|^{\frac{1}{2}}\left(\mathbf{s}^t\mathbf{R}^{-1}\mathbf{s}\right)^{\frac{\alpha+d}{2}}} \tag{5}$$

[1] Note the presence of the scaling factor $2^{-\frac{\alpha}{2}}$ in [3, Prop. 2.5.5]. We chose here to omit this factor to remain consistent with the scalar definition (2).

We skip here the details of the proof.

Let us now turn to the study of the joint distribution of the location and velocity of the harmonic oscillator submitted to a Lévy-stable excitation.

3. HARMONIC OSCILLATOR EXCITED BY LÉVY-STABLE NOISE

3.1 Notations and normalization

A damped harmonic oscillator submitted to a random force is described by the fundamental equation of dynamics that, after normalization, reads

$$\ddot{x}(t) = -2\zeta\omega_0\dot{x}(t) - \omega_0^2x(t) + \xi(t) \tag{6}$$

where ω_0 is the undamped pulsation and ζ the damping ratio, assumed here strictly positive (we exclude the unstable pure oscillator). The external force $\xi(t)$ will be assumed to be a random process with independent and identically distributed Lévy-stable samples , i.e. $\forall t \neq t'$, $\xi(t)$ and $\xi(t')$ are independent and have the same characteristic function (2). Without loss of generality, we assume a null location parameter μ and a unit scaling parameter σ. As a consequence, for any set of epochs $t_1 \le t_2 \le \dots$, the increments $\int_{t_i}^{t_{i+1}}\xi(t)dt$ are independent with characteristic function (2), scale parameter $\sigma = (t_{i+1}-t_i)^{\frac{1}{\alpha}}$ and location parameter $\mu = 0$; we note that the integral $\int_{t_i}^{t_{i+1}}\xi(t)dt$ is a Lévy motion, see [3, Chap 3].

In the case where the noise $\xi(t)$ is Gaussian, the distribution of the phase-space is well-known to be bivariate Gaussian, thus showing two remarkable properties: it is elliptically invariant (and thus centro-symmetric), and with independent components. In the case of a Lévy-stable noise $\xi(t)$ with parameter $0 < \alpha < 2$, it is shown without proof in [1] that both previous properties are lost. As a consequence of Maxwell-Hershell's theorem [5, Prop. 4.11], we know that both properties can in fact hold simultaneously in the Gaussian case only. Nevertheless, like in the Gaussian case, the joint distribution of the harmonic oscillator with Lévy-stable noise is jointly Lévy-stable again, as shown in [1, 2]. Here, we will compute explicitly the spectral measure that characterizes this bivariate Lévy-stable distribution. As a consequence, this proves that the bivariate distribution of the components (x,v) can neither have independent components, nor be elliptically invariant. However, it remains centro-symmetric.

3.2 Solution of the equation

Using the Laplace transform, the solution of equation (6) is found to be

$$\mathbf{x}(t) = \int_{-\infty}^{t}\mathbf{G}(t-t')\,\xi(t')\,dt' \tag{7}$$

where $\mathbf{G}(t) = [G_x(t)\,;\,G_v(t)]^t$ is the vector of the Green functions of the location $G_x(t)$ and of the velocity $G_v(t) = \frac{d}{dt}G_x(t)$, that reads

$$\mathbf{G}(t) = \frac{\mathrm{e}^{-\zeta\omega_0 t}\,\mathbb{1}_{\mathbb{R}_+}(t)}{\omega_c}\left[\begin{array}{c}\sin(\omega_c t)\\ -\zeta\omega_0\sin(\omega_c t) + \omega_c\cos(\omega_c t)\end{array}\right]$$

where the (complex) pulsation ω_c of the oscillator reads

$$\omega_c = \omega_0\sqrt{1-\zeta^2} \tag{8}$$

and $\mathbb{1}_A$ is the indicator function of the set A. In the underdamped regime $\zeta < 1$, the solution is a damped sinusoidal function whereas it is exponentially decreasing in the overdamped regime $\zeta > 1$. The critical case $\zeta = 1$ is recovered by taking the limit $\zeta \to 1$ in either of the preceding regimes.

