# NOISY CYCLO-STATIONARY BSS USING FREQUENCY-DOMAIN PSEUDO-CORRELATION

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## ABSTRACT

In this paper, we propose a new approach for blind separation of noisy linear instantaneous mixtures of cyclo-stationary sources using pseudo-correlation matrices in the frequency domain. This approach is an extension of a new method based on spectral decorrelation that we recently proposed and which assumes that all the cyclo-stationary sources and the stationary noise signals are mutually uncorrelated. Contrary to most of noisy BSS algorithms, our approach provides good performance in the determined case even if the noise signals are colored and/or non-Gaussian and of different variances. The simulation results show the much better performance of our approach in comparison to a classical BSS algorithm.

## 1. INTRODUCTION

This paper deals with the Blind Source Separation (BSS) problem for noisy, linear instantaneous mixtures of cyclo-stationary sources (which represent an important class of non-stationary sources and include telecommunication signals).

Considering $K$ noisy mixtures $x_i(t)$ of $M$ discrete-time sources $s_j(t)$, $K \geq M$, this problem can be modeled by:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{A}$ is a $K \times M$ real mixing matrix and $\mathbf{x}(t) = [x_1(t), ..., x_K(t)]^T$, $\mathbf{s}(t) = [s_1(t), ..., s_M(t)]^T$ and $\mathbf{n}(t) = [n_1(t), ..., n_K(t)]^T$ are respectively the observation, source and noise vectors, and $^T$ stands for transpose. BSS aims at restoring source signals $\mathbf{s}(t)$ from their mixtures by estimating the *pseudo-inverse* of the matrix $\mathbf{A}$, denoted by $\mathbf{A}^+$, provided that $\mathbf{A}$ is of full rank (equal to $M$). To this end, the approaches based on Independent Component Analysis (ICA) and exploiting higher-order statistics assume the sources $s_j(t)$ are mutually independent and non-Gaussian and the noise signals $n_i(t)$ are independent from them [3]. Other approaches, based on second-order statistics, assume the sources are only mutually uncorrelated, but autocorrelated and/or non-stationary, and the noises are only uncorrelated to them [1, 2, 4]. These BSS approaches may be split into two principal classes depending on their assumptions about noise:

1. The higher-order approaches like *JADE* [3], or second-order approches like *SOBI* [2], which start by a whitening step using the correlation matrix $\mathbf{R}_\mathbf{x}(\tau) = E\left[\mathbf{x}(t)\mathbf{x}^H(t-\tau)\right]$ computed at $\tau = 0$ (where $\mathbf{x}^H(t)$ represents the Hermitian transpose of $\mathbf{x}(t)$). These approaches suppose the noise signals $n_i(t)$ are **stationary** and of the **same variance** $\sigma^2$ and the mixture is strictly **over-determined** (i.e. $K > M$), so that the noise variance $\sigma^2$ can be estimated and then used to estimate the whitening matrix. In the second step, which determines a unitary separating matrix, the noises $n_i(t)$ are supposed **white** for the second-order approaches and **Gaussian** for the higher-order ones.

2. The second-order approaches like *SOBI-RO* [1] (used for stationary and autocorrelated signals) and *SEONS* [4] (called also *SONS*[1], used for non-stationary and/or autocorrelated signals), which start by a Robust Whitening using correlation matrices $E\left[\mathbf{x}(t)\mathbf{x}^H(t-\tau_k)\right]$ computed this time at $\tau_k \neq 0$. These approaches require the noise signals $n_i(t)$ to be **white** (although non-stationary and/or of possibly different variances, which is a first advantage of these approaches), so that their correlation matrices $E\left[\mathbf{n}(t)\mathbf{n}^H(t-\tau_k)\right]$ in whitening step are zero for $\tau_k \neq 0$. The second advantage is that these approaches also work in the case of **determined mixtures** (i.e. $K = M$).

