NEW CONSTRAINED LEAST SQUARES APPROACH FOR RANGE-BASED POSITIONING

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ABSTRACT

The problem of finding the location of a target based on range measurements from an array of receivers is addressed. In the linear least squares (LLS) approach for range-based positioning, an extra range variable is usually introduced. In this paper, we derive a LLS algorithm with exploiting the known relation between the source position and range variable, which results in a simple constrained optimization problem. The optimality of the proposed algorithm at sufficiently small noise conditions is demonstrated by the theoretical analysis as well as computer simulations.

1. INTRODUCTION

Wireless localization refers to the task of finding the position of a target of interest based on measurements from an array of spatially separated sensors with *a priori* known locations. It has been one of the central problems in many fields such as radar, sonar, telecommunications, mobile communications and sensor networks [1]–[2]. Time-of-arrival (TOA), time-difference-of-arrival (TDOA), received signal strength (RSS) and angle-of-arrival (AOA) are commonly used measurements for source localization. Basically, TOAs, TDOAs and RSSs provide the distance information between the source and receivers while AOAs are the source bearings relative to the receivers. Nevertheless, finding the source position is not a trivial task because the location-bearing measurements are nonlinear functions of the target coordinates. In this work, we focus on range-based positioning with the use of TOA or RSS information.

Positioning algorithms can be classified as nonlinear and linear approaches. The first category deals with the nonlinear equations directly constructed from the range measurements, which includes the nonlinear least squares and maximum likelihood estimators [3]–[4]. Although they can achieve optimal localization accuracy, global convergence of these schemes may not be guaranteed because their optimization cost functions are multi-modal. On the other hand, the second approach provides global solutions because it converts the nonlinear equations to be linear. Linear least squares (LLS) [5]–[11] and subspace [12]–[14] methods are representative examples for the second category. In the LLS approach for range-based positioning, an extra range variable is required in order to produce linear equations. Recently, Zhu and Ding [11] have proposed a computationally efficient closed-form LLS-based position estimator which involves a quadratic equation to relate the extra variable and the position to be estimated. However, its estimation accuracy is suboptimal because the known relationship between the source position and range parameter is not exploited. In this work, our main contributions are to improve [11] by making use of the known relationship according to constrained optimization and prove that the performance of the proposed estimator can achieve Cramér-Rao lower bound (CRLB) at sufficiently small error conditions.

The rest of the paper is organized as follows. In Section 2, an improved version of [11] by utilizing the known relation between the range variable and position coordinates is derived. The resultant constrained optimization problem can be easily solved by the method of Lagrange multipliers. The localization accuracy of the constrained LLS algorithm is analyzed in Section 3, which shows that its performance can achieve CRLB when the measurement noises are sufficiently small. Simulation results are presented in Section 4 to evaluate the proposed method. Finally, conclusions are drawn in Section 5.

2. PROPOSED POSITION ESTIMATOR

Consider an array of $L \geq 3$ receivers in a two-dimensional (2-D) space. Note that extension to three-dimensional space is straightforward. Let $\mathbf{x} = [x\ y]^T$ be the source position to be determined and $\mathbf{x}_l = [x_l\ y_l]^T$, $l = 1, 2, \cdots, L$, be the known coordinates of the lth receiver. The range measurements are

$$r_l = \|\mathbf{x} - \mathbf{x}_l\|_2 + n_l, \quad l = 1, 2, \cdots, L \tag{1}$$

where $\|\cdot\|_2$ is the 2-norm operator and $\{n_l\}$ are the measurement errors. For simplicity but without loss of generality, we assume that $\{n_l\}$ are zero-mean white Gaussian processes with variances $\{\sigma_l^2\}$.

