

# AVERAGE CONSENSUS WITH MINIMUM ENERGY CONSUMPTION: OPTIMAL TOPOLOGY AND POWER ALLOCATION

S. Sardellitti<sup>1</sup>, S. Barbarossa<sup>1</sup>, A. Swami<sup>2</sup>

<sup>1</sup>INFOCOM Dept., University of Rome "La Sapienza",  
Via Eudossiana 18, 00184 Rome, Italy

<sup>2</sup>Army Research Laboratory, Adelphi, MD 20783, USA  
email: Stefania.Sardellitti@uniroma1.it, sergio@infocom.uniroma1.it, a.swami@ieee.org

## ABSTRACT

Consensus algorithms have generated a lot of interest due to their simplicity in computing globally relevant statistics exploiting only local communications among sensors. However, the inherent iterative nature of consensus algorithms makes them prone to a possibly large energy consumption. Because of the strong energy constraints of wireless sensor networks, it is then of interest to minimize energy consumption necessary to achieve consensus, within a prescribed accuracy requirement. In this work, we propose a method for optimizing the network topology and power allocation over each link, in order to minimize energy consumption, while ensuring that the network reaches a global consensus. Interestingly, we show how to introduce a relaxation in the topology optimization that converts a combinatorial problem into a convex-concave fractional problem. The results show how the sparsity of the resulting network depends on the propagation model.

## 1. INTRODUCTION

Average consensus algorithms have received considerable attention in recent years because of their simplicity: Every node in the network is eventually able to reach globally relevant statistics of the data, by only exchanging information with nearby nodes, without the need for a centralized fusion center [1]. The price paid for this simplicity and the underlying decentralized philosophy is that consensus algorithms are inherently iterative, which causes a repeated expenditure of energy in the iterated exchange of data among the nodes. This must be contrasted with a centralized strategy where there is a sink node that, after collecting the observations from all the sensors (perhaps over multiple hops), is virtually able to compute the desired statistic in one shot. To make a distributed approach useful in a sensor network context, it is then necessary to minimize the energy consumption necessary to reach consensus. Clearly, the network topology plays a fundamental role in determining the convergence rate. It is well known in fact that, as network connectivity increases, so does the convergence rate. However, having a densely connected network requires a high power consumption, to guarantee reliable links between the nodes. In principle, having a fully connected network is equivalent to having as many sink nodes as sensors, so that the convergence time of fully connected networks is minimum. But the power consumption necessary to maintain a fully connected network is also maximum. On the other hand, a network with minimal connectivity requires small power consumption to maintain the topology, but it is also slow to converge. So, it is of interest

to look at what topologies provide the best trade-off between convergence time and power consumption necessary to establish the links with the desired reliability.

If the links are symmetric, or, equivalently, the graph describing the network topology is undirected, the convergence rate can be measured through the so called *algebraic connectivity*, defined as the second smallest eigenvalue of the graph Laplacian [2]. For this reason, there has been work on maximizing the algebraic connectivity of an undirected graph, by a suitable choice of the weights associated with each edge, for a given topology [3] [4]. In [5] it was shown how some network topologies, like small world graphs, for example, can greatly increase the convergence rate. On the other hand, enforcing a small world, or scale-free, graph is not an easy task, in a wireless network. In [6] it was shown how to add edges from a given set to a graph in order to maximize its algebraic connectivity. There has also been work on topology optimization in order to minimize the power consumption necessary to guarantee connectivity, e.g., [7, 8], but without any specific reference to the running application. Conversely, it is now well known that an efficient design of wireless sensor networks requires the exploitation of their specificity, or applications, which make them intrinsically different from communication networks.

Since what really matters is energy consumption rather than minimizing convergence time (although there are important applications where the latter could be more important), it was shown in [9] that one should actually minimize the product of the global power spent to enforce a given topology and the convergence time. As shown in [9], there typically is an optimum power that minimizes the energy necessary to achieve consensus within a prescribed accuracy. The results obtained in [9] assumed a random geometric graph model, with no knowledge of the node position. In such a case, the task was to find out the (common) transmit power to be associated to each node to minimize energy consumption. However, if the node positions are known, we have the potential of improving the performance considerably by optimizing the power budget over each link. In this work, we extend the approach of [9] to arbitrary networks, where the positions are known, and we show how to optimize the network topology *jointly* with the power allocation across each link, in order to minimize the energy necessary to reach consensus within a predefined accuracy.