The new approach presented in this paper is an extension to the noisy case of one of the two "spectral decorrelation" methods that we recently proposed [6]. These methods exploit second-order statistics of the signals in the frequency domain and are used for blind separation of noiseless, determined, real mixtures of non-stationary, zero-mean and mutually uncorrelated real sources. The **first method** exploits both correlation and pseudo-correlation matrices of the mixtures in the frequency-domain while the **second method** only uses correlation matrices. An extension of the second method to the noisy case has been recently proposed in [8]. Contrary to the approaches from the literature mentioned above, it allows the noise signals $n_i(t)$ to be colored and/or non-Gaussian and only mutually uncorrelated to each other and to the sources. However in [8] the noises were supposed stationary and of the same vari-

[1]For **S**econd (or **SE**cond) **O**rder **N**on-**S**tationary approach.

ance, and the mixture was supposed strictly over-determined.

In this paper, we propose an extension to the noisy case of our **first** spectral decorrelation method. It principally exploits the properties of the **pseudo-correlation matrix** of the noise vector in the frequency domain and only assumes that the noise signals $n_i(t)$ are stationary and mutually uncorrelated to each other and to the source signals $s_j(t)$. Thus, like for our extension proposed in [8], and contrary to the classical approaches, the noise signals $n_i(t)$ are not necessarily white and/or Gaussian, which is a first advantage. Moreover, these noise signals are not necessarily of the same variance and the mixture may be determined, like for the classical approaches belonging to the second class presented above.

The remainder of this paper is organized as follows. A review of our basic (noiseless) first spectral decorrelation method is presented in Section 2. In Section 3, we describe the extension proposed in this new work. Simulation results are presented in Section 4, before we conclude in Section 5.

## 2. BASIC METHOD FOR NOISELESS MIXTURES

As presented in [6], our first spectral decorrelation method deals with noiseless determined mixtures (i.e. $K = M$ and $\mathbf{n}(t) = 0$). It assumes the sources are real, **non-stationary** and mutually uncorrelated. It processes the mixtures in the frequency domain. In fact, in a determined, noiseless context, we have

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t). \tag{2}$$

with $\mathbf{x}(t) = [x_1(t), ..., x_M(t)]^T$, $\mathbf{s}(t) = [s_1(t), ..., s_M(t)]^T$. Computing the Fourier transform, we obtain

$$\mathbf{X}(\omega) = \mathbf{A}\mathbf{S}(\omega), \tag{3}$$

where $\mathbf{X}(\omega) = [X_1(\omega), ..., X_M(\omega)]^T$, $\mathbf{S}(\omega) = [S_1(\omega), ..., S_M(\omega)]^T$; $S_j(\omega)$ and $X_i(\omega)$, $(i, j) \in [1, M]^2$, are respectively the Fourier transforms[2] of $s_j(t)$ and $x_i(t)$. Thus, the frequency-domain observations $X_i(\omega)$ are linear instantaneous mixtures of the frequency-domain sources $S_j(\omega)$.

Using the correlation matrix $\mathbf{R}_X(\omega) = E\left[\mathbf{X}(\omega)\mathbf{X}^H(\omega)\right]$ and the **pseudo-correlation** matrix $\mathbf{Q}_X(\omega) = E\left[\mathbf{X}(\omega)\mathbf{X}^T(\omega)\right]$, where $\mathbf{X}^H(\omega)$ and $\mathbf{X}^T(\omega)$ are respectively the Hermitian transpose and the transpose of $\mathbf{X}(\omega)$, our first method in [6] is based on the following theorem.

**Theorem 1**: *Let $s_j(t)$ $(j = 1, 2, \cdots, M)$ be $M$ real, zero-mean and mutually uncorrelated signals. If there is a frequency $\omega_1$ such that $E[|S_j(\omega_1)|^2] \neq 0, \forall j$, and*

$$\frac{E[S_i^2(\omega_1)]}{E[|S_i(\omega_1)|^2]} \neq \frac{E[S_j^2(\omega_1)]}{E[|S_j(\omega_1)|^2]}, \quad \forall\, i \neq j, \tag{4}$$