Squaring both sides of (1), we obtain:

$$2\mathbf{x}_l\mathbf{x} = \|\mathbf{x}_l\|_2^2 - r_l^2 + \|\mathbf{x}\|_2^2 + 2d_l n_l + n_l^2, \quad l = 1, 2, \cdots, L \tag{2}$$

where $d_l = \|\mathbf{x} - \mathbf{x}_l\|_2$. Introducing an extra variable $R = \|\mathbf{x}\|_2^2$ yields the following set of linear equations in matrix form [11]:

$$\mathbf{A}\mathbf{x} = \mathbf{b} + \mathbf{h}R + \mathbf{m} \tag{3}$$

where

$$\mathbf{A} = 2\begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_L^T \end{bmatrix} \tag{4}$$

$$\mathbf{b} = \begin{bmatrix} \|\mathbf{x}_1\|_2^2 - r_1^2 \\ \|\mathbf{x}_2\|_2^2 - r_2^2 \\ \vdots \\ \|\mathbf{x}_L\|_2^2 - r_L^2 \end{bmatrix} \tag{5}$$

 - ISSN 2076-1465

$$\mathbf{m} = \begin{bmatrix} 2d_1 n_1 + n_1^2 \\ 2d_2 n_2 + n_2^2 \\ \vdots \\ 2d_L n_L + n_L^2 \end{bmatrix} \tag{6}$$

and $\mathbf{h} = \mathbf{1}_L$ with $\mathbf{1}_L$ being a $L \times 1$ vector of all elements 1.

To enforce the known relationship between R and $\mathbf{x}$ in the LLS approach, a constrained optimization problem is formulated. The corresponding position estimate, denoted by $\widehat{\mathbf{x}}$, is:

$$\widehat{\mathbf{x}} = \arg\min_{\widetilde{\mathbf{x}}} J(\widetilde{\mathbf{x}}, R) \tag{7}$$

$$\text{s.t.} \quad \widetilde{\mathbf{x}}^T \widetilde{\mathbf{x}} = R \tag{8}$$

where

$$J(\widetilde{\mathbf{x}}, R) = (\mathbf{A}\widetilde{\mathbf{x}} - \mathbf{b} - \mathbf{h}R)^T \mathbf{W} (\mathbf{A}\widetilde{\mathbf{x}} - \mathbf{b} - \mathbf{h}R) \tag{9}$$

The $\widetilde{\mathbf{x}} = [\tilde{x}\ \tilde{y}]^T$ is the optimization variable for $\mathbf{x}$ and $\mathbf{W}$ is the weighting matrix. That is to say, [11] performs the unconstrained optimization on (9) while our improvement is to minimize (9) subject to the constraint of (8). If $\{\sigma_l^2\}$ are known, we should use the optimum $\mathbf{W}$, which is computed as $\mathbf{W} = [\mathbb{E}\{\mathbf{m}\mathbf{m}^T\}]^{-1} \approx [\mathbf{BQB}]^{-1}$ [7]–[8] where $\mathbb{E}$ represents the expectation operator, $\mathbf{Q} = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \cdots, \sigma_L^2)$, and $\mathbf{B} = 2\,\mathrm{diag}(r_1, r_2, \cdots, r_L)$ which are obtained by ignoring $\{n_l^2\}$ in $\mathbf{m}$ and employing the approximation of $d_l = r_l, l = 1, 2, \cdots, L$. Otherwise, an appropriate choice for the weighing matrix is $\mathbf{W} = \mathbf{I}_L$ where $\mathbf{I}_L$ denotes a $L \times L$ identity matrix.

The problem of (7)–(8) is in fact equivalent to minimizing the Lagrangian:

$$\begin{aligned} L(\widetilde{\mathbf{x}}, \lambda, R) = (\mathbf{A}\widetilde{\mathbf{x}} - \mathbf{b} - \mathbf{h}R)^T \mathbf{W} (\mathbf{A}\widetilde{\mathbf{x}} - \mathbf{b} - \mathbf{h}R) \\ + \lambda(\widetilde{\mathbf{x}}^T \widetilde{\mathbf{x}} - R) \end{aligned} \tag{10}$$

where λ is the Lagrange multiplier. The position estimate in terms of the unknown λ and R is easily determined as:

$$\widehat{\mathbf{x}} = (\mathbf{A}^T \mathbf{W} \mathbf{A} + \lambda \mathbf{I}_2)^{-1} \mathbf{A}^T \mathbf{W} (\mathbf{b} + \mathbf{h}R) \tag{11}$$

