# ESTIMATING FREQUENCY CHARACTERISTICS OF QUANTIZATION NOISE FOR PERFORMANCE EVALUATION OF FIXED POINT SYSTEMS

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**ABSTRACT**

Word-length optimization of signal processing algorithms is a necessary and crucial step for implementation. System level performance evaluation happens to be the most time consuming step during word-length optimization. Analytical techniques have been proposed as an alternative to simulation based approach to accelerate this step. The inability to handle all types of operators analytically and the increasing diversity and complexity of signal processing algorithms demand a mixed evaluation approach where both simulation and analytical techniques are used for performance evaluation of the whole system. The interoperability between simulation and analytical techniques requires study of noise sources and noise propagation characteristics. While the noise power and noise PDF have been studied, the output noise power distribution has not been studied. This paper addresses the problem of power spectral density estimation of the noise analytically. This paper also proposes to use the spectral density estimate for noise power calculation by having an approximate filter thereby accelerating the process of performance evaluation.

## 1. INTRODUCTION

Signal processing algorithms are increasingly finding their way into modern electronic gadgets. While developing the algorithm to fulfil objectives is quite a challenge, optimal implementation of these algorithms such that the implementation cost is kept minimal consists of many tough challenges. One such challenge is to choose optimal computational kernels having the right word-lengths to perform the signal processing computations. It is an obvious choice to use fixed point kernels over their floating point counter parts to save area, power and time.

Finite precision word-lengths introduce quantization errors into the system. These errors have been studied with the help of Widrow quantization [1] model which treats such noise as small signal perturbations more popularly referred to as quantization noise. The amount of quantization noise introduced into the system brings to the forefront an optimization problem which makes a choice between the cost and accuracy of the system.

The process of optimizing word-lengths is typically an iterative process where an optimization heuristic is used to determine optimality by comparing the system cost and the corresponding system performance. While good optimization heuristics can reduce the number of iterations, efficient techniques for cost and performance evaluation reduce the time taken for each optimization iteration. The cost of the system is generally defined as a function of area occupied, total power consumed and the delay in execution by the system. Cost evaluation is typically the sum of individual component costs in the system. The performance of any signal processing system is evaluated in terms of a system specific. Typically, parameters such as output noise power, signal to noise ratio (SNR), bit error rate (BER) are used for this purpose. Evaluating performance requires the knowledge of the system functionality and is hence not trivial.

A hierarchical methodology has been developed [2] for efficient optimization and performance evaluation of large systems. At the heart of this methodology is the single-noise source model, which is used to sub-divide the optimization problem into smaller optimization problems. In this paper, we further develop the single-noise source model to incorporate the noise frequency response estimation.

The paper is organised as follows, the next section provides motivation and previous related work. Section 3 describes the hierarchical framework, the role of single-noise source model and the importance of estimation of the frequency response. Section 4 develops the necessary mathematical framework to deal with non-linear time-variant but stationary sub-systems. Section 5 applies the proposed framework for output noise spectrum analysis of a non-linear second order Volterra filter and presents the results obtained thereof. In Section 6, the paper concludes with a summary of contributions and discuss future work.

## 2. MOTIVATION

Simulation based techniques and analytical models have been proposed to measure performance of fixed-point systems. While the simulation based techniques [3, 4] are applicable universally, long execution times is a deterrent to use them always. On the other hand, the analytical models provide closed form expressions for evaluating noise power for different choice of word-lengths. The noise power can then be used to evaluate the performance metric of the system. However, analytical models have not been developed thus far for all types of systems (e.g. un-smooth operators such as decision operators) and are applicable only on a subset of systems.