It can be shown that the bivariate characteristic function $\Phi_{\mathbf{X}}$ is given by

$$\Phi_{\mathbf{X}}(\mathbf{u}) = \exp\left(-\int_{\mathbb{R}_+} \left|\mathbf{u}^t\mathbf{G}(t)\right|^{\alpha}\left(1+\imath\beta\eta_\alpha\left(\mathbf{u}^t\mathbf{G}(t)\right)\right)dt\right) \quad (9)$$

This result can be obtained by considering the limit of the discretized integrals (7), that let appear the independent increments with scale parameter $\delta t^{\frac{1}{\alpha}}$ (see e.g. [1] or [3, chap. 3] and [2] for a more rigorous proof). With the Green functions above mentioned, one can check that since $\zeta \neq 0, \forall\, 0 < \alpha < 2$, $\int_{\mathbb{R}_+} \|\mathbf{G}(t)\|^{\alpha} dt < +\infty$ and moreover that $\int_{\mathbb{R}_+} \mathbf{G}(t)\log\|\mathbf{G}(t)\|\, dt < +\infty$. Thus, the couple $(x(t), v(t))$ is Lévy-stable distributed for all t, as appears by choosing the integration set $\mathscr{D} = \mathbb{R}_+$, the integration variable $\mathbf{y} = t$, the Lebesgue measure $M(dt) = dt$, the location parameter $\mu_0 = \mathbf{0}$ and a constant skewness β in (3).

In the sequel, we will concentrate on the symmetric case $\beta = 0$. In order to compute explicitly the spectral measure given formally in [3, §3.2], we normalize the Green vector $\mathbf{G}$ and make the appropriate changes of variable to write the integral in (3) using the angular variable θ. Some algebra leads to

$$\begin{aligned}\lambda(\theta) = &\frac{1}{\left(\omega_0^2\cos^2\theta + \sin^2\theta + 2\zeta\omega_0\cos\theta\,\sin\theta\right)^{\frac{\alpha}{2}+1}}\\ &\times\frac{1}{2}\mathrm{e}^{-\frac{\zeta\omega_0}{\omega_c}\arctan\left(\frac{\omega_c}{\zeta\omega_0+\tan\theta}\right)} D(\theta)\end{aligned} \quad (10)$$

with, in the underdamped case,

$$\begin{aligned}D(\theta) \quad = \quad &\frac{\exp\left(-\frac{\pi\alpha\zeta\omega_0}{2\omega}\right)}{2\sinh\left(\frac{\pi\alpha\zeta\omega_0}{2\omega}\right)}\\ &+\mathbb{1}_{\left(-\arctan(\zeta\omega_0);\,\frac{\pi}{2}\right)\cup\left(\pi-\arctan(\zeta\omega_0);\,\frac{3\pi}{2}\right)}(\theta)\end{aligned}$$

and in the overdamped case,

$$D(\theta) = \mathbb{1}_{\left(-\arctan(\zeta\omega_0-\omega),\,\frac{\pi}{2}\right)\cup\left(\pi-\arctan(\zeta\omega_0-\omega),\,\frac{3\pi}{2}\right)}(\theta).$$

The critical case is obtained by taking the limit $\zeta \to 1$. Note that the support of $\lambda(\theta)$ is not the entire circle $\mathbb{S}_2$ in both critical and overdamped regimes: clearly, in both cases, the coordinates (x, v) cannot be elliptically distributed.

4. PROPERTIES OF THE COORDINATES OF THE HARMONIC OSCILLATOR EXCITED BY LÉVY-STABLE NOISE

Clearly, the measure Λ has a density and thus does not concentrate on the intersection of the axes with the sphere; thus the components are not independent. To assess quantitatively the dependence between these components, different measures can be used. In their study, Sokolov et al. propose various ad hoc measures of dependence, such as a covariation measure (similar to a covariance, but with fractional order moments since Lévy-stable distributions do not admit covariance). Here, we will concentrate on the other property of the distribution in the Gaussian case: ellipticity.

We note that the spectral measure λ satisfies $\lambda(\theta) = \lambda(\theta \pm \pi)$: in other words, the centro-symmetry property observed when the input of the harmonic oscillator is Gaussian is preserved in the Lévy-stable context. However, ellipticity is lost. Indeed, it is clear that none of the expressions (10) of the spectral measure can be put under the form (4) in any regime; thus the joint distribution of the coordinates (x, v) is not elliptical. However, we propose to characterize more precisely the deviation from ellipticity.