## 2. CONSENSUS ALGORITHMS

Let us consider a wireless network with  $n$  sensors represented by an undirected graph  $G = \{V, E\}$  where  $V$  denotes the set of

$n$  vertices  $v_i$  and  $E \subseteq V \times V$  is the set of bidirectional edges  $e_{ij} = e_{ji}$  connecting  $v_i$  and  $v_j$ . Furthermore, let be  $\mathbf{A}$  the  $n \times n$ -dimensional symmetric adjacency matrix of the graph  $G$  with elements  $a_{ij} = 1$  if  $e_{ij} \in E$  and  $a_{ij} = 0$  otherwise. According to this notation and assuming no self-loops, i.e.,  $a_{ii} = 0 \ \forall i = 1, \dots, n$ , the degree of the node  $v_i$  is defined as  $\deg(v_i) = \sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji}$ . Let  $\mathcal{N}_i$  denote the set of neighbors of node  $i$ , so that  $|\mathcal{N}_i| = \deg(v_i)$ . The Laplacian matrix of the graph  $G$  is the  $n \times n$  symmetric matrix  $\mathbf{L}(G)$  defined as

$$\ell_{ij} = \ell_{ji} = \begin{cases} \deg(v_i) & \text{if } j = i \\ -a_{ij} & \text{if } j \neq i \end{cases}. \quad (1)$$

Reaching consensus over a common measurement or decision can be seen as the minimization of the disagreement between the states  $x_i$  of the interacting nodes. One of the nice properties of the Laplacian is that the disagreement can be measured as a quadratic form built on the Laplacian [10]:

$$J(\mathbf{x}) := \frac{1}{2} \sum_{i=1}^n \sum_{j \in \mathcal{N}_i} (x_i - x_j)^2 = \mathbf{x}^T \mathbf{L} \mathbf{x}. \quad (2)$$

The minimization of this quadratic form can be achieved in a decentralized strategy, using a simple steepest gradient technique. In continuous time, we may achieve the minimization through the following dynamical system [11]

$$\dot{\mathbf{x}}(t) = -\mathbf{L} \mathbf{x}(t), \quad (3)$$

with  $\mathbf{x}(0) := \mathbf{x}_0$ , where  $\mathbf{x}_0$  is the  $n$ -size column vector whose entries are the measurements of each node<sup>1</sup>. The convergence of (3) depends on the eigenvalues of  $\mathbf{L}$ . In particular, the convergence of (3) to a consensus is guaranteed if  $\mathbf{L}$  has a null eigenvalue of multiplicity one, which corresponds to having a connected graph. Under such a circumstance, the convergence rate is dictated by the second smallest eigenvalue of  $\mathbf{L}$ , namely  $\lambda_2(\mathbf{L})$ . More specifically, the dynamic system converges to consensus exponentially, i.e.,  $\|\dot{\mathbf{x}}(t) - \omega^*\| \leq O(e^{-rt})$ , with  $r \propto \lambda_2(\mathbf{L})$  and  $\omega^*$  the  $n$ -dimensional vector with all entries equal to the common consensus value.

The Laplacian matrix  $\mathbf{L}$  depends, in turn, on the topology of  $G$ , i.e., the power used by any node to exchange messages with other nodes and on the propagation model. More specifically, given two nodes  $i$  and  $j$ , at a distance  $r_{ij}$ , we assume that there is a link between them if the received power at node  $j$  exceeds a threshold, i.e., if  $p_{Rj} > p_{min}$ , in which case  $a_{ij} = 1$ ; otherwise  $a_{ij} = 0$ . In this paper, we assume the following propagation model:

$$p_{Rj} = \frac{p_{ij}}{1 + (r_{ij}/r_0)^\eta} \quad (5)$$

where  $r_0$  is a scaling factor representing a reference distance,  $\eta$  is the channel loss exponent,  $p_{ij}$  is the power used by node  $i$  to transmit messages to node  $j$ . In (5), the distance  $r_0$  typically corresponds to the so called Fraunhofer distance, such