To find λ and R, we substitute (11) into the equality constraint of (8) to get a quadratic equation:

$$aR^2 + bR + c = 0 \tag{12}$$

where

$$a = (\mathbf{A}^T \mathbf{W} \mathbf{h})^T (\mathbf{A}^T \mathbf{W} \mathbf{A} + \lambda \mathbf{I}_2)^{-2} \mathbf{A}^T \mathbf{W} \mathbf{h} \tag{13}$$

$$b = 2(\mathbf{A}^T \mathbf{W} \mathbf{b})^T (\mathbf{A}^T \mathbf{W} \mathbf{A} + \lambda \mathbf{I}_2)^{-2} \mathbf{A}^T \mathbf{W} \mathbf{h} - 1 \tag{14}$$

and

$$c = (\mathbf{A}^T \mathbf{W} \mathbf{b})^T (\mathbf{A}^T \mathbf{W} \mathbf{A} + \lambda \mathbf{I}_2)^{-2} \mathbf{A}^T \mathbf{W} \mathbf{b} \tag{15}$$

If λ is available, we can use the root selection procedure in [11] to solve R. Its main feature is that the solution ambiguity can be removed according to the least squares criterion which is similar to the first stage of the two-step LLS method in [7]. Moreover, since there are two unknown parameters of λ and R in (12), polynomial rooting which is employed in the constrained scheme of [8], is not suitable for our problem. As $J(\widetilde{\mathbf{x}}, R)$ may be multimodal, we suggest to combine bisection search and grid search methods to solve for λ. The estimation procedure of the proposed positioning algorithm is summarized using the following steps:

(i) Grid search. Denote $[-k, k]$ as the admissible range of λ where k is a positive integer and there are a number of grid points in this region. We calculate two roots of R with (12) for each grid point of $\hat{\lambda} \in [-k, k]$. The root selection procedure of [11] is employed to choose one proper root as the range variable estimate, denoted by $\hat{R}$, for each $\hat{\lambda}$. From the investigated groups of $\{\hat{\lambda}, \hat{R}\}$, we find the pair which minimizes the cost function of $J(\widetilde{\mathbf{x}}, R)$, denoted by $\{\hat{\lambda}_{\min}, \hat{R}_{\min}\}$. In doing so, a reduced range of $\hat{\lambda}_{\min} \in [k_1, k_2]$ is obtained where $k_1 >> -k$ and $k_2 << k$.

(ii) Bisection search. Assuming the cost function is unimodal for $\lambda \in [k_1, k_2]$, the bisection method is employed to compute the optimum estimates of λ and R which make $J(\widetilde{\mathbf{x}}, R)$ minimal.

(iii) Substituting the optimum $\hat{\lambda}$ and $\hat{R}$ into (11), $\hat{\mathbf{x}}$ is obtained.

It is worthy to point out that the proposed constrained LLS algorithm is different from the the constrained approach in [7]–[10],[15]. In our formulation as well as [11], $\mathbf{x}$ is considered as a function of R in (3), which is in the right hand side of the equations. On the other hand, the latter methodology formulates R as an unknown parameter to be estimated and it is placed at the same side as $\mathbf{x}$ in its corresponding equations. In theory, the two approaches should provide the same estimation performance. Nevertheless, our approach works properly even for a non-uniform linear array of receivers while the technique of [8]–[9],[15] fail in this case because there exists singular problems in the corresponding system of equations.

