STEADY-STATE ANALYSIS OF A QUANTIZED AVERAGE CONSENSUS ALGORITHM USING STATE-SPACE DESCRIPTION

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ABSTRACT

Following our recently developed method we provide a proof of convergence of the average consensus algorithm with quantized communication links as proposed by Censi and Murray. Using a state-space framework for describing distributed algorithms, we can derive accurate bounds on the drift from the mean for algorithms with noisy links, either caused by an external noise or by quantization. We then test these bounds for several network topologies and compare with simulations.

1. INTRODUCTION

Reaching the capabilities of serial processing hardware and increased usage of inter-connected devices over the past several years have led to increased research interest in parallel and distributed algorithms. While considering a network of, more or less, computationally constrained devices, simple gossip-based message-passing algorithms, were proposed, solving also more complex problems.

Distributed averaging algorithms, e.g. push-sum [5], average consensus [7] or consensus propagation [6], have been studied from different points of view using matrix theory, theory of Markov chains, control theory, and more. Since the real environments introduce significant constraints on the performance and accuracy, quantized algorithms have also appeared in the literature [1, 3, 4] and quantization errors have been widely studied [2, 8].

In our previous paper [8] we proposed a framework which naturally arose from the connection between the local, node-based, and the global, network-based, algorithm, leading to a state-space description of distributed algorithms. With this formalism we were able to study the impact of computational and communicational imperfections on the behaviour of distributed averaging algorithms.

Organization of the paper: In Section 2 we briefly recall the applied formalism. In Section 3 we derive the convergence behaviour of quantized consensus and show the form of the steady-state. In Section 4 a-priori bounds for several network topologies are compared with bounds proposed by Censi and Murray [3] and with simulation results.

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2. FRAMEWORK

First, we briefly revise the framework which we are going to use throughout this paper.

If not stated otherwise we always consider a network to be a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of $|\mathcal{V}|$ vertices (nodes) and $\mathcal{E}$ is a set of $|\mathcal{E}|$ edges (links). The graph is described by its adjacency matrix

$$(\mathbf{A}_\mathcal{G})_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{if } (i,j) \notin \mathcal{E}. \end{cases} \tag{1}$$

The graph is supposed to have no self-loops ($e = (v, v) \notin \mathcal{E}$) and each element of $\mathcal{E}$ is unique. Moreover, we implicitly consider the graph $\mathcal{G}$ to be strongly-connected.

2.1 Linear homogeneously distributed algorithms

A distributed algorithm is said to be *homogenously distributed*(HDA) when each node processes messages and calculates data in the same manner. There is no qualitative difference between processing nodes, meaning that they can be interchanged without any impact on the global behaviour of the algorithm.

A linear homogenously distributed algorithm is then an HDA with update and communication strategy being linear functions. We can formalize it in the following definition.

Definition 1 (Linear HDA). *If* update *strategy and the* communication *strategy in the node v are linear functions, then the algorithm is said to be* ***linear homogeneously distributed*** *and can be described as follows*[1]*:*

$$\mathbf{x}_v(k+1) = \alpha_v \sum_{u \in Rec_v(k)} \mathbf{y}_u(k) + \beta_v \mathbf{x}_v(k), \tag{2}$$

$$\mathbf{y}_v(k+1) = \gamma_v \sum_{u \in Rec_v(k)} \mathbf{y}_u(k) + \delta_v \mathbf{x}_v(k), \tag{3}$$

where $\alpha_v \in \mathbb{R}^{n_X \times n_Y}$ (receptivity), $\beta_v \in \mathbb{R}^{n_X \times n_X}$ (self-transmissivity), $\theta_v \in \mathbb{R}^{n_X \times n_U}$ (absorptivity), $\gamma_v \in \mathbb{R}^{n_Y \times n_Y}$ (transmissivity), $\delta_v \in \mathbb{R}^{n_Y \times n_X}$ (distributivity) and $\vartheta_v \in \mathbb{R}^{n_Y \times n_U}$ (emissivity) are constant matrices.

[1,2] We omit here the measurements and the algorithm in the node; for a more extensive definition see [8].