<sup>1</sup>The discrete-time counterpart of (3) is [10]

$$\mathbf{x}[k+1] = \mathbf{x}[k] - \tilde{\epsilon} \mathbf{L} \mathbf{x}[k] := \mathbf{W} \mathbf{x}[k], \quad (4)$$

where  $\tilde{\epsilon}$  is a coefficient chosen in order to ensure that no eigenvalue of  $\mathbf{W}$  is greater than one.

that for  $r_{ij} \gg r_0$  we are in the antenna far-field, where the power attenuation is inversely proportional to  $r_{ij}^\eta$ , whereas for  $r_{ij} \ll r_0$ , we are in the antenna near-field, where the power is approximately equal to the transmitted one. The unity term in the denominator of (5) is used to avoid the impossible situation in which the received power could be greater than the transmitted one.

### 3. MINIMIZING ENERGY TO ACHIEVE CONSENSUS

The overall energy spent to reach consensus can be measured as the product of the transmit powers necessary to establish the links between neighboring nodes and the convergence time, namely

$$\mathcal{E} = K \frac{\sum_{i=1}^n \sum_{j \in \mathcal{N}_i} p_{ij}}{\lambda_2(\mathbf{L}(\mathbf{p}_T))}, \quad (6)$$

where  $K$  is a constant factor depending only on the desired accuracy, and  $\mathbf{p}_T := \text{vec}(p_{ij}, i, j = 1, \dots, n, i \neq j)$  is the vector of transmit powers with  $p_{ij} = p_{ji}$ .

Our goal in this paper is to find out the optimal network topology and choose  $\mathbf{p}_T$  so as to minimize the energy consumption in (6), while ensuring network connectivity, i.e.,  $\lambda_2(\mathbf{L}(\mathbf{p}_T)) > 0$ . Unfortunately, the search for the optimal topology involves a combinatorial strategy that makes the solution of the problem extremely hard, especially for networks with a large number of nodes. To make the problem analytically tractable and obtain a numerically appealing solution, we relax the constraint that the coefficients  $a_{ij}$  are either zero or one and assume, instead, the following expression

$$a_{ij} = \frac{1}{1 + (r_{ij}/r_{c_{ij}})^\alpha} \quad (7)$$

where  $\alpha$  is a positive coefficient and  $r_{c_{ij}}$  is the coverage radius, depending on the transmit power. In particular, given the propagation model (5), we have:

$$r_{c_{ij}} = r_0 \left( \frac{p_{ij}}{p_{min}} - 1 \right)^{1/\eta}. \quad (8)$$

We are now able to formulate our optimization problem as follows:

$$\begin{aligned} \min_{\mathbf{p}_T} \quad & \frac{\mathbf{p}_T^T \mathbf{1}}{\lambda_2(\mathbf{L}(\mathbf{p}_T))} \\ \text{s.t.} \quad & \epsilon \leq \lambda_2(\mathbf{L}(\mathbf{p}_T)) \\ & \mathbf{1} p_{min} \leq \mathbf{p}_T \end{aligned} \quad (9)$$

where  $\mathbf{1}$  is the column vector of all ones and  $\epsilon$  is an arbitrarily small positive constant used to prevent the algebraic connectivity from going to zero, i.e., ensuring that the network is connected. In (9), using (8) in (7), the coefficients  $a_{ij}$  composing the Laplacian are written explicitly in terms of the propagation parameters as:

$$a_{ij}(p_{ij}) = \frac{r_0^\alpha (p_{ij} - p_{min})^{\alpha/\eta}}{r_0^\alpha (p_{ij} - p_{min})^{\alpha/\eta} + r_{ij}^\alpha p_{min}^{\alpha/\eta}}. \quad (10)$$

In principle, the last inequality in (9) makes any link feasible. But this does not mean that the final network will be fully connected, because some coefficients  $a_{ij}$  might equal zero, thus implying that the link between node  $i$  and  $j$  is not active. The first important result, for the solution of (9), is the following

**Theorem 1** Given the propagation model (5) and the positions (7)-(8), the minimization problem

$$\begin{aligned} \min_{\mathbf{p}_T} \quad & \frac{\mathbf{p}_T^T \mathbf{I}}{\lambda_2(\mathbf{L}(\mathbf{p}_T))} \\ \text{s.t.} \quad & \varepsilon \leq \lambda_2(\mathbf{L}(\mathbf{p}_T)) \\ & \mathbf{1} p_{\min} \leq \mathbf{p}_T \end{aligned} \quad (11)$$

is a convex-concave fractional problem if  $\eta \geq \alpha$ .