3. PERFORMANCE ANALYSIS

The variance of the proposed range-based positioning algorithm is derived as follows. It is assumed that $\{\sigma_l^2\}$ are known and the optimum weighting matrix $\mathbf{W}$ is employed. The constrained optimization problem of (7)–(8) can be transformed to an unconstrained minimization problem by substituting the constraint into the cost function. In doing so, the cost function is now:

$$J(\widetilde{\mathbf{x}}) = \left(\mathbf{A}\widetilde{\mathbf{x}} - \mathbf{b} - \mathbf{h}\widetilde{\mathbf{x}}^T\widetilde{\mathbf{x}}\right)^T \mathbf{W} \left(\mathbf{A}\widetilde{\mathbf{x}} - \mathbf{b} - \mathbf{h}\widetilde{\mathbf{x}}^T\widetilde{\mathbf{x}}\right) \tag{16}$$

Applying the variance formula for unconstrained optimization problems which is valid at sufficiently small noise conditions, the covariance of $\hat{\mathbf{x}}$, denoted by $\mathbf{C_x}$, is [12]:

$$\mathbf{C_x} \approx [\mathbb{E}\{\mathbf{H}(J(\widetilde{\mathbf{x}}))\}]^{-1} \, \mathbb{E}\{\nabla(J(\widetilde{\mathbf{x}}))\nabla^T J(\widetilde{\mathbf{x}})\} \, [\mathbb{E}\{\mathbf{H}(J(\widetilde{\mathbf{x}}))\}]^{-1} \tag{17}$$

where $\mathbf{H}(J(\widetilde{\mathbf{x}})) = \partial^2 J(\widetilde{\mathbf{x}})/\partial\widetilde{\mathbf{x}}\partial\widetilde{\mathbf{x}}^T$ and $\nabla(J(\widetilde{\mathbf{x}})) = \partial J(\widetilde{\mathbf{x}})/\partial\widetilde{\mathbf{x}}$ are the corresponding Hessian matrix and gradient vector evaluated at the true location.

The expected value of Hessian matrix for $J(\widetilde{\mathbf{x}})$ is expressed as:

$$\mathbb{E}\left\{\frac{\partial^2 J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}\widetilde{\mathbf{x}}^T}\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} = \left[\mathbb{E}\left\{\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\right)\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} \quad \mathbb{E}\left\{\frac{\partial}{\partial\tilde{y}}\left(\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\right)\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}}\right] \tag{18}$$

We start with differentiating (16) with respect to $\widetilde{\mathbf{x}}$

$$\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}} = 2\left(\mathbf{A}-2\mathbf{h}\widetilde{\mathbf{x}}^T\right)^T\mathbf{W}\left(\mathbf{A}\widetilde{\mathbf{x}}-\mathbf{b}-\mathbf{h}\widetilde{\mathbf{x}}^T\widetilde{\mathbf{x}}\right) \tag{19}$$

Then differentiating (19) with respect to $\tilde{x}$, we get

$$\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\right) = -4[1\ 0]^T\mathbf{h}^T\mathbf{W}\left(\mathbf{A}\widetilde{\mathbf{x}}-\mathbf{b}-\mathbf{h}\widetilde{\mathbf{x}}^T\widetilde{\mathbf{x}}\right) + 2(\mathbf{A}-2\mathbf{h}\widetilde{\mathbf{x}}^T)^T\mathbf{W}\left(\mathbf{A}[1\ 0]^T-2\mathbf{h}x\right) \tag{20}$$

Substituting the true source location of $\mathbf{x}$ into (20) yields

$$\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\right)\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} = -4[1\ 0]^T\mathbf{h}^T\mathbf{W}\mathbf{m} + 2(\mathbf{A}-2\mathbf{h}\mathbf{x}^T)^T\mathbf{W}\left(\mathbf{A}[1\ 0]^T-2\mathbf{h}x\right) \tag{21}$$

Taking expectation on both sides of (21) and applying the approximation of $\mathbf{m}\approx 2\mathbf{d}\odot\mathbf{n}$ with $\mathbf{d}=[d_1\ d_2\cdots d_L]^T$ and $\mathbf{n}=[n_1\ n_2\cdots n_L]^T$ where $\odot$ is the element-by-element product, which is valid for small error scenarios, we have:

$$\mathbb{E}\left\{\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\right)\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} \approx 2(\mathbf{A}-2\mathbf{h}\mathbf{x}^T)^T\mathbf{W}\left(\mathbf{A}[1\ 0]^T-2\mathbf{h}x\right) \tag{22}$$