2.2 Global algorithm

If we further aggregate parameters

$$\boldsymbol{\alpha} \triangleq diag(\alpha_1,\ldots,\alpha_v,\ldots,\alpha_{|\mathcal{V}|}) \in \mathbb{R}^{(|\mathcal{V}|n_X)\times(|\mathcal{V}|n_Y)}, \quad (4)$$
$$\boldsymbol{\beta} \triangleq diag(\beta_1,\ldots,\beta_v,\ldots,\beta_{|\mathcal{V}|}) \in \mathbb{R}^{(|\mathcal{V}|n_X)\times(|\mathcal{V}|n_X)}, \quad (5)$$
$$\boldsymbol{\gamma} \triangleq diag(\gamma_1,\ldots,\gamma_v,\ldots,\gamma_{|\mathcal{V}|}) \in \mathbb{R}^{(|\mathcal{V}|n_Y)\times(|\mathcal{V}|n_Y)}, \quad (6)$$
$$\boldsymbol{\delta} \triangleq diag(\delta_1,\ldots,\delta_v,\ldots,\delta_{|\mathcal{V}|}) \in \mathbb{R}^{(|\mathcal{V}|n_Y)\times(|\mathcal{V}|n_X)}, \quad (7)$$

and vectors $\mathbf{x}_v(k)$ and $\mathbf{y}_v(k)$, we can make connections between *local* (node-based) and *global* (network-based) algorithm.

Proposition 1 (Global algorithm). *Using the set of parameters (4)–(7), a global algorithm (the aggregation of the algorithms of all the nodes) can be formulated by*[2]*:*

$$\mathbf{x}(k+1) = \boldsymbol{\alpha}\left(\mathbf{A}_{\mathcal{G}'(k)} \otimes \mathbf{I}_{n_Y}\right)\mathbf{y}(k) + \boldsymbol{\beta}\mathbf{x}(k), \quad (8)$$
$$\mathbf{y}(k+1) = \boldsymbol{\gamma}\left(\mathbf{A}_{\mathcal{G}'(k)} \otimes \mathbf{I}_{n_Y}\right)\mathbf{y}(k) + \boldsymbol{\delta}\mathbf{x}(k). \quad (9)$$

If we aggregate data $\mathbf{x}(k)$ *and* $\mathbf{y}(k)$, *i.e.* $\boldsymbol{\Gamma}(k) \triangleq \begin{pmatrix}\mathbf{x}(k)\\ \mathbf{y}(k)\end{pmatrix}$, *we can rewrite the equations in a compact state-space form as*

$$\boldsymbol{\Gamma}(k+1) = \underbrace{\begin{pmatrix}\boldsymbol{\beta} & \boldsymbol{\alpha}(\mathbf{A}_{\mathcal{G}'(k)} \otimes \mathbf{I}_{n_Y})\\ \boldsymbol{\delta} & \boldsymbol{\gamma}(\mathbf{A}_{\mathcal{G}'(k)} \otimes \mathbf{I}_{n_Y})\end{pmatrix}}_{\mathbf{P}(k)}\boldsymbol{\Gamma}(k). \quad (10)$$

Considering noises on $\mathbf{x}(k)$ and $\mathbf{y}(k)$, due to imprecise computation and limited communication, we model this by adding a noise term, thus having a noisy process

$$\boldsymbol{\Gamma}^\star(k+1) = \mathbf{P}(k)\boldsymbol{\Gamma}^\star(k) + \mathbf{R}\boldsymbol{\zeta}(k), \quad (11)$$

where $\mathbf{R}$ is any linear transformation of the noise term $\boldsymbol{\zeta}(k)$[3].