Complex signal processing systems are often described with well known signal processing sub-systems. The existing techniques for performance evaluation such as [5] require the system to be flattened all the way to the operator level in order to consolidate the noise contributions to the output. This is not only cumbersome but it is also sometimes impossible as there can be sub-systems made of operators which do not have an analytical model. Hence a hybrid approach to measure performance that can exploit the good of both simulation and analytical approaches is formulated. Also, the

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existing optimization techniques concentrate mostly on optimizing word-lengths at the operator level and treat the word-length of each operator as an optimization variable. The hierarchical methodology proposed in [2] uses a divide and conquer approach which makes it easier to scale the optimization algorithm on bigger systems. The single-noise source model which is at the heart of this approach captures the effect of quantization noise at the sub-system level for use in the system level optimization.

In a hierarchical approach, it is important that the noise generated in the sub-systems is propagated across hierarchies and eventually to the system output. The sub-systems through which the noise is propagated could be sensitive to noise distribution and frequency spectrum. Hence, instead of a simplistic metric like noise power, the quantization noise spectral density and distribution needs to be studied at the output of every sub-system. In the case of sub-systems made of arithmetic operators, it has already been shown experimentally that the noise distribution is a parametric summation of uniform and Gaussian distributions [6]. In this paper, we focus on characterizing the spectral characteristics of the sub-system generated quantization noise.

A technique to shape the noise spectral characteristics for LTI systems has been proposed in [7]. In this approach, the fixed point design optimization constraints are specified by means of spectral characteristics of noise. The proposed algorithm determines the word-lengths such that the spectral constraints are met while minimizing area. The proposed technique is interesting as it relates word-length choice to the spectral density function. However, the problem consideration in this approach is converse of the problem considered in this paper. Also, the proposed technique has not been applied on non-linear or time-variant systems. Thus, the problem of spectral estimation of the quantization noise spectral density remains largely unexplored.

This paper demonstrates a simple technique to estimate the power spectral density of the output quantization noise of the sub-system and proposes a linear filter model to model the effect of quantization at various sub-system hierarchical levels. The proposed model is shown to be applicable to non-linear, time variant systems.

## 3. THE HIERARCHICAL APPROACH

The hierarchical approach in [2], a divide-and-conquer strategy is used to solve the multi-variable optimization problem. The system is hierarchically divided at the boundaries of predefined sub-systems recursively. The problem of system optimization in this hierarchical approach is formulated as

$$\min\left(C\left(\mathbf{P}\right)\right) \quad \text{such as} \quad \lambda\left(\mathbf{P}\right) \geq \lambda_{obj} \tag{1}$$

where, $\mathbf{P} = [P_0, P_1, \ldots, P_{N_b-1}]$ is a vector of sub-system noise powers at any given hierarchy. Each of $P_i$ represents the total noise power at the output of the $i^{th}$ sub-system. This technique uses the sub-system noise power as the optimization variable.

Each sub-system is optimized by solving the original optimization problem which is defined at the word-length level stated as

$$\min(C_i(\mathbf{W_{D_i}})) \quad \text{such as} \quad f_{P_i}(\mathbf{W_{D_i}}) < P_i \tag{2}$$

where $W_{D_i}$ is the vector of word-lengths and $C_i$, $f_{P_i}$ are respectively the cost and performance functions of the $i^{th}$ sub-system. It is easy to see from comparison between Equations 1 and 2 that the sub-system noise power replaces the noise due to word-length quantization in the global optimization problem.

To evaluate the performance at the system level, the entire system is simulated at the system level with double precision floating point accuracy only once to collect the data required for use with the noise model. The fixed-point system is then replaced by the single-noise source and the double precision floating point system. Any greedy optimization technique is used to optimize the sub-system optimization problem stated in Equation 2. The same could be applied recursively up to the top-level design abstraction to solve the problem defined by Equation 1.

While the proposed divide-and-conquer approach reduces the complexity of the problem by using just the noise power, performance evaluation at the system level is not a simple function of the noise power. It is clear from first principles of signal processing that the performance of the total system performance would depend on the sub-system noise spectral and distribution characteristics. Hence, from the perspective of performance evaluation, study of noise spectral density and distribution is important. While the word-length quantization effects have been extensively studied in terms of quantization noise power, little has been done to understand and analyse other properties of quantization noise at the sub-system level.