We omit the details of the proof here, because of lack of space. The full proof is in [12] and is based on the fact that  $\lambda_2(\mathbf{L}(\mathbf{p}_T))$  can be shown to be a concave function of  $\mathbf{p}_T$ .

Proving that the problem (9) is a convex-concave fractional problem is a basic step in finding a numerical solution, as several methods are available to solve quasi-convex optimization problems, see e.g. [13, 14]. In this paper, we consider the nonlinear parametric problem proposed in [14], expressed as

$$h(\mu) = \min\{\mathbf{p}_T^T \mathbf{1} - \mu \lambda_2(\mathbf{L}(\mathbf{p}_T)) : \mathbf{p}_T \in \Gamma\} \quad (12)$$

where  $\mu$  is a real positive parameter while  $\Gamma = \{\mathbf{p}_T : \mathbf{p}_T \geq \mathbf{1} p_{\min}, \lambda_2(\mathbf{p}_T) \geq \varepsilon\}$ . The solution of this problem can be obtained using the following result proved in [14, 15]:

**Theorem 2** : Let  $\psi(x) = \frac{f(x)}{g(x)}$  be a continuous function  $\forall x \in \Theta$  where  $\Theta$  is a nonempty compact set of  $\mathbb{R}^n$ , the function  $f(x)$  is convex and  $g(x)$  is concave with  $g(x) > 0 \forall x \in \Theta$ . Then  $y^* \in \Theta$  is an optimal solution of

$$\min\{\psi(x) = \frac{f(x)}{g(x)} : x \in \Theta\} \quad (13)$$

if and only if  $y^*$  is an optimal solution of the following parametric problem

$$\min\{f(x) - \psi(y^*)g(x) : x \in \Theta\} . \quad (14)$$

Within this setup, Dinkelbach developed an algorithm for solving non-linear fractional program in the case where numerator and denominator of the objective function are, respectively, a convex and a concave function [14]. Given the parametric problem

$$h(\mu) = \min\{f(x) - \mu g(x) : x \in \Theta\} \quad (15)$$

*Dinkelbach's algorithm* is based on the following observations. Choosing as initial  $\mu$  a value  $\psi(x_1) = \mu$ , we observe that  $h(\mu) < 0$  if and only if  $f(x) - \mu g(x) < 0$  for some point  $x \in \Theta$ . Thus, there are two possible solutions to (15):

- $h(\mu) \geq 0$ : then  $f(x) - \mu g(x) \geq 0$  and  $\frac{f(x)}{g(x)} \geq \mu$  for all  $x \in \Theta$  then  $y^* = x_1$  and  $\mu = \mu_{opt}$ .
- $h(\mu) < 0$ : then solving (15) a point  $\tilde{x} \in \Theta$  is found with  $\psi(\tilde{x}) < \mu$ .

With this in mind, the algorithm is as follows:

1. Set  $i = 1$  and let  $x_i$  be a feasible point of  $\Theta$ , with  $\mu_i = \psi(x_i)$ ;
2. Set  $\mu = \mu_i$  and find  $x_{i+1} \in \Theta$  that solves (15);
3. If  $h(\mu_i) = 0$ , stop and  $x_{i+1}$  is optimal, otherwise set  $i = i + 1$ ,  $\mu_i = \psi(x_i)$  and go to step 2.

The above considerations suggest us to substitute the convex-concave optimization problem in (11) with the following one

$$\begin{aligned} \min_{\mathbf{p}_T} \quad & \mathbf{p}_T^T \mathbf{1} - \mu \lambda_2(\mathbf{L}(\mathbf{p}_T)) \\ \text{s.t.} \quad & \varepsilon \leq \lambda_2(\mathbf{L}(\mathbf{p}_T)) \\ & \mathbf{1} p_{\min} \leq \mathbf{p}_T \end{aligned} \quad (16)$$