Theorem 1. *Considering a case where* $\mathbf{P}$ *is constant in time. Then, the term* $\Delta\boldsymbol{\Gamma}^\star(k) \triangleq \boldsymbol{\Gamma}^\star(k) - \boldsymbol{\Gamma}(k)$ *is the noise added to* $\boldsymbol{\Gamma}(k)$ *and satisfies*

$$\Delta\boldsymbol{\Gamma}^\star(k+1) = \mathbf{P}\Delta\boldsymbol{\Gamma}^\star(k) + \mathbf{R}\boldsymbol{\zeta}(k). \quad (12)$$

Thus, the first and second moment order of $\Delta\boldsymbol{\Gamma}^\star$ *are given by*

$$\mu_{\Delta\boldsymbol{\Gamma}^\star} = \mu_\zeta(\mathbf{I}-\mathbf{P})^{-1}\mathbf{R}, \quad \sigma^2_{\Delta\boldsymbol{\Gamma}^\star} = tr(\mathbf{W}), \quad (13)$$

where $\mathbf{W}$ *is the solution of the Lyapunov equation* $\mathbf{W} = \mathbf{P}\mathbf{W}\mathbf{P}^\top + \mathbf{R}\boldsymbol{\Psi}_\zeta\mathbf{R}^\top$.

Proof: See [8]. ■

[3] $\boldsymbol{\zeta}(k)$ contains $\boldsymbol{\xi}(k)$ (noise on $\mathbf{x}(k)$) and/or $\boldsymbol{\xi}'(k)$ (noise on $\mathbf{y}(k)$).

3. STEADY-STATE FOR QUANTIZED AVERAGE CONSENSUS

In our previous paper [8] we analyzed a feed-back type algorithm proposed by Censi and Murray [3] (We will refer to this type of algorithm as *"Censi's algorithm"* in the text) in terms of Theorem 1. We showed by simulations that Eq. (13) holds. However, nor Censi neither we proved explicitly that the algorithm converges in the mean to a consensus for any quantization scheme.

After finding a general solution for the steady-state, we can easily compute bounds on the *drift* from the mean.

3.1 Average consensus over quantized channels

As shown in [8] the parameters describing Censi's algorithm [3] in our framework, Eq. (10), are as follows:

$$\boldsymbol{\alpha} = \mathbf{I}_{|\mathcal{V}|} \otimes \underbrace{\begin{pmatrix}\frac{\eta}{\Delta}\\ \frac{\eta}{\Delta}\\ 0\end{pmatrix}}_{\mathbf{L}}, \quad (14)$$

$$\boldsymbol{\beta} = \tfrac{\eta}{\Delta}\mathbf{D} \otimes \overbrace{\begin{pmatrix}-1 & 0 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}^{\mathbf{K}} + \mathbf{I}_{|\mathcal{V}|} \otimes \begin{pmatrix}1 & 0 & 0\\ 2 & -1 & -1\\ -1 & 1 & 1\end{pmatrix} \quad (15)$$
$$\boldsymbol{\delta} = \tfrac{\eta}{\Delta}\mathbf{D} \otimes \underbrace{\begin{pmatrix}-1 & 0 & 0\end{pmatrix}}_{\mathbf{M}} + \mathbf{I}_{|\mathcal{V}|} \otimes \underbrace{\begin{pmatrix}2 & -1 & -1\end{pmatrix}}_{\mathbf{N}}, \quad (16)$$
$$\boldsymbol{\gamma} = \tfrac{\eta}{\Delta}\mathbf{I}_{|\mathcal{V}|}, \quad (17)$$

and quantization noise on the links is transformed according to the matrix

$$\mathbf{R} = \begin{pmatrix}\mathbf{I}_{|\mathcal{V}|} \otimes \begin{pmatrix}0\\1\\0\end{pmatrix}\\ \mathbf{I}_{|\mathcal{V}|}\end{pmatrix}. \quad (18)$$

With Censi's notation the update data consists of $\mathbf{x}^\top(k) = (\mathbf{x}_1(k), \mathbf{x}_2(k), \mathbf{x}_3(k))^\top \equiv (x(k), y(k), c(k))^\top$ and communication data $\mathbf{y}^\top(k) \equiv y^\top(k)$. Since the algorithm broadcasts messages, the adjacency matrix is constant in time, i.e. $\mathbf{A}_{\mathcal{G}'(k)} = \mathbf{A}_\mathcal{G} \equiv \mathbf{A}$ (in the following text). Matrix $\mathbf{D}$ is the degree matrix, $\mathbf{I}_{|\mathcal{V}|}$ the identity matrix of size $|\mathcal{V}|$; Δ is the maximum degree and η is the step size.