Similarly, repeating the derivation in (21)–(22) with the variable $\tilde{y}$ yields

$$\mathbb{E}\left\{\frac{\partial}{\partial\tilde{y}}\left(\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\right)\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} \approx 2(\mathbf{A}-2\mathbf{h}\mathbf{x}^T)^T\mathbf{W}\left(\mathbf{A}[0\ 1]^T-2\mathbf{h}y\right) \tag{23}$$

Substituting (22)–(23) into (18), we get

$$\mathbb{E}\left\{\frac{\partial^2 J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}\widetilde{\mathbf{x}}^T}\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} \approx 2(\mathbf{A}-2\mathbf{h}\mathbf{x}^T)^T\mathbf{W}\left(\mathbf{A}-2\mathbf{h}\mathbf{x}^T\right) \tag{24}$$

In a similar manner, we have:

$$\begin{aligned}\mathbb{E}&\left\{\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\left(\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\right)^T\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} \\ &= 4\left(\mathbf{A}-2\mathbf{h}\mathbf{x}^T\right)^T\mathbf{W}\mathbb{E}\{\mathbf{m}\mathbf{m}^T\}\mathbf{W}\left(\mathbf{A}-2\mathbf{h}\mathbf{x}^T\right) \\ &\approx 4\left(\mathbf{A}-2\mathbf{h}\mathbf{x}^T\right)^T\mathbf{W}\left(\mathbf{A}-2\mathbf{h}\mathbf{x}^T\right)\end{aligned} \tag{25}$$

With the use of (24) and (25), it is shown that

$$\mathbb{E}\left\{\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\left(\frac{\partial J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}}\right)^T\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} = 2\mathbb{E}\left\{\frac{\partial^2 J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}\widetilde{\mathbf{x}}^T}\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}} \tag{26}$$

Then the covariance matrix of $\mathbf{x}$ is:

$$\begin{aligned}\mathbf{C}_{\mathbf{x}} &\approx 2\left[\mathbb{E}\left\{\frac{\partial^2 J(\widetilde{\mathbf{x}})}{\partial\widetilde{\mathbf{x}}\widetilde{\mathbf{x}}^T}\right\}\Big|_{\widetilde{\mathbf{x}}=\mathbf{x}}\right]^{-1} \\ &= \left[(\mathbf{A}-2\mathbf{h}\mathbf{x}^T)^T\mathbf{B}^{-1}\mathbf{Q}^{-1}\mathbf{B}^{-1}\left(\mathbf{A}-2\mathbf{h}\mathbf{x}^T\right)\right]^{-1}\end{aligned} \tag{27}$$

Using

$$\mathbf{B}^{-1}(\mathbf{A}-2\mathbf{h}\mathbf{x}^T) = -\begin{bmatrix}\frac{x-x_1}{d_1} & \frac{y-y_1}{d_1} \\ \frac{x-x_2}{d_2} & \frac{y-y_2}{d_2} \\ \vdots & \vdots \\ \frac{x-x_L}{d_L} & \frac{y-y_L}{d_L}\end{bmatrix} \tag{28}$$

$\mathbf{C}_{\mathbf{x}}$ can be expressed as

$$\mathbf{C}_{\mathbf{x}} \approx \begin{bmatrix}\sum_{l=1}^{L}\frac{(x-x_l)^2}{\sigma_l^2 d_l^2} & \sum_{l=1}^{L}\frac{(x-x_l)(y-y_l)}{\sigma_l^2 d_l^2} \\ \sum_{l=1}^{L}\frac{(x-x_l)(y-y_l)}{\sigma_l^2 d_l^2} & \sum_{l=1}^{L}\frac{(y-y_l)^2}{\sigma_l^2 d_l^2}\end{bmatrix}^{-1} \tag{29}$$

which is identical to the CRLB for range-based positioning [8], indicating the optimality of $\widehat{\mathbf{x}}$.