*and if we note $\mathbf{V}$ a complex matrix whose columns are the eigenvectors of the matrix $\mathbf{R}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_1)$, then the separating matrix $\mathbf{A}^{-1}$ is given, up to a permutation and a real diagonal matrix, by:*

$$\mathbf{A}^{-1} = \Re\left\{\mathbf{V}^T\right\}. \tag{5}$$

The implementation of this method requires one to estimate the matrices $\mathbf{R}_X(\omega)$ and $\mathbf{Q}_X(\omega)$. Since generally only one realization of the mixtures is available and the frequency-domain mixtures are often non-ergodic (because they are non-stationary), our method may be **implemented in practice** only for **cyclo-stationary sources**. In this case, by splitting the mixtures into several time frames, each one containing an integral number of cyclo-stationarity periods, we obtain several realizations of the mixtures which may be used for estimating the expected values. Thus, denoting $N_c$ the cyclo-stationarity period of the temporal mixtures $x_i(t)$, the matrices $\mathbf{R}_X(\omega)$ and $\mathbf{Q}_X(\omega)$ are estimated as follows:

1. Split the mixed signals $x_i(t)$ into $L$ frames, denoted $x_{i,l}(t)\,(l = 1, 2, \cdots, L)$, whose length $F$ is an integral multiple of $N_c$ ($F = kN_c$).

2. Compute the Fourier transform of each frame $x_{i,l}(t)$, denoted by $X_{i,l}(\omega)\,(l = 1, 2, \cdots, L)$. Define the vector of frequency-domain observations $\mathbf{X}_l(\omega)$ by: $\mathbf{X}_l(\omega) = [X_{1,l}(\omega), X_{2,l}(\omega), \cdots, X_{M,l}(\omega)]^T$.

3. Estimate the matrices $\mathbf{R}_X(\omega)$ and $\mathbf{Q}_X(\omega)$ by averaging $\mathbf{X}_l(\omega)\mathbf{X}_l^H(\omega)$ and $\mathbf{X}_l(\omega)\mathbf{X}_l^T(\omega)$ over the $L$ frames:

$$\begin{cases} \widehat{\mathbf{R}}_X(\omega) = \frac{1}{L}\sum_{l=1}^{L} \mathbf{X}_l(\omega)\mathbf{X}_l^H(\omega) \\ \widehat{\mathbf{Q}}_X(\omega) = \frac{1}{L}\sum_{l=1}^{L} \mathbf{X}_l(\omega)\mathbf{X}_l^T(\omega) \end{cases} \tag{6}$$

In [6], we show that the possible candidates for satisfying the identifiability condition (4) are the frequencies $\omega_1 = k\omega_{c_l}/2$, where $k$ is an integer and $\omega_{c_l}$ is the least common multiplier of $M-1$ source cyclo-stationarity frequencies. Moreover, as used in some classical time-domain BSS algorithms (see for example [1] and [4]), it is also possible to jointly diagonalize several matrices $\mathbf{R}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_1)$ corresponding to several frequencies $\omega_1 = k\omega_{c_l}/2$. This extension generally improves the performance of our method because in this case the identifiability condition (4) is needed to be satisfied only for one of these frequencies.

## 3. EXTENSION TO NOISY MIXTURES

In this section, we consider the noisy, determined[3] linear instantaneous mixture defined by Eq. (1), with $K = M$. Our working hypotheses are:

- the sources $s_j(t)$ are real, zero-mean, **cyclo-stationary** and **mutually uncorrelated**,
- the noises $n_i(t)$ are real, zero-mean, **stationary** (not necessarily of same variance, and may be colored) and **mutually uncorrelated**,
- $s_j(t)$ et $n_i(t)$ are mutually uncorrelated $\forall i, j$.

[2]The Fourier transform of a discrete-time stochastic process $u(t)$ is a stochastic process $U(\omega)$ defined by $U(\omega) = \sum_{t=-\infty}^{\infty} u(t)e^{-j\omega t}$ [7].

[3]Note that if we have an over-determined mixture (i.e. $K > M$), knowing the number of sources $M$, we can transform it into a determined mixture either by considering only $M$ observations among $K$, or by applying a Principal Component Analysis (PCA) and holding only the first $M$ principal components.