### 3.1 Single Noise Source Model

The single-noise source model allows the use of output noise power as given by Equation 1. To explain this central theme, consider a sub-system $B$ in a hierarchically defined system $\mathbb{B}$ with input $x$ and output $s$ as shown in Figure 1. The noises $b_x$ and $b_s$ are associated with signals $x$ and $s$ respectively. The power $P_i$ in Equation 1 corresponds to the noise power of the signal $s$. The total quantization noise $b_s$ at the output of the system consists of two components $b_{sg}$ and $b_{st}$ for the noise generated within the system and noise transmitted through the system respectively.

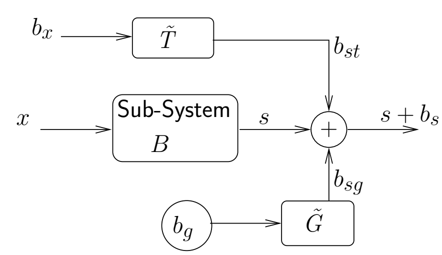


Figure 1: Single Noise Source Model

The noise $b_x(n)$ associated with the input signal $x(n)$ is independent of the signal. The effect of this noise at the output $b_{st}$ is calculated by passing it through the noise propagation filter $\tilde{T}$ which modifies the power spectrum of $b_x$ like the sub-system $B$. The noise $b_{sg}$, generated in the sub-system $B$ is modeled by passing the single-noise source $b_g$ through the noise generation filter $\tilde{G}$. The noise generation filter shapes the spectral characteristics of the noise to represent the effect of quantization noise generated within the sub-system $B$. It has been shown in [6] that the output PDF is not uniform and is in fact closer to being a Gaussian (due to central limit theorem). Hence, noise source $b_g$ is modeled as white Gaussian.

The noise model used for propagation of quantization noise through arithmetic operators [5] essentially linearizes the noise propagation for all types of arithmetic nonlinearities under the conditions described by the Widrow model [1]. Following the Widrow model for quantization noise, all the operator noise sources are known to be uniform and white. The noise propagation models however tend to be time varying in nature.

Taking both noise propagated and generated into consideration and given that the noise models are linear, the total noise at the output of sub-system $B_i$ is given by

$$b_s(n) = \tilde{g}(n) \odot b_g(n) + \tilde{t}(n) \odot b_x(n). \tag{3}$$

Where $\tilde{g}(n)$ and $\tilde{t}(n)$ are the impulse response of the filters $\tilde{G}$ and $\tilde{T}$ from Figure 1 respectively. As long as the output noise is stationary, it is possible to compute its noise power density spectrum $S_{b_s b_s}(e^{j\omega})$. Deriving an expression for analytical evaluation of the noise power spectrum at the output is the primary issue addressed in this paper.

### 3.2 Accelerating Performance Evaluation

While evaluating the sub-system noise, the noise power, its spectral density and distribution are estimated. The noise spectrum is represented by an equivalent linear filter whose magnitude response is a close approximation of the estimated spectrum. The linear filters replace the original sub-system while preserving the topology of the signal network. The idea here is to provide a framework which can be used for CAD-based automation for mixed performance evaluation technique.

The noise-equivalent is used in order to eliminate the necessity of a full simulation to estimate the performance degradation with every combination of word-lengths. However, the original signal is required for evaluating systems which cannot be handled analytically (such as decision operators). For handling such cases, the signal at the output of the sub-system which is the sub-system response to the input signal is obtained by means of a floating point simulation and stored in memory. The stored signal along with the filters successfully mimic the behavior of the original sub-system, thus providing an analytical means to perform the performance estimation of the fixed point system without having to simulate the sub-system. This method provides a seamless framework for interoperability between usage of simulation and analytical modeling for performance evaluation of the entire system. This technique also enables traversing hierarchies and utilizes the sub-system filter model to construct its equivalent counterparts at any higher hierarchical level.