First, from (4), assuming ellipticity, the characteristic function in the hyperspherical domain should be proportional to $\left(a^2\cos^2\theta + b^2\sin^2\theta + 2\rho ab\cos\theta\,\sin\theta\right)^{-\frac{\alpha}{2}-1}$ where the corresponding matrix $\mathbf{R}^{-1} = \begin{bmatrix} a^2 & \rho ab \\ \rho ab & b^2\end{bmatrix}$ is definite positive. Looking at (10) shows that the spectral measure is indeed of this form up to a non-constant "modulation" term, the "elliptical" term being

$$\lambda_{\mathrm{ed}} \propto \frac{1}{(\mathbf{s}^t\mathbf{M}_{\mathrm{ed}}\mathbf{s})^{\frac{\alpha}{2}+1}} \quad \text{with} \quad \mathbf{M}_{\mathrm{ed}} = \begin{bmatrix} \omega_0^2 & \zeta\omega_0 \\ \zeta\omega_0 & 1\end{bmatrix}. \quad (11)$$

However, the matrix $\mathbf{M}_{\mathrm{ed}}$ is positive definite if and only if $\zeta < 1$. For $\zeta \geq 1$, the result is not contradictory with the necessary positivity of λ: indeed, in this case, when $\lambda_{\mathrm{ed}} \leq 0$ the "modulation" term vanishes. In conclusion, it is clear that the distribution of (x, v) is not elliptical neither in the overdamped context, nor in the critical case. Obviously, due to the "modulation" term , the distribution of (x, v) is not elliptical in the overdamped case either; but in this case, λ_{ed} can be viewed as a spectral measure and the "elliptical" part of the distribution may be studied.

In all cases, one can also examine further the elliptical-like part λ_{ed} of the spectral measure. From (11) one can determine the symmetry axes of λ_{ed}. The symmetry directions $\theta_{\mathrm{ed}}^{\mathrm{m}}$ and $\theta_{\mathrm{ed}}^{\mathrm{M}}$, given by the eigenvectors of $\mathbf{M}_{\mathrm{ed}}$, are

$$\theta_{\mathrm{ed}}^{\mathrm{m,M}} = \arctan\left(\frac{1-\omega_0^2}{2\zeta\omega_0} \pm \sqrt{1+\left(\frac{1-\omega_0^2}{2\zeta\omega_0}\right)^2}\right). \quad (12)$$

At the opposite, for elliptical characteristic functions Φ, the unit vector $\mathbf{s} = \mathbf{s}_m$ (resp. $\mathbf{s} = \mathbf{s}_M$) that minimizes (resp. maximizes) $\Phi(\mathbf{s})$ coincides with the long (resp. short) axis of the constant characteristic function ellipsoids. For the harmonic oscillator, although not elliptically distributed, one can then define directions as

$$\theta_{\mathrm{ho}}^{\mathrm{m,M}} = \operatorname*{argmax}_{\theta}\left(\pm\int_{\mathbb{S}_2}|\mathbf{u}^t(\theta)\,\mathbf{s}|^{\alpha}d\mathbf{s}\right) \quad \text{with } \mathbf{u} \in \mathbb{S}_2. \quad (13)$$

A measure of the distance between the axes θ_{ho} and θ_{ed} should also be considered as a measure of the distance of the distribution to ellipticity. However, one must keep in mind that one can find a zero distance while the distribution is not elliptical. Moreover, since (x, v) is not elliptical, there is no reason why the axes $\theta_{\mathrm{ho}}^{\mathrm{m,M}}$ should be symmetry axes (indeed they are not). And there is no reason why these axes should be orthogonal either.