By mapping Eq. (1) in the frequency domain we obtain:

$$\mathbf{X}(\omega) = \mathbf{A}\mathbf{S}(\omega) + \mathbf{N}(\omega) \tag{7}$$

where $\mathbf{N}(\omega) = [N_1(\omega), ..., N_M(\omega)]^T$, $N_i(\omega)$ are the Fourier transforms of the noises $n_i(t)$.

Thanks to the properties verified by the Fourier transforms of mutually uncorrelated temporal signals, the pseudo-correlation matrices $\mathbf{Q_S}(\omega) = E\left[\mathbf{S}(\omega)\mathbf{S}^T(\omega)\right]$ and $\mathbf{Q_N}(\omega) = E\left[\mathbf{N}(\omega)\mathbf{N}^T(\omega)\right]$ are diagonal, and the matrices $E\left[\mathbf{S}(\omega)\mathbf{N}^T(\omega)\right]$ and $E\left[\mathbf{N}(\omega)\mathbf{S}^T(\omega)\right]$ are zero, $\forall\,\omega$. In fact, if $u(t)$ et $v(t)$ are two mutually uncorrelated signals, i.e. if $E\left[u(t_1)v(t_2)\right] = 0, \forall t_1, t_2$, then their Fourier transforms $U(\omega) = \sum_{t=-\infty}^{\infty} u(t)e^{-j\omega t}$ and $V(\omega) = \sum_{t=-\infty}^{\infty} v(t)e^{-j\omega t}$ verify $E\left[U(\omega_1)V(\omega_2)\right] = 0, \forall\,\omega_1, \omega_2$, because we have

$$E\left[U(\omega_1)V(\omega_2)\right] = \sum_{t_1=-\infty}^{\infty}\sum_{t_2=-\infty}^{\infty} E\left[u(t_1)v(t_2)\right]e^{-j(\omega_1 t_1 + \omega_2 t_2)},$$

and $E\left[u(t_1)v(t_2)\right] = 0, \forall t_1, t_2$.

Thus, due to (7), by computing the pseudo-correlation matrix of the vector $\mathbf{X}(\omega)$ we obtain:

$$\begin{aligned}
\mathbf{Q_X}(\omega) &= E\left[\mathbf{X}(\omega)\mathbf{X}^T(\omega)\right] \\
&= \mathbf{A}E\left[\mathbf{S}(\omega)\mathbf{S}^T(\omega)\right]\mathbf{A}^T + E\left[\mathbf{N}(\omega)\mathbf{N}^T(\omega)\right] \\
&= \mathbf{A}\mathbf{Q_S}(\omega)\mathbf{A}^T + \mathbf{Q_N}(\omega).
\end{aligned} \tag{8}$$

The extension of the method presented in Section 2 to the noisy case is based on the exploitation of the properties of the pseudo-correlation matrix $\mathbf{Q_N}(\omega)$ which reads:

$$\mathbf{Q_N}(\omega) = \begin{pmatrix} E\left[N_1^2(\omega)\right] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & E\left[N_M^2(\omega)\right] \end{pmatrix}. \tag{9}$$

The noises $n_i(t)$ being stationary, we can use the following proposition.

**Proposition**: *Let $u(t)$ be a real stationary signal with Fourier transform $U(\omega)$. Then, $E[U^2(\omega)] = 0$, $\forall\,\omega \neq k\pi$, where $k$ is an integer.*

*Proof*: See Appendix A.

Following this proposition, $E\left[N_i^2(\omega)\right] = 0$, $\forall\,\omega \neq k\pi$, and consequently:

$$\mathbf{Q_N}(\omega) = E\left[\mathbf{N}(\omega)\mathbf{N}^T(\omega)\right] = \mathbf{0}_M,\ \forall\,\omega \neq k\pi, \tag{10}$$

where $\mathbf{0}_M$ is the null matrix of dimension $M \times M$. Then, Eq. (8) becomes:

$$\mathbf{Q_X}(\omega) = \mathbf{A}\mathbf{Q_S}(\omega)\mathbf{A}^T,\ \forall\,\omega \neq k\pi. \tag{11}$$

Since the matrix $\mathbf{Q_S}(\omega)$ is diagonal $\forall\,\omega$ and reads

$$\mathbf{Q_S}(\omega) = \begin{pmatrix} E\left[S_1^2(\omega)\right] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & E\left[S_M^2(\omega)\right] \end{pmatrix}, \tag{12}$$

a new identifiability theorem for the mixing matrix $\mathbf{A}$ in the noisy case can be formulated as follows.