## 4. ESTIMATING FREQUENCY RESPONSE

Consider a sub-system $B$ which is a non-linear time variant system as shown in Figure 1 with noise generation and noise propagation system functions $\tilde{T}$ and $\tilde{G}$ respectively. Both of these filters are similar for analysis in the context of this paper except that, the noise power spectrum of the input to the generation filter $b_g(n)$ is always known to be white while the input noise $b_x(n)$ does not need to be white. Thus the results obtained in the following Equations are equally applicable to both system functions.

The noise propagation model for all arithmetic operator-based functions (including non-linear) can be linearized [5]. Hence without any loss of generality, the propagation and generation equivalent system functions is considered to be linear and time-varying system.

### 4.1 Windowing for Stationarity

Frequency spectrum of the noise is meaningful only when the noise is stationary. However, not all signals are stationary. One way of handling this problem is by windowing the signals such that they are locally stationary [8]. A pathological scenario is where there are too many such windows and the windowing overhead out weighs the benefits. In such extreme cases, one has to revert back to simulation based techniques.

The windowing is performed such that all relevant signals in the given window may be considered stationary. This sense of pseudo-stationarity or temporally stationarity enables estimation of the noise power spectrum. It is not always easy to find windows where the so called pseudo-stationarity would exist. In such cases, the definition of pseudo needs to be approximated such that the performance degradation is still conservative.

### 4.2 Noise Power Spectrum

The noise power spectral distribution at the sub-system output is the magnitude response obtained by calculating the Fourier transform of the autocorrelation function.

The output of any time varying system $H$ is obtained by the convolution operation of the system function with the input signal. Consider a time window $W$ wherein all the coefficients of the varying system $H$ are stationary. The output noise $b_y(n)$ contributed by a noise $b_x(n)$ at the input given by

$$b_y(n) = \sum_{k=-\infty}^{\infty} h(k,n) b_x(n-k) \tag{4}$$

Here, $h(k,n)$ can be thought of as the *pseudo impulse response* of the time varying system $H$ at any discrete point in time $n$ at the $k^{th}$ tap. The autocorrelation function $R_{b_y b_y}$ of the noise $b_y(n)$ at the output of such a system can be expanded and simplified as

$$\begin{aligned} R_{b_y b_y}(m) &= E\{b_y(n) b_y(n+m)\} \\ &= \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} E\{h(k,n)h(r,n+m)\ldots \\ &\qquad b_x(n-k) b_x(n+m-r)\} \end{aligned} \tag{5}$$

Applying Fourier transform on both sides of Equation 5, the power spectral density of the noise at output can be written and expanded as

$$\begin{aligned} S_{b_y b_y}(e^{jw}) = \sum_{m=-\infty}^{\infty} E\{ \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} h(k,n)h(r,n+m)\ldots \\ b_x(n-k) b_x(n+m-r)\} e^{-j\omega m} \end{aligned} \tag{6}$$

So,

$$\begin{aligned} S_{b_y b_y}(e^{jw}) = E\{ \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} h(k,n)h(r,n+m)\ldots \\ b_x(n-k) b_x(n+m-r) e^{-j\omega m}\} \end{aligned} \tag{7}$$

By taking $m = p + r - k$ in Equation 6, the expression power spectral density can be written as

$$
\begin{aligned}
S_{b_y b_y}(e^{jw}) = E\{ & \sum_{p=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} h(k,n) e^{j\omega k} \ldots \\
& b_x(n-k) b_x(n+p-k) e^{-j\omega p} \ldots \\
& h(r, n+p-k) e^{-j\omega r} \} \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
S_{b_y b_y}(e^{jw}) = E\{ & \sum_{k=-\infty}^{\infty} h(k,n) e^{j\omega k} \ldots \\
& \sum_{r=-\infty}^{\infty} h(r, n+p-k) e^{-j\omega r} \} \ldots \\
& \underbrace{E\{ \sum_{p=-\infty}^{\infty} b_x(n-k) b_x(n+p-k) e^{-j\omega p} \}}_{S_{b_x b_x}(e^{j\omega})} \qquad (9)
\end{aligned}
$$