One can also adopt another point of view to assess the non-ellipticity of the law. Again, for an elliptical distribution, the characteristic function (9) (with $\beta = 0$) must reduce to the form (4). The idea is then to measure the deviation of $\int_{\mathbb{R}_+} |\mathbf{u}^t \mathbf{G}(t)|^\alpha \, dt$ from the function $(\mathbf{u}^t \mathbf{R} \mathbf{u})^{\frac{\alpha}{2}}$. Due to the relation $\int_{\mathbb{R}_+} |(a\mathbf{u})^t \mathbf{G}(t)|^\alpha \, dt = |a|^\alpha \int_{\mathbb{R}_+} |\mathbf{u}^t \mathbf{G}(t)|^\alpha \, dt$, one can restrict $\mathbf{u}$ to be on the unit sphere, $\mathbf{u} = \mathbf{s}(\theta)$. Taking the power $\frac{2}{\alpha}$ of both quantities, the goal is then to measure the distance between the function

$$f(\theta) = \left(\int_{\mathbb{R}_+} \left| \mathbf{s}(\theta)^t \mathbf{G}(t) \right|^\alpha dt \right)^{\frac{2}{\alpha}} \tag{14}$$

and the quadratic form $g(\theta) = \mathbf{s}(\theta)^t \mathbf{R} \mathbf{s}(\theta)$ where we constraint $\mathbf{R}$ to be symmetric. A possible measure of discrepancy that yields an explicit expression for $\mathbf{R}$ is the L^2 distance between f and g: writing $\int_0^{2\pi} (f(\theta) - g(\theta))^2 d\theta$ in terms of the entries of $\mathbf{R}$ and equating to zero the derivatives in the entries of this matrix yields

$$\mathbf{R} = \begin{pmatrix} f_0 + 2f_c & 2f_s \\ 2f_s & f_0 - 2f_c \end{pmatrix} \tag{15}$$

with

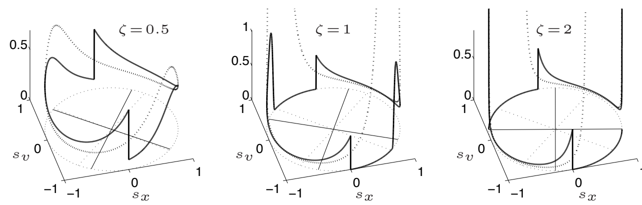
$$\left\{ \begin{array}{lcl} f_0 & = & \dfrac{1}{\pi} \displaystyle\int_0^\pi f(\theta) d\theta \\ f_c & = & \dfrac{1}{\pi} \displaystyle\int_0^\pi f(\theta) \cos(2\theta) d\theta \\ f_s & = & \dfrac{1}{\pi} \displaystyle\int_0^\pi f(\theta) \sin(2\theta) d\theta \end{array} \right. \tag{16}$$

Simple algebra shows that for $\alpha = 2$, matrix $\mathbf{R}$ coincides with twice the covariance matrix of the Gaussian components[2]. In the general case, a natural way to assess the non-ellipticity of the couple (x, v) consists in comparing (10) with (5) in terms of any metric we wish. For example, one can study again the axes of this "best" elliptic approximation of the true distribution,

$$\theta_R^{\mathrm{m,M}} = \arctan \left(-\frac{f_c}{f_s} \pm \sqrt{1 + \left(\frac{f_c}{f_s} \right)^2} \right). \tag{17}$$

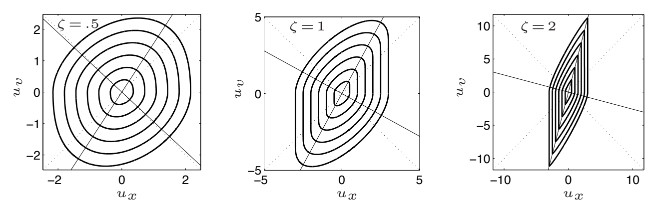
Searching the best least squares elliptic approximation of the spectral density or of the characteristic function is a difficult task. However, a measure of discrepancy from ellipticity can be built as the distance between the characteristic function and that using the elliptic approximation just presented, or any distance involving these quantities.