It can be noted that  $\lambda_2(\mathbf{L}(\mathbf{p}_T))$  is a concave function of  $\mathbf{p}_T$  (see Theorem 1). Hence, the objective function, as a sum of convex functions, is a convex function. The constraint functions are convex. Then, problem (16) is a convex parametric problem whose unique solution is a function of the parameter  $\mu$  that controls the trade-off between the global transmit power and the convergence time. The problem in (16) can be solved numerically using *cvx*, a Matlab software for disciplined convex programming [16]. Indeed, the use of *cvx* is possible through a preliminary change of variables to avoid the nonlinear dependence of the Laplacian matrix on the power vector  $\mathbf{p}_T$ . To this end, the problem in (16) has been reformulated as

$$\begin{aligned} \min_{\mathbf{y}} \quad & \phi(\mathbf{y}) - \mu \lambda_2(\mathbf{L}(\mathbf{y})) \\ \text{s.t.} \quad & \varepsilon \leq \lambda_2(\mathbf{L}(\mathbf{y})) \\ & \mathbf{0} \leq \mathbf{y} < \mathbf{1} \end{aligned} \quad (17)$$

where the  $n(n-1)/2$  entries  $y_{ij} = y_{ji}$  of the vector  $\mathbf{y} := \text{vec}(y_{ij}, i, j = 1, \dots, n, i \neq j)$  are given by  $y_{ij}(p_{ij}) = a_{ij}(p_{ij})$ , so that according to (10), we obtain

$$p_{ij} = q(y_{ij}) = p_{\min} + k_1 \left( \frac{y_{ij}}{1 - y_{ij}} \right)^{\eta/\alpha}, \quad (18)$$

with

$$k_1 = p_{\min} \frac{r_{ij}^\eta}{r_0^\eta} \quad \text{and} \quad \phi(\mathbf{y}) = \sum_{i=1}^n \sum_{j=1}^n q(y_{ij}). \quad (19)$$

Let us now verify the convexity of problem (17). To study the behavior of  $\phi(\mathbf{y})$ , we compute the second order derivative of  $q(y_{ij})$ , obtaining

$$\frac{d^2 q(y_{ij})}{dy_{ij}^2} = k_1 \frac{\eta}{\alpha} \left( \frac{y_{ij}}{1 - y_{ij}} \right)^{\eta/\alpha - 2} \frac{1}{(1 - y_{ij})^4} \left( \frac{\eta}{\alpha} - 1 + 2y_{ij} \right). \quad (20)$$

This shows that

$$\frac{d^2 q(y_{ij})}{dy_{ij}^2} \geq 0 \Leftrightarrow \eta - \alpha + 2\alpha y_{ij} \geq 0 \quad (21)$$

or

$$y_{ij} \geq \frac{1}{2} \left( 1 - \frac{\eta}{\alpha} \right). \quad (22)$$

We note that, if  $\eta \geq \alpha$ , then  $\frac{d^2 q(y_{ij})}{dy_{ij}^2} \geq 0$  for  $0 \leq y_{ij} < 1$ , so

that  $\phi(\mathbf{y})$ , as a sum of convex functions, is convex. Finally, the algebraic connectivity  $\lambda_2(\mathbf{L}(\mathbf{y}))$  is a concave function of  $\mathbf{y}$ , as can be proved following the same steps as in Theorem 1. Then the optimization problem in (17) is a convex parametric problem equivalent to the original problem in (16), since the change of variables  $p_{ij} = q(y_{ij})$  in (18) ensures a

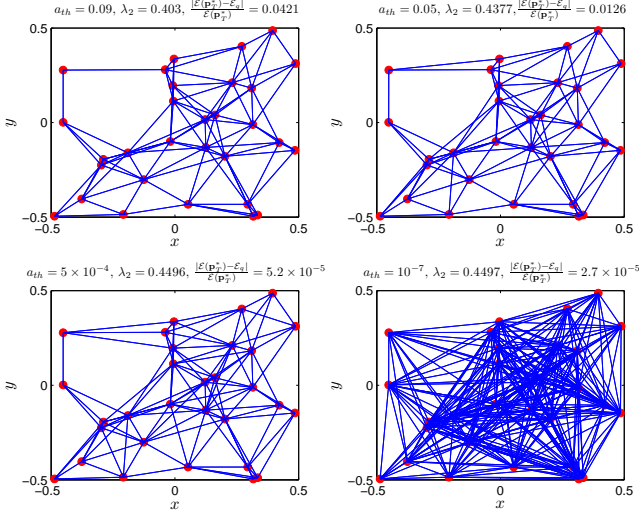


Figure 1: Nodes configuration on the plane and optimal active links for  $r_0 = 0.19$  and several quantization thresholds.

one-to-one mapping  $q : \mathbb{R} \rightarrow \mathbb{R}$  for  $0 \leq y_{ij} < 1$  with image covering the problem domain in (16) (see [17][p. 130]).