**Theorem 2**: *Let $s_j(t)$ $(j = 1, 2, \cdots, M)$ be $M$ real, zero-mean and mutually uncorrelated signals. If there are two frequencies $\omega_1 \neq k_1\pi$ and $\omega_2 \neq k_2\pi$ such that $E[S_j^2(\omega_1)] \neq 0, \forall\, j$, and*

$$\frac{E[S_i^2(\omega_2)]}{E[S_i^2(\omega_1)]} \neq \frac{E[S_j^2(\omega_2)]}{E[S_j^2(\omega_1)]}, \quad \forall\, i \neq j, \tag{13}$$

*and if we note $\mathbf{V}$ a complex matrix whose columns are the eigenvectors of the matrix $\mathbf{Q}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_2)$, then the separating matrix $\mathbf{A}^{-1}$ is given, up to a permutation and a real diagonal matrix, by:*

$$\mathbf{A}^{-1} = \Re\left\{\mathbf{V}^T\right\}. \tag{14}$$

*Proof*: See Appendix B.

For implementing this extended algorithm, the sources are supposed to be cyclo-stationary, so that the matrices $\mathbf{Q}_X(\omega_1)$ and $\mathbf{Q}_X(\omega_2)$ can be estimated as explained in Section 2 and Eq. (6). Once more, it can be shown that the possible candidates for satisfying the identifiability condition (13) are the frequencies $\omega_1 = k_1\omega_{c_l}/2$ and $\omega_2 = k_2\omega_{c_l}/2$, where $k_1 \neq k_2$ are two integers and $\omega_{c_l}$ is the least common multiplier of $M-1$ source cyclo-stationarity frequencies. It is also possible to jointly diagonalize several matrices $\mathbf{Q}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_2)$ corresponding to several couples of frequencies ($\omega_1 = k_1\omega_{c_l}/2$ , $\omega_2 = k_2\omega_{c_l}/2$), as mentioned in Section 2. In this case, it is sufficient that the identifiability condition (13) is satisfied only for one of these couples $(\omega_1, \omega_2)$.

Once an estimate of the separating matrix denoted $\widehat{\mathbf{A}}^{-1}$ is obtained, a noisy estimate of the source vector $\mathbf{s}(t)$, denoted $\widehat{\mathbf{s}}_n(t)$, can be obtained using Eq. (1) as follows:

$$\begin{aligned}
\widehat{\mathbf{s}}_n(t) &= \widehat{\mathbf{A}}^{-1}\mathbf{x}(t) \\
&= \widehat{\mathbf{A}}^{-1}\mathbf{A}\mathbf{s}(t) + \widehat{\mathbf{A}}^{-1}\mathbf{n}(t) \\
&\simeq \mathbf{s}(t) + \widehat{\mathbf{A}}^{-1}\mathbf{n}(t).
\end{aligned} \tag{15}$$

Since this extension of our **Spec**tral **Dec**orrelation method only uses the **P**seudo-**C**orrelation matrices, it will be called ***SpecDec-PC*** in the following.

## 4. SIMULATION RESULTS

We consider a determined noisy mixture of two random cyclo-stationary autocorrelated sources $s_j(t)\,(j = 1, 2)$. In our first experiment, the sources are two 8192-sample artificial signals generated using $s_j(t) = r_j(t)\mu_j(t)$, where $r_j(t)$ are random stationary signals obtained by filtering two i.i.d. Gaussian, zero-mean and mutually independent signals by two different 31th-order FIR filters, $\mu_1(t) = \sin(\omega_0 t)$ and $\mu_2(t) = \cos(\omega_0 t)$ with $\omega_0 = \pi/8$. It can be easily shown that the sources $s_j(t)$ are cyclo-stationary with a cyclo-stationarity frequency $\omega_c = 2\omega_0$. In the second experiment, the sources are two real-world cyclo-stationary telecommunication signals already used in [6]. The first signal is a recorded GMSK-modulated burst signal, used in the European digital cellular communication system, called GPS, whereas the second one is a very noisy QAM16-modulated signal. Both signals were shifted to the central frequency 20MHz and resampled at 80 million samples per second. Each cyclo-stationarity period contains $N_c = 4$ samples and we use 9984 samples of the signals.