The idea behind this approach is that the communication quantization error is fed-back into the system, thus preserving convergence to a steady-state which can, however, differ from the true average of the initial state.

In general, having no assumptions on $\boldsymbol{\zeta}(k)$, except being bounded, it can model any quantization noise on links as well as any independent disturbance in transmission.

3.2 Steady-state of Censi's algorithm

We will now show that under some assumptions on the network topology and the noise on the links, Censi's algorithm always converges to a steady-state.

Theorem 2. *Using the state-space Equation* (12) *with parameters* (14)–(17)*, and satisfying the conditions:*

1. *The graph $\mathcal{G}$ is strongly connected,*
2. *The noise on the links $\boldsymbol{\zeta}(k)$ is bounded and converges to $\overline{\boldsymbol{\zeta}}$,*

the Censi's algorithm (Section 3.1) asymptotically converges to a steady-state

$$\overline{\Delta\boldsymbol{\Gamma}^\star} = \underbrace{\mathbf{P}^\infty \Delta\boldsymbol{\Gamma}^\star(0)}_{\text{Mean value}} + \underbrace{+\left[\left(\begin{matrix}\mathbf{I}_{|\mathcal{V}|}\otimes\begin{pmatrix}0\\0\\1\end{pmatrix}\\ \mathbf{0}_{|\mathcal{V}|}\end{matrix}\right) + \left(\begin{matrix}\left[\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\mathbf{A}\right]\otimes\mathbf{L}\\ \frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\frac{\eta}{\Delta}\mathbf{A}\end{matrix}\right)\right]\overline{\boldsymbol{\zeta}}}_{d\,-\,\text{drift from the mean}} \tag{19}$$

where $\mathbf{P}^\infty = \begin{pmatrix}\mathbf{P}_{\infty,1} & \mathbf{P}_{\infty,2}\\ \mathbf{P}_{\infty,3} & \mathbf{P}_{\infty,4}\end{pmatrix}$ *with*

$$\mathbf{P}_{\infty,1} = \left[\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\left(\frac{\eta^2}{\Delta^2}\mathbf{A}\mathbf{D} - \left(\frac{\eta}{\Delta}\mathbf{D}-\mathbf{I}\right)^2\right)\right]\otimes\mathbf{K} + \left[\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\mathbf{A}\right]\otimes\mathbf{L}\mathbf{N} \tag{20}$$

$$\mathbf{P}_{\infty,2} = \left[\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\mathbf{A}\right]\otimes\mathbf{L} \tag{21}$$

$$\mathbf{P}_{\infty,3} = \left[\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\left(\frac{\eta^2}{\Delta^2}\mathbf{A}\mathbf{D} - \left(\frac{\eta}{\Delta}\mathbf{D}-\mathbf{I}\right)^2\right)\right]\otimes\mathbf{M} + \left[\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\frac{\eta}{\Delta}\mathbf{A}\right]\otimes\mathbf{N} \tag{22}$$

$$\mathbf{P}_{\infty,4} = \frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\frac{\eta}{\Delta}\mathbf{A} \tag{23}$$

and $\overline{\boldsymbol{\zeta}}$ being determined by the statistics of the noise $\boldsymbol{\zeta}(k)$.

Proof: Taking Equation (12), for time K we can write:

$$\Delta\boldsymbol{\Gamma}^\star(K) = \mathbf{P}^K\Delta\boldsymbol{\Gamma}^\star(0) + \sum_{k=0}^{K-1}\mathbf{P}^k\mathbf{R}\boldsymbol{\zeta}(K-1-k).$$