Thus the noise spectrum at the output can be written as

$$
S_{b_y b_y}(e^{jw}) = \tilde{H}_{k,r}(e^{j\omega}) S_{b_x b_x}(e^{j\omega}) \qquad (10)
$$

where $\tilde{H}_{k,r}(e^{j\omega})$ is defined as

$$
\tilde{H}_{k,r}(e^{j\omega}) = E\{ \underbrace{\sum_{k=-\infty}^{\infty} h(k,n) e^{j\omega k}}_{\tilde{H}_k(e^{-j\omega})} \underbrace{\sum_{r=-\infty}^{\infty} h(r, n+p-k) e^{-j\omega r} \}}_{\tilde{H}_r(e^{j\omega})} \qquad (11)
$$

The $h(k,n)$ and $h(r,n)$ terms in Equation 11 are the time-varying coefficients of the system under consideration. By expanding the Equation 11 with $(k,r) \in \ (-\infty, \infty)$, and utilizing the available symmetry, the expression for the power spectrum of the transfer function can be written as follows

$$
\tilde{H}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \sum_{\Delta=0}^{\infty} (E\{h_k h_{k+\Delta}\} + E\{h_{k+\Delta} h_k\}) cos(\Delta\omega) \qquad (12)
$$

Clearly, the computation of $\tilde{H}$ requires the correlation evaluated at 0 $\forall (k, \Delta)$ in Equation 12. This is calculated only once from one single floating point simulation of the system and is independent of the fixed point format. Hence, it is used in every iteration of the optimization process thereby delivering the acceleration during optimization over simulation based approaches.

In case of LTI systems, the filter coefficients are constant. Hence it is possible to write the spectrum of the filter as

$$
\tilde{H}(e^{j\omega}) = \sum_{i=-\infty}^{\infty} \sum_{\Delta=0}^{\infty} (h_i h_{i+\Delta} + h_{i+\Delta} h_i) cos(\Delta\omega) \qquad (13)
$$

From Equation 13, it can be seen that the expression for $\tilde{H}(e^{j\omega})$ degenerates to $|\tilde{H}_i(e^{j\omega})|^2$ which is a classical result obtained in the case of LTI systems, thus also verifying the expression for the spectrum in the LTI case.

### 4.3 Complexity Analysis

The proposed technique attempts to provide a closed form expression as given in Equation 13 to represent the noise coloring filter. The effort for computation of the frequency response depends on the number of coefficients whose auto correlation and cross correlations are to be found. When there are $N_d$ number of delays in the noise expression, the total number of correlation terms $N_c$ that need to be calculated is ${N_d}^2$.

Each of the delay terms in the noise expression is associated with a coefficient. These coefficient terms are typically dependent on the input signal and hence need to be calculated for every input sample $N_s$. The effort involved in computing each correlation at 0 is $O(N_s)$. Hence the total time for evaluating the filter expression $T_{Expr}$ is given by

$$
T_{Expr} = {N_d}^2 . t_{Corr} + N_d . t_{Coeff}. \qquad (14)
$$

Where $t_{Corr}$ is the time for computing one correlation term and $T_{Coeff}$ is the average time to evaluate each coefficient expression. The multiplication and addition operations required during correlation and the time for coefficient expression are proportional to the number of samples. Hence, it is possible to estimate the time required for computing correlation terms and coefficient terms can be written as

$$
t_{Corr} = \alpha_{Corr} . N_s \qquad (15)
$$

$$
t_{Coeff} = \alpha_{Coeff} . N_s \qquad (16)
$$

Where $\alpha_{Corr}$ and $\alpha_{Coeff}$ are proportional constants. Though there is a contribution of the $N_d$ term in the complexity estimation, it is of the order of few tens of elements. Whereas the order of $N_s$ is in millions of samples. Thus, it can be concluded that the computational complexity to determine the analytical expression for the system is $O(N_s)$.