In both experiments, the mixing matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} \\ a_{21} & 1 \end{pmatrix}, \qquad (16)$$

where $a_{12}$ and $a_{21}$ are two random variables uniformly distributed on [0,1]. We consider two cases for the additive noises $n_i(t)\,(i = 1,2)$. In the first case, the noise signals are white and uniformly distributed. In the second case, they are colored and obtained by filtering two uniform white noises by two 31th-order FIR filters. In both cases, the signal to noise ratio is equal to $10dB$ at the first sensor and $15dB$ at the second one[4].

The separation performance for each source $s_j(t)$ is measured using the *performance index,* defined, in the case of $M$ sources, by

$$\forall j \in [1,M],\ \mathscr{I}_j = max_i 10\log_{10}\left(\frac{g_{ij}^2}{\sum_{k\in[1,M]}^{k\neq j} g_{ik}^2}\right), \qquad (17)$$

where $g_{ij}$, $(i,j) \in [1,M]^2$, are the entries of the M-dimensional square matrix $\mathbf{G} = \widehat{\mathbf{A}}^{-1}\mathbf{A}$, called the *performance matrix*. The global separation performance for $M$ sources is then measured by the *global performance index*, defined by

$$\mathscr{I} = \left(\sum_{j=1}^{M} \mathscr{I}_j\right)/M. \qquad (18)$$

Since the artificial sources used in the first experiment are cyclo-stationary with a cyclo-stationarity period $N_c = 2\pi/\omega_c = 8$, we split the temporal mixtures into 512 frames of length $F = 16$ $(= 2N_c)$. We use the first and the 9th multiples of the frequency $\omega_c/2$ as the frequencies $\omega_1$ and $\omega_2$. The cyclo-stationarity period of the telecommunication sources used in the second experiment being equal to 4, their mixtures are split into 1248 frames of length $F = 8$ and the first and the 6th multiples of $\omega_c/2$ are used as the frequencies $\omega_1$ and $\omega_2$ [5].

In the following, we compare the performance of our method with that of the *SEONS* algorithm [4]. According to the simulation results presented in [4], *SEONS* outperforms some other classical BSS algorithms like *SOBI-RO* [1], *SOBI* [2] and *JADE* [3] in the determined case[6]. Moreover, in the noisy over-determined case, the tests in [8] show that *SEONS* is more efficient than *SOBI*. Note that all the three methods *SEONS*, *SOBI* and *SOBI-RO* exploit time correlation of the sources but *SEONS* also exploits their non-stationarity. Hence, it is a good candidate for the comparison with our method.

Thus, using the ICALAB toolbox [5], we tested the *SEONS* algorithm with 16 frames of 512 samples in the first experiment and 19 frames of 512 samples in the second one and using 5 covariance matrices on each frame. The mean and the standard deviation of the global performance index $\mathscr{I}$ for our method and *SEONS* using 50 MonteCarlo simulations corresponding to 50 different values of the mixing matrix entries $a_{12}$ and $a_{21}$ (and 50 different realizations of the random signals $r_j(t)$ in the first experiment), with and without an additive noise vector $\mathbf{n}(t)$ are reported in Table 1.

This table deserves the following comments:

- **Noiseless mixtures**: Our *SpecDec-PC* method is very efficient and outperforms *SEONS* even in the noiseless case (about $6dB$ in the first experiment and $15dB$ in the second one). This result may be explained by the assumption of piece-wise stationarity made by *SEONS* which is not verified by the cyclo-stationary signals used in our tests.
- **Noisy mixtures**: Our method always outperforms *SEONS* especially in the presence of colored noise. This result is not surprising because *SEONS* assumes that the noise is white, while our method does not need this assumption.