4. SIMULATION RESULTS

Simulations have been carried out to evaluate the performance of the constrained algorithm by comparing with the unconstrained version of [11] and CRLB. Firstly, we consider a 2-D geometry of 4 receivers with known coordinates at (0,0), (0,10),(10,0) and (10, 10). The errors $\{n_i\}$ have identical variances of $\sigma_i^2=\sigma^2$. For [11], we investigate the cases of $\mathbf{W}=\mathbf{I}_4$ and optimum weighting matrix which are referred to as LLS and WLLS methods, respectively, while the proposed estimator only examines the latter. In the proposed scheme, we assign constant $k=100$, 20 grid points and 30 iterative steps in the bisection search. The mean square position error (MSPE) is employed as the performance measure and all the results are averages of 1000 independent runs.

Figure 1 shows the MSPEs versus $\sigma^2\in[0.0001\ 1]$ at $\mathbf{x}=[2\ 8]^T$ where the source is located inside the square bounded by the receiver coordinates. It is seen that the accuracy of the proposed scheme attains the CRLB when $\sigma^2\le 0.01$ while both versions of [11] are suboptimal in the full range of σ^2, although the weighted one performs better. In the second test, we plot the average MSPEs when the source position is uniformly chosen within the square bounded by the receivers for each trial in order to see the average performance. The results are shown in Figure 2 and we again observe the optimality of the constrained estimator. On the other hand, both versions of [11] perform similarly in the average scenario.

The above two tests are repeated for the source located outside the region bounded by the four receivers. First, we consider a source fixed at $\mathbf{x}=[-2\ 8]^T$ and then investigate the scenario when the source is uniformly distributed in an area of 100 which is outside the square. The results are shown in Figures 3 and 4, respectively, and similar findings are obtained. Nevertheless, the MSPE gap between the proposed method and [11] is larger in Figure 4 when comparing with Figure 2.

Additionally, the second and fourth tests are repeated for the three-dimensional (3-D) geometry cases with 8 receivers with known coordinates at (0,0,0), (10,0,0), (10,10,0), (0,10,0), (0,0,10), (10,0,10), (10,10,10) and (0,10,10). The results are plotted in Figures 5 and 6, respectively, and similar findings are also obtained. Note that for Figure 6, the source is uniformly distributed in a volume of 1000 which is outside the cube for each trial.

To summary, according to all of the above results, it seems that employing the weighting matrix only in the existing LLS method [11] does not help improving its performance while the proposed constrained weighted method is optimum when the noise is sufficiently small.

5. CONCLUSION

A constrained linear least squares algorithm for source localization based on range measurements has been devised and analyzed. Basically, it is an improved version of [11] by incorporating the constraint which relates the unknown source location and extra range variable introduced in the linearization process. It is proved that the accuracy of the proposed estimator can achieve Cramér-Rao lower bound for small uncorrelated Gaussian disturbances.

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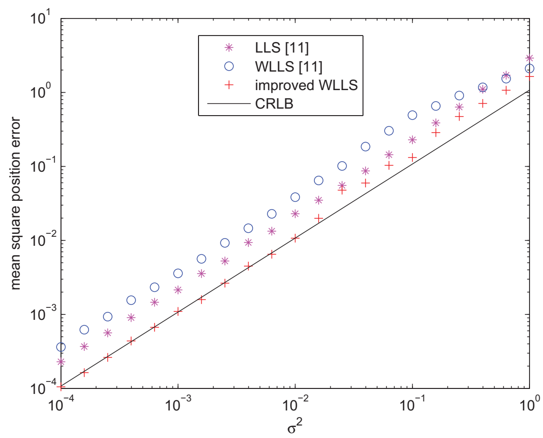

Figure 1: Mean square position error versus σ^2 at $\mathbf{x} = [2\ 8]^T$

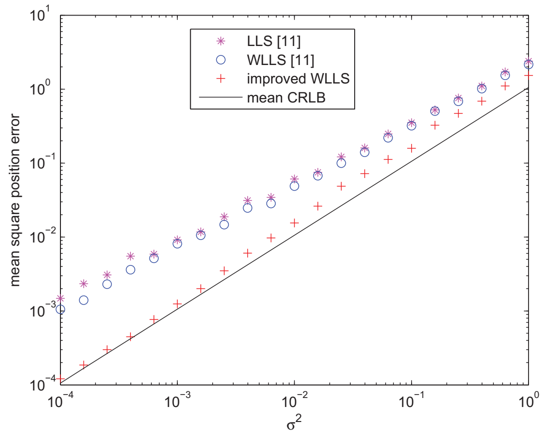

Figure 2: Mean square position versus σ^2 with random source position inside boundary