In a simulation based approach the performance evaluation takes $O(N_s)$ time and it has to be repeated for every iteration. In the proposed analytical method, though the time to arrive at the analytical expression is $O(N_s)$, it is performed only once which can be regarded as a pre-processing step and hence better than any simulation based approach.

## 5. EXPERIMENT AND RESULTS

### 5.1 Volterra Filter

While it is trivial to obtain the transfer function of an LTI system, an equivalent propagation and generation filter is considered for a non-linear second-order Volterra filter. Given input $x(n)$, the output $y(n)$ for the Volterra filter is given by

$$
\begin{aligned}
y(n) = & a_{11} x^2(n) + a_{22} x^2(n-1) + \ldots \\
& a_{21} x(n) x(n-1) + a_1 x(n) + a_2 x(n-1). \qquad (17)
\end{aligned}
$$

By replacing the operators with their equivalent noise model, the noise output expression for the Volterra filter is

$$
\begin{aligned}
b_y = & \underbrace{\{2a_{11}x(n) + a_1 + a_{12}x(n-1)\}}_{\alpha_1} b_x(n) + \\
& \underbrace{\{2a_{22}x(n-1) + a_2 + a_{12}x(n)\}}_{\alpha_2} b_x(n-1) + b_g(n) \qquad (18)
\end{aligned}
$$

where $b_g$ groups together all the noise generated inside the system. In this example, all the noise generated within the system happens to be white and is not very interesting to study. Also, in the course of study, no quantization effects are introduced within the filter. Therefore, only the propagation filter is presented. The propagation filter is given as

$$\tilde{h}_i(n) = \alpha_1(n)b_x(n) + \alpha_2(n)b_x(n-1) \quad (19)$$

The terms $\alpha_1(n)$ and $\alpha_2(n)$ are time varying. Thus, by expanding the transfer function Equation 12, the spectrum of the filter becomes

$$|H(e^{j\omega})|^2 = (R_{\alpha_1,\alpha_1}(0) + R_{\alpha_2,\alpha_2}(0)) + (R_{\alpha_1,\alpha_2}(0) + R_{\alpha_2,\alpha_1}(0))cos(\omega) \quad (20)$$

where $R_{\alpha_i,\alpha_j}(0)$ are the cross correlation and auto correlation functions.

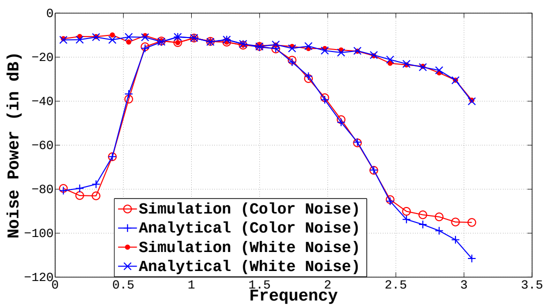


Figure 2: Output noise power spectrum estimation for Volterra

As input to the Volterra filter, a sine wave perturbed by Gaussian noise is used as the input to the system. The correlation coefficients and the impulse response of the equivalent filter is calculated. The magnitude response of the filter computed analytically is compared with the noise spectrum at the output of Volterra filter which is obtained by fixed-point simulation.