## 5. CONCLUSION AND PERSPECTIVES

In this paper, we proposed a new BSS approach, called *SpecDec-PC*, for noisy mixtures of cyclo-stationary sources, based on exploitation of *Pseudo-Correlation matrices* in the frequency domain. Our assumptions about source and noise signals are much less restrictive than those made by classical BSS methods. In fact, our method is able to handle the determined case in the presence of stationary noises which may be colored and/or non-Gaussian and of different variances. Our simulations confirmed the better performance of our approach compared to the *SEONS* algorithm for separating cyclo-stationnary signals especially with colored noise. A more detailed statistical performance test seems however necessary and will be done in the future. Moreover, we expect the performance of our method improves when considering several matrices $\mathbf{Q}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_2)$ defined for different values of $\omega_1$ and $\omega_2$, which are diagonalized simultaneously like in [1] and [4].

### Appendix A: Proof of Proposition

Let $u(t)$ be a real stationary signal with Fourier transform $U(\omega)$. We want to show that $E[U^2(\omega)] = 0$, for $\omega \neq k\pi$. Using the definition of the Fourier transform, we can write

$$E[U^2(\omega)] = \sum_{t_1=-\infty}^{\infty}\sum_{t_2=-\infty}^{\infty} E[u(t_1)u(t_2)]e^{-j\omega(t_1+t_2)}. \qquad (19)$$

Since $u(t)$ is stationary, its autocorrelation function only depends on $t_2 - t_1$: $E[u(t_1)u(t_2)] = f(t_2 - t_1)$. Denoting the

[4]More precisely, the variances of the noises $n_1(t)$ et $n_2(t)$, denoted respectively $\sigma_1^2$ and $\sigma_2^2$, are chosen such that $10\log_{10}(1/\sigma_1^2) = 10dB$ and $10\log_{10}(1/\sigma_2^2) = 15dB$, knowing that $s_j(t)$ are normalized to have unit power.

[5]This choice leads to the best results but other choices provide acceptable results too.

[6]The moderate performance of *SOBI* and *JADE* in the determined case is not surprising because, as mentioned in Section 1, in the noisy case these methods are specially adapted to over-determined mixtures.

| | | Without noise | | White noise | | Colored noise | |
|---|---|---|---|---|---|---|---|
| | Algorithm | $\mathscr{I}$(dB) | $\sigma_{\mathscr{I}}$(dB) | $\mathscr{I}$(dB) | $\sigma_{\mathscr{I}}$(dB) | $\mathscr{I}$(dB) | $\sigma_{\mathscr{I}}$(dB) |
| **Artificial signals** | *SpecDec-PC* | **44.2** | 7.1 | **41.6** | 6.5 | **40.0** | 6.5 |
| | *SEONS* | **38.2** | 7.0 | **35.7** | 8.6 | **23.4** | 8.8 |
| **Real signals** | *SpecDec-PC* | **52.0** | 2.2 | **40.0** | 5.9 | **38.0** | 10.4 |
| | *SEONS* | **37.0** | 1.1 | **34.0** | 2.5 | **30.9** | 10.1 |

Table 1: Mean and standard deviation, in dB, of the global performance index $\mathscr{I}$, obtained using 50 MonteCarlo simulations.

auxiliary variable $t = t_2 - t_1$,

$$E[U^2(\omega)] = \sum_{t_1=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} f(t)e^{-j\omega(2t_1+t)}$$
$$= \sum_{t_1=-\infty}^{\infty} e^{-j2\omega t_1}\sum_{t=-\infty}^{\infty} f(t)e^{-j\omega t}. \quad (20)$$

The inner sum represents the power spectral density of $u(t)$, denoted by $F(\omega)$. Thus, we can write

$$E[U^2(\omega)] = F(\omega)\sum_{t_1=-\infty}^{\infty} e^{-j2\omega t_1}. \quad (21)$$