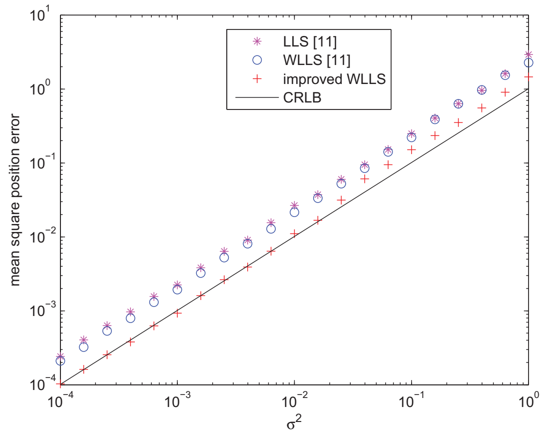

Figure 3: Mean square position error versus σ^2 at $\mathbf{x} = [-2\ 8]^T$

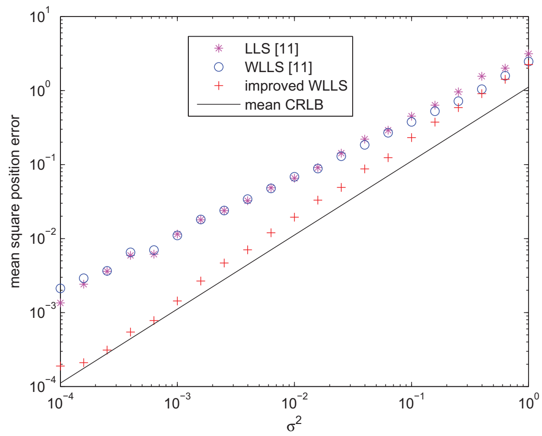

Figure 4: Mean square position versus σ^2 with random source position outside boundary

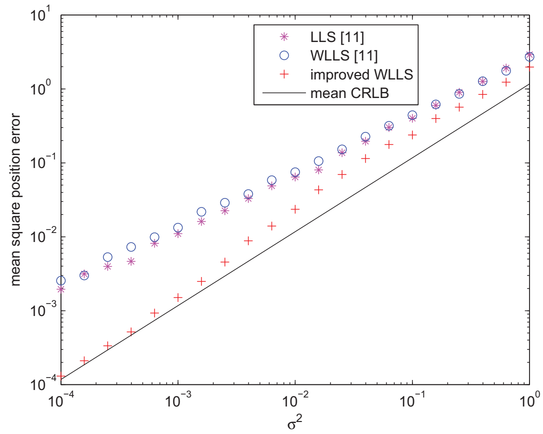

Figure 5: Mean square position versus σ^2 with random source position inside boundary in 3-D space

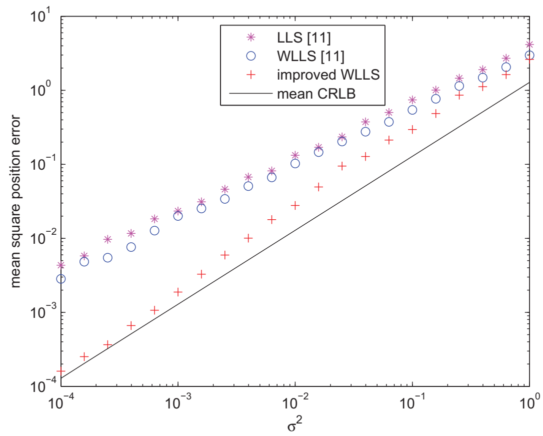

Figure 6: Mean square position versus σ^2 with random source position outside boundary in 3-D space