Figure 1 depicts the spectral measure for the three regimes together with its "elliptical" part λ_{ed} when the input noise of the oscillator is Cauchy ($\alpha = 1$ and $\beta = 0$). The lines represent the axes θ_{ho} and the (symmetry) axes θ_{ed} of the elliptical part. The angles are also reported in table 1. Figure 2 gives the level curves of the characteristic function, or more precisely the curves $L_\lambda(C) = \left\{ \mathbf{u} = [u_x \quad u_v]^t \in \mathbb{R}^2, \int_{\mathbb{S}_d} |\mathbf{u}^t \mathbf{s}|^\alpha \lambda(\mathbf{s}) d\mathbf{s} = C \right\}$ for the true spectral measure and for its elliptical part λ_{ed}, while the axes θ_{ho} and θ_{ed} are also represented. It clearly appears in these figures that if centro-symmetry is conserved, ellipticity is lost. Visually, it is clear in the critical and overdamped cases (divergence of the elliptical part of the spectral density and/or of the spectral density itself). One can also see the presence of a "modulation" factor applied to λ_{ed} in eq. (10). Even the symmetry with respect to axes is lost. Finally, the axes θ_{ho} have no reasons to be orthogonal; in fact they are not as one can check both in the figures and in the table.



(a) (b) (c)

Figure 1: Density of the spectral measure $\lambda(\mathbf{s})$, with $\mathbf{s} = [s_x \quad s_v]^t \in \mathbb{S}_2$ (solid lines in bold), that characterizes the location-velocity Lévy-stable distribution compared to the "elliptical" part $\lambda_{\mathrm{ed}}(\mathbf{s})$ eq. (11) (dotted line in bold). In all cases, $\alpha = 1$ and $\omega_0 = 1$. Curve (a) depicts the underdamped case for $\zeta = .5$, curve (b) the critical case ($\zeta = 1$) and (c) represents the overdamped case with $\zeta = 2$. The circle in dotted line is for indication only. The axes in solid lines represent the direction $\theta_{\mathrm{ho}}^{\mathrm{m,M}}$ eq. (13) while the axes in dotted lines represent $\theta_{\mathrm{ed}}^{\mathrm{m,M}}$ eq. (12).



(a) (b) (c)

Figure 2: Level curves $L_\lambda(C)$ for several values of C. In all cases, $\alpha = 1$ ($\beta = 0$) and $\omega_0 = 1$. Curve (a) depicts the underdamped case ($\zeta = .5$), curve (b) the critical case ($\zeta = 1$) and (c) represents the overdamped case with $\zeta = 2$. The axes $\theta_{\mathrm{ho}}^{\mathrm{m,M}}$ eq. (13) (solid line) and $\theta_{\mathrm{ed}}^{\mathrm{m,M}}$ eq. (12) (dotted line) are shown in the figure.

Figure 3 represents the spectral density, compared to the best elliptical approximation λ_R given from eqs. (5)-(15), together with the θ_{ho} axes and the (symmetry) axes θ_R. It gives the level curves representation of the true characteristic function, of that obtained from the elliptical part (when it makes sense, i.e. in the underdamped case) and of that obtained from the elliptical approximation. These figures confirm visually the discrepancy of the phase space distribution to ellipticity. As expected from the form of λ, it is more pronounced in the critical regime and even more in the underdamped regime.

[2] The scale factor 2 comes from our definition (4), that is consistent with definition (2).

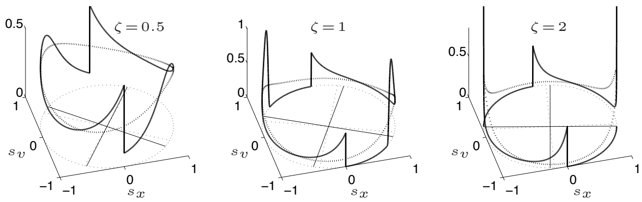


Figure 3: Density of the spectral measure $\lambda(\mathbf{s})$ (solid lines in bold) compared to the elliptical density $\lambda_R(\mathbf{s})$ eqs. (5) (dotted line in bold). The parameters are those of figure 1 and the axes are $\theta_{\mathrm{ho}}^{\mathrm{m,M}}$ eq. (13) (solid line) while the axes in dotted line are now $\theta_R^{\mathrm{m,M}}$ eq. (17).