Thus we have proved that the optimization problem in (17) is convex; the convexity guarantees that a solution exists, is unique, and can be found via efficient numerical tools. Furthermore,  $\mu$  can be optimized using Dinkelbach's algorithm by assuming  $\mathbf{0} \leq \mathbf{y} \leq \mathbf{1} - \varepsilon'$  with  $\varepsilon'$  an infinitesimal positive constant so that the feasible set of (17) is a compact set of  $\mathbb{R}^{n(n-1)/2}$ .

#### 4. NUMERICAL RESULTS

In our simulations, we considered a sensor network composed of  $n = 30$  nodes, randomly deployed within a unit square. In all the simulations, we assumed  $\varepsilon = 10^{-3}$ ,  $p_{\min} = 1$  and  $\eta = \alpha$ . The scaling distance  $r_0$  was chosen to guarantee network connectivity, although only in a probabilistic sense. More specifically, we set

$$r_0 = r_0(n) = \sqrt{\frac{1.1 \log(n)}{\pi n}} \quad (23)$$

to guarantee that the graph is connected with probability one, as the number of nodes tends to infinity [18]. Recall that we use the relaxation expression of the coefficients  $a_{ij}$  given in (10), thus, in practice, the coefficients  $a_{ij}$  resulting from the optimization algorithm are compared to a threshold  $a_{th}$ , so that if  $a_{ij} \leq a_{th}$  the link between nodes  $i$  and  $j$  is suppressed. Of course, this thresholding operation will affect the final result in terms of convergence time and energy consumption. To check the sensitivity of our algorithm with respect to the threshold value, we report the active links obtained for different values of  $a_{th}$  and  $\eta = 6$ . From Fig. 1, we notice that, below a certain value of  $a_{th}$ , the algebraic connectivity remains practically constant. In the same figure, we also report the normalized difference  $|\mathcal{E}(\mathbf{p}_T^*) - \mathcal{E}_q(\mathbf{p}_T^*)| / \mathcal{E}(\mathbf{p}_T^*)$  between the minimum energy  $\mathcal{E}(\mathbf{p}_T^*)$  and the energy pertaining to the graph with quantized coefficients  $\mathcal{E}_q$ . We can see that this difference becomes very small, even with very small threshold values. This means that the algorithm, in spite of the

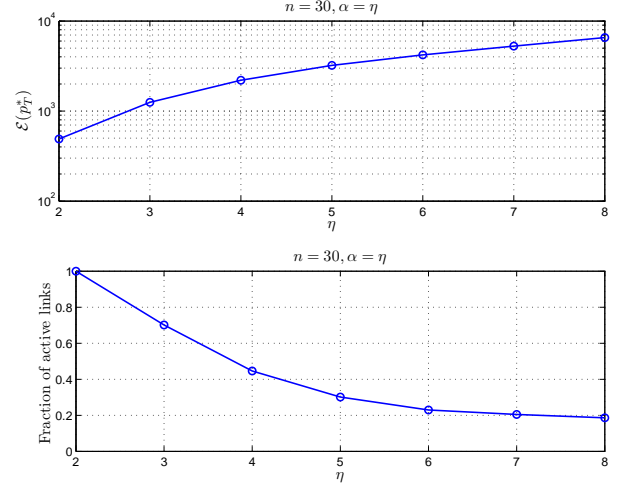


Figure 2: Optimal energy consumption and fraction of active links versus  $\eta$  for  $a_{th} = 0.09$ .

relaxation step, is rather stable in finding the links that can be turned off, with no appreciable performance loss. This is indeed a crucial property, as it provides a topology control mechanism, although passing through an intermediate relaxation step. From Fig. 1, for example, we can say that the topology in the top left side is practically equivalent to the almost full topology of the bottom right plot, from the point of view of energy consumption and convergence time.