After inserting parameters (14)-(17) in $\mathbf{P}$ and applying $K \geq 2$ multiplications we obtain

$$\mathbf{P}^K = \begin{pmatrix}\mathbf{P}_{K,1} & \mathbf{P}_{K,2}\\ \mathbf{P}_{K,3} & \mathbf{P}_{K,4}\end{pmatrix}, \tag{24}$$

where

$$\mathbf{P}_{K,1} = \left[\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-2}\left(\frac{\eta^2}{\Delta^2}\mathbf{A}\mathbf{D} - \left(\frac{\eta}{\Delta}\mathbf{D}-\mathbf{I}\right)^2\right)\right]\otimes\mathbf{K} + \left[\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-2}\mathbf{A}\right]\otimes\mathbf{L}\mathbf{N}$$

$$\mathbf{P}_{K,2} = \left[\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-1}\mathbf{A}\right]\otimes\mathbf{L}$$

$$\mathbf{P}_{K,3} = \left[\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-2}\left(\frac{\eta^2}{\Delta^2}\mathbf{A}\mathbf{D} - \left(\frac{\eta}{\Delta}\mathbf{D}-\mathbf{I}\right)^2\right)\right]\otimes\mathbf{M} + \left[\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-2}\frac{\eta}{\Delta}\mathbf{A}\right]\otimes\mathbf{N}$$

$$\mathbf{P}_{K,4} = \left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-1}\frac{\eta}{\Delta}\mathbf{A}.$$

where $\mathbf{K},\mathbf{L},\mathbf{M},\mathbf{N}$ are as in (14)-(17).

Since the term $\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^K$ is the only term growing with K, we only need to show that this term converges as $K\to\infty$. As the term is nothing else than the so-called *Perron matrix* [7], which, for strongly connected graphs, has a trivial *maximum* eigenvalue $\lambda_n = 1$ with corresponding eigenvector $\mathbf{v}_n = 1/\sqrt{|\mathcal{V}|}\mathbf{1}$, we obtain

$$\lim_{K\to\infty}\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^K = \lim_{K\to\infty}(\mathbf{U}\boldsymbol{\Lambda}\mathbf{U})^K = \\ = \mathbf{U}\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{0}\end{pmatrix}\mathbf{U}^\top = \mathbf{v}_n\mathbf{v}_n^\top = \frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top. \tag{25}$$

Thus, we proved that $\mathbf{P}^\infty$ converges.

Now we prove also the convergence for the drift from the mean.

First, we assume that the disturbance noise $\boldsymbol{\zeta}(k)$ asymptotically converges to a value $\overline{\boldsymbol{\zeta}}$, i.e. $\left|\boldsymbol{\zeta}(k_0)-\overline{\boldsymbol{\zeta}}\right| < \epsilon$. As shown by Censi [3], for deterministic quantization this assumption hold only in mean, i.e. converged states tend to oscillate around a common mean. However, if the links are disturbed by an independent noise, or a probablistic quantization scheme is used, e.g. [1], this assumption holds accurately.

Secondly, we will show that $\mathbf{P}^k\mathbf{R}\to\mathbf{0}$ for some $k \geq k_0 \gg 0$, i.e., for the drift from the mean we can write

$$\sum_{k=0}^{K-1}\mathbf{P}^k\mathbf{R}\boldsymbol{\zeta}(K-k-1) = \sum_{k=0}^{k_0-1}\mathbf{P}^k\mathbf{R}\overline{\boldsymbol{\zeta}} + \underbrace{+\sum_{k=k_0}^{K-1}\mathbf{P}^k\mathbf{R}\boldsymbol{\zeta}(K-k-1)}_{=0}. \tag{26}$$

Now let's determine $\sum_{k=0}^{k_0-1}\mathbf{P}^k\mathbf{R}$. Multiplying $\mathbf{P}$ by $\mathbf{R}$ we obtain

$$\mathbf{P}\mathbf{R} = \left(\begin{matrix}\mathbf{I}_{|\mathcal{V}|}\otimes\begin{pmatrix}0\\-1\\1\end{pmatrix} + \mathbf{A}\otimes\mathbf{L}\\ \frac{\eta}{\Delta}\mathbf{A}-\mathbf{I}_{|\mathcal{V}|}\end{matrix}\right), \tag{27}$$

respectively by taking the power of $K \geq 2$

$$\mathbf{P}^K\mathbf{R} = \frac{\eta}{\Delta}\begin{pmatrix}\left[(\mathbf{A}-\mathbf{D})\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-2}\mathbf{A}\right]\otimes\mathbf{L}\\ \left[(\mathbf{A}-\mathbf{D})\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-2}\mathbf{A}\right]\frac{\eta}{\Delta}\mathbf{A}\end{pmatrix}. \tag{28}$$