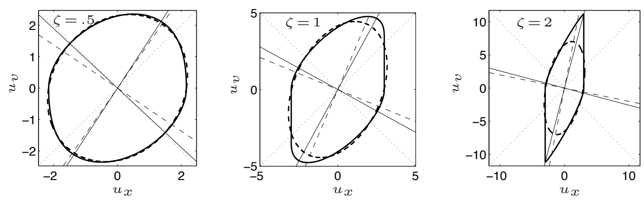


Figure 4: Comparison of the level curves L_λ (solid line), $L_{\lambda_{\mathrm{ed}}}$ (dotted line, when it makes sense, i.e. in the underdamped case), and L_{λ_R} (dashed line). The parameters and axes are those of figure 3.

Figure 5 represents the quadratic error between $\int_{\mathbb{S}_d} |\mathbf{u}^t\mathbf{s}|^\alpha \lambda(\mathbf{s}) d\mathbf{s}$ and $(\mathbf{u}^t\mathbf{R}\mathbf{u})^{\frac{\alpha}{2}}$ where $\mathbf{u}$ varies on the unit circle. In some sense, it measures an error between the level curves $L_\lambda(C)$. This error is plotted as a function of α and of ζ respectively. These figures confirm the previous observations. The loss of ellipticity increases as the input force distribution deviates from Gaussiannity. Similarly, the less the oscillator is damped, the more the distribution is elliptical.

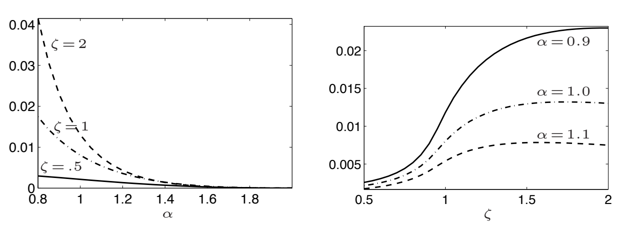


Figure 5: Quadratic error between $\int_{\mathbb{S}_d} |\mathbf{u}^t\mathbf{s}|^\alpha \lambda(\mathbf{s}) d\mathbf{s}$ and $(\mathbf{u}^t\mathbf{R}\mathbf{u})^{\frac{\alpha}{2}}$ where $\mathbf{u}$ varies on the unit circle; (a) as a function of α ($\zeta = .5$ for the solid line, $\zeta = 1$ for the dashed-dotted line, and $\zeta = 2$ for the dashed line), and (b) as a function of ζ ($\alpha = .9$ for the solid line, $\alpha = 1$ for the dashed-dotted line, and $\alpha = 1.1$ for the dashed line). The other parameters and axes are those of figure 3.

	underdamped $\zeta = .5$	critical $\zeta = 1$	overdamped $\zeta = 2$
$\theta_{\mathrm{ho}}^{\mathrm{m}}$	57.3°	61.8°	75.0°
$\theta_{\mathrm{ho}}^{\mathrm{M}}$	−43.4°	−29.1°	−14.5°
$\theta_{\mathrm{ed}}^{\mathrm{m}}$	45.0°	45.0°	45.0°
$\theta_{\mathrm{ed}}^{\mathrm{M}}$	−45.0°	−45.0°	−45.0°
θ_R^{m}	55.5°	67.3°	78.7°
θ_R^{M}	−34.5°	−22.7°	−11.3°

Table 1: Angles corresponding to the maximal directions/symmetry axes: $\theta_{\mathrm{ho}}^{\mathrm{m,M}}$ eq. (13) for the spectral density λ eqs. (10), $\theta_{\mathrm{ed}}^{\mathrm{m,M}}$ eq. (12) for the "elliptical" part λ_{ed} eq. (11) and $\theta_R^{\mathrm{m,M}}$ eq. (17) for the elliptical density λ_R eq. (5). The parameters are those of figure 1

5. CONCLUSION

We have shown that the spectral measure of the harmonic oscillator with Lévy-stable noise can be expressed explicitly in all its regimes. This derivation allows to deduce that (i) the mutual distribution of the location and velocity is again Lévy-stable distributed (ii) except in the Gaussian case, the ellipticity of this distribution is lost (iii) several measures can be used to characterize this discrepancy to ellipticity (axes, error between the true distribution and its elliptical approximation). Further studies include the general n-dimensional case as in [2]: we are interested in the determination of the minimal set of constraints that an n-dimensional linear system should meet so that ellipticity of the output distribution is ensured.

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