It is also interesting to look at the impact of the path loss exponent on energy consumption and optimal network topology. In Fig. 2 we report, respectively in the upper and lower subplots, the optimal energy consumption and the percentage of active links  $\frac{\sum_{i=1}^n |\mathcal{N}_i|}{n(n-1)}$ , versus  $\eta = \alpha$ . The fraction of active links gives us a measure of the sparsity of the final network. It is interesting to observe, from Fig. 2 (bottom plot) that, for low path loss exponents, the network tends to be fully connected, because evidently, for those values of  $\eta$ , the most critical contribution to energy consumption is convergence time. Hence, the resulting topology is the one that minimizes convergence time. Conversely, as  $\eta$  increases, the power consumption tends to assume more and more importance and the resulting optimal topology becomes more and more sparse. In all cases, clearly the energy consumption increases as  $\eta$  increases (top plot).

Finally, in Fig. 3 we compare the energy consumption obtained using a common transmit power (solid lines) or the optimal power allocation  $\mathbf{p}_T^*$  (colored dots). The results have been averaged over 100 independent random deployments of  $n = 30$  nodes, uniformly placed on the unit square. For the same random node configurations, we consider a random geometric graph where each node transmits with the same per link power  $p$  to the nodes lying within its coverage area. We compute the average energy consumption  $\bar{\mathcal{E}}(p)$  by averaging

the measure  $\mathcal{E}(p) = \frac{p \sum_{i=1}^n n_i(p)}{\lambda_2(p)}$  for different values of  $p$ , where  $n_i(p)$  denotes the random number of neighbors of node  $i$ , when transmitting at power  $p$ . We plot this curve versus the average network power (i.e., the mean of the sum over



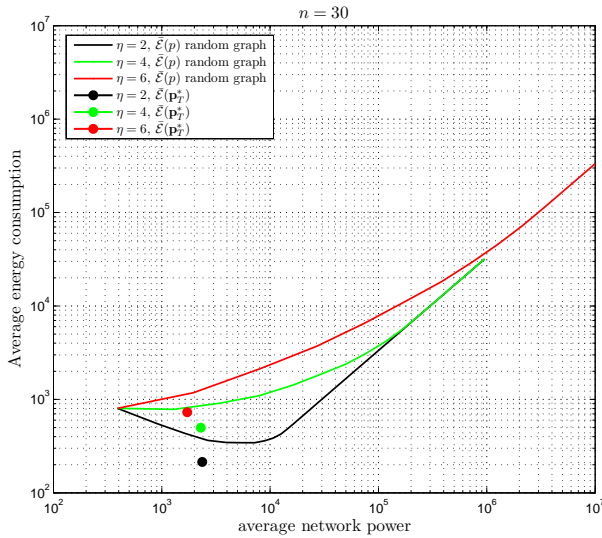


Figure 3: Average network energy consumption versus the average network power.

all the nodes of the per link powers). We observe from Fig. 3 that, also using a common transmit power, there typically is an optimal value that minimizes energy consumption. That value is the result of a trade-off between transmit power and convergence time. As expected, for any path loss exponent, the proposed joint optimization of transmit power and topology leads to a smaller energy consumption with respect to the common transmit power case.

## 5. CONCLUSIONS

In this paper we have proposed a method to optimize the network topology that, for a given arbitrary position of the nodes, minimizes the energy consumption necessary to reach average consensus. In particular, through an appropriate relaxation technique, we have converted a topology optimization problem, which is typically a combinatorial problem, into a convex-concave fractional program that admits a unique solution, obtainable through efficient numerical algorithms. As a by-product of our optimization procedure, we get both network configuration and optimal power allocation across all links. After applying the proposed relaxation, the topology is obtained by applying a thresholding mechanism. In principle, this operation may induce non negligible errors. Nevertheless, for the application at hand, we showed, through numerical results, that the thresholding operation does not affect the network performance appreciably. Interestingly, it turns out that, at low path loss exponents, the best topology tends to be fully connected. Conversely, at higher values of the path loss exponent, the optimal topology tends to be more and more sparse, so as to reduce as much as possible the number of active links. The problem formulation required to have a number of degrees of freedom equal to the number of possible links. In this way we end up with, potentially, different powers on each link. Hence, we do not exploit the broadcast capabilities of wireless networks. This is an issue that could be further investigated, together with the choice of the appropriate radio access technique, to see whether there can be further margins of improvements.

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