Moreover, since $\sum_{t_1=-\infty}^{\infty} e^{-j2\omega t_1}$ is the discrete-time Fourier transform of the constant 1, evaluated at $2\omega$, we have

$$\sum_{t_1=-\infty}^{\infty} e^{-j2\omega t_1} = 2\pi\sum_{k=-\infty}^{\infty}\delta(2\omega - 2k\pi). \quad (22)$$

Hence, Eq. (21) can be rewritten as

$$E[U^2(\omega)] = 2\pi F(\omega)\sum_{k=-\infty}^{\infty}\delta(2\omega - 2k\pi), \quad (23)$$

which yields

$$E[U^2(\omega)] = 0,\ \forall\ \omega \neq k\pi. \quad (24)$$

**Appendix B: Proof of Theorem 2**

From (11), for two frequencies $\omega_1 \neq k_1\pi$ and $\omega_2 \neq k_2\pi$ we have:

$$\mathbf{Q}_X(\omega_1) = \mathbf{A}\mathbf{Q}_S(\omega_1)\mathbf{A}^T \quad (25)$$

and

$$\mathbf{Q}_X(\omega_2) = \mathbf{A}\mathbf{Q}_S(\omega_2)\mathbf{A}^T. \quad (26)$$

If $\mathbf{Q}_S(\omega_1)$ is nonsingular, *i.e.* if $E[S_j^2(\omega_1)] \neq 0\ \forall j$, then left multiplying (26) by the inverse of (25) yields

$$\mathbf{Q}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_2) = \mathbf{A}^{T^{-1}}\mathbf{Q}_S^{-1}(\omega_1)\mathbf{Q}_S(\omega_2)\mathbf{A}^T. \quad (27)$$

Since following (12), $\mathbf{Q}_S^{-1}(\omega_1)\mathbf{Q}_S(\omega_2)$ is a diagonal matrix, the above equation is nothing but an eigenvalue decomposition of the matrix $\mathbf{Q}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_2)$. If the $M$ eigenvalues are distinct (*i.e.* if the algebraic multiplicity of each eigenvalue equals one), then the dimension of the eigenspace corresponding to each eigenvalue equals one. Moreover, it is clear that the eigenvalues may be arranged as diagonal entries of a diagonal matrix in an arbitrary order. Hence, if the matrix $\mathbf{Q}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_2)$ has $M$ distinct eigenvalues (which are the diagonal entries of $\mathbf{Q}_S^{-1}(\omega_1)\mathbf{Q}_S(\omega_2)$), *i.e.* if we have

$$\frac{E[S_i^2(\omega_2)]}{E[S_i^2(\omega_1)]} \neq \frac{E[S_j^2(\omega_2)]}{E[S_j^2(\omega_1)]},\ \forall\, i \neq j, \quad (28)$$

and if $\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ is an eigenvalue decomposition of $\mathbf{Q}_X^{-1}(\omega_1)\mathbf{Q}_X(\omega_2)$, then the columns of $\mathbf{V}$ are equal to the columns of $\mathbf{A}^{T^{-1}}$ up to scaling factors and a permutation, so that

$$\mathbf{V} = \mathbf{A}^{\mathbf{T}^{-1}}\mathbf{D}\mathbf{P_1}, \quad (29)$$

where $\mathbf{D}$ is a complex diagonal matrix and $\mathbf{P_1}$ is a permutation matrix. It follows that

$$\mathbf{V}^T = \mathbf{P_1}^T\mathbf{D}^T\mathbf{A}^{-1} = \mathbf{P}\mathbf{D}\mathbf{A}^{-1}, \quad (30)$$

where $\mathbf{P} = \mathbf{P_1}^T$ is a permutation matrix too. Moreover, $\mathbf{A}$ and $\mathbf{P}$ being two real matrices, we can write

$$\Re\{\mathbf{V}^T\} + j\Im\{\mathbf{V}^T\} = \mathbf{P}\left(\Re\{\mathbf{D}\} + j\Im\{\mathbf{D}\}\right)\mathbf{A}^{-1}, \quad (31)$$

so that

$$\Re\{\mathbf{V}^T\} = \mathbf{P}\left(\Re\{\mathbf{D}\}\right)\mathbf{A}^{-1}. \quad (32)$$

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