When the perturbation noise input into the Volterra filter is white, the output noise spectrum traces the frequency response. In the Figure 2, the output noise spectrum calculated analytically and the spectrum obtained by simulation match very closely. In the Volterra filter example considered, the presence of $cos(\omega)$ term in the magnitude response as shown in Equation 20 is clearly seen as the spectrum tapers towards the x-axis at higher frequencies. In the case of a colored noise, the noise used to perturb the sinusoid is a coloured band-pass noise. The output noise spectrum obtained analytically matches the spectrum obtained by simulation. It has to be noted that the effect of the $cos(\omega)$ term is visible even in the coloured noise case.

The outcome of the experiments suggests that, the frequency response of the filter derived analytically by adopting the procedure described in this paper can faithfully represents the frequency characteristics of the system.

## 6. CONCLUSION

A technique to estimate the noise power spectrum of arithmetic operator based systems and thereby accelerating the performance evaluation of fixed point systems is presented. Computation of the power spectrum in this technique requires the auto-correlation functions of the filter coefficients evaluated at 0 and is hence not computation intensive. The original sub-system is replaced by the double precision floating point system and linear time-invariant filters whose magnitude spectrum is a close approximation of the noise power spectrum. The floating point data and the filters are used for all noise power calculation during every iteration during optimization. The idea is to be able to seamlessly use this technique in a complex and hierarchically defined signal processing system to find equivalent approximations at all hierarchical levels. Using this technique accelerates the performance evaluation process in situations where both simulation and analytical techniques are used. This technique also helps in noise power budgeting in situations where a divide-and-conquer algorithm is used to for optimization.

This paper contributes by providing the necessary framework and a generic filter coefficient formula for computing the spectrum of any arithmetic based signal processing system. The acceleration obtained is due to the fact that, a fixed point evaluation of the system is not required during every iteration of the optimization algorithm. Though the proposed method requires windowing in case of non-stationary signals, it provides an alternative to complete simulation. As long as it is possible to define pseudo-stationary windows over the signal, this alternative stands to gains over a pure simulation approach.

## REFERENCES


[1] B. Widrow, I. Kollár, and M.-C. Liu, "Statistical Theory of Quantization," *IEEE Trans. on Instrumentation and Measurement*, vol. 45, no. 2, pp. 353–61, Apr. 1996.

[2] K. Parashar, D. Menard, R. Rocher, and O. Sentieys, "A hierarchical Methodology for Word-Length Optimization of Signal Processing Systems," in *23rd International Conference on VLSI Design*, Bangalore, India, Jan. 2010.

[3] H. Keding, M. Willems, M. Coors, and H. Meyr, "FRIDGE: A Fixed-Point Design And Simulation Environment," in *IEEE/ACM conference on Design, Automation and Test in Europe 1998 (DATE 98)*, Paris, France, march 1998, pp. 429 – 435.

[4] W. Sung and K. Kum, "Simulation-Based Word-Length Optimization Method for Fixed-Point Digital Signal Processing Systems," *IEEE Transactions on Signal Processing*, vol. 43, no. 12, pp. 3087–3090, Dec. 1995.

[5] D. Menard, R. Rocher, and O. Sentieys, "Analytical Fixed-Point Accuracy Evaluation in Linear Time-Invariant Systems," *IEEE Transactions on Circuits and Systems I: Regular Papers,*, vol. 55, no. 1, November 2008.

[6] D. Menard, R. Serizel, R. Rocher, and O. Sentieys, "Accuracy Constraint Determination in Fixed-Point System Design," *EURASIP Journal on Embedded Systems*, vol. 2008, p. 12, 2008.

[7] G. Constantinides, P. Cheung, and W. Luk, "Roundoff-noise shaping in filter design," in *IEEE Symposium on Circuits and Systems, ISCAS*. Geneva, Switzerland: Institute of Electrical and Electronics Engineers, May 28-31 2000, pp. IV57–IV60.

[8] C. Fancourt and J.C.Principe, "Competitive principal component analysis for locally stationary time series," *Signal Processing, IEEE Transactions on*, vol. 46, no. 11, pp. 3068 –3081, Nov 1998.