From (25) we find

$$\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)\mathbf{v} = 1\mathbf{v} \\ \Rightarrow (\mathbf{A}-\mathbf{D})\mathbf{v} = \mathbf{0}, \tag{29}$$

and therefore for $K \geq k_0$

$$\lim_{K\to\infty} \mathbf{P}^K\mathbf{R} = \frac{\eta}{\Delta}\begin{pmatrix} (\mathbf{A}-\mathbf{D})\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-2}\mathbf{A}\otimes\mathbf{L} \\ (\mathbf{A}-\mathbf{D})\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^{K-2}\frac{\eta}{\Delta}\mathbf{A} \end{pmatrix} =$$
$$= \frac{\eta}{\Delta}\left(\underbrace{(\mathbf{A}-\mathbf{D})\,\mathbf{v}}_{=0}\mathbf{v}^\top \begin{matrix}\mathbf{A}\otimes\mathbf{L} \\ \frac{\eta}{\Delta}\mathbf{A}\end{matrix}\right) = \frac{\eta}{\Delta}\begin{pmatrix}\mathbf{0}_{3|\mathcal{V}|\times|\mathcal{V}|} \\ \mathbf{0}_{|\mathcal{V}|\times|\mathcal{V}|}\end{pmatrix} = \mathbf{0}. \tag{30}$$

It means that for K large enough, i.e. $K \geq k_0$, $\mathbf{P}^K\mathbf{R}\to\mathbf{0}$ and since $\mathbf{P}^k\mathbf{R}$ is bounded and decreasing for $\forall k$, also the series $\sum_{k=0}^{\infty}\mathbf{P}^k\mathbf{R}$ must exists.

We can then directly show that, for $K\to\infty$

$$\lim_{K\to\infty}\sum_{k=0}^{K-1}\mathbf{P}^k\mathbf{R} = \mathbf{R}+\mathbf{P}\mathbf{R}+\mathbf{P}^2\mathbf{R}+\cdots =$$
$$= \begin{pmatrix}\mathbf{I}_{|\mathcal{V}|}\otimes\begin{pmatrix}0\\1\\0\end{pmatrix} \\ \mathbf{I}_{|\mathcal{V}|}\end{pmatrix} + \begin{pmatrix}\mathbf{I}_{|\mathcal{V}|}\otimes\begin{pmatrix}0\\-1\\1\end{pmatrix}+\mathbf{A}\otimes\begin{pmatrix}\frac{\eta}{\Delta}\\\frac{\eta}{\Delta}\\0\end{pmatrix} \\ \frac{\eta}{\Delta}\mathbf{A}-\mathbf{I}_{|\mathcal{V}|}\end{pmatrix} +$$
$$+\frac{\eta}{\Delta}\begin{pmatrix}(\mathbf{A}-\mathbf{D})\sum_{k=0}^{\infty}\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^k\mathbf{A}\otimes\begin{pmatrix}\frac{\eta}{\Delta}\\\frac{\eta}{\Delta}\\0\end{pmatrix} \\ (\mathbf{A}-\mathbf{D})\sum_{k=0}^{\infty}\left(\frac{\eta}{\Delta}(\mathbf{A}-\mathbf{D})+\mathbf{I}\right)^k\frac{\eta}{\Delta}\mathbf{A}\end{pmatrix}. \tag{31}$$

Thus, for $0<\eta<1$, $K\geq k_0$

$$\sum_{k=0}^{K-1}\mathbf{P}^k\mathbf{R} = \begin{pmatrix}\mathbf{I}_{|\mathcal{V}|}\otimes\begin{pmatrix}0\\1\\0\end{pmatrix} \\ \mathbf{I}_{|\mathcal{V}|}\end{pmatrix} + \begin{pmatrix}\mathbf{I}_{|\mathcal{V}|}\otimes\begin{pmatrix}0\\-1\\1\end{pmatrix}+\mathbf{A}\otimes\begin{pmatrix}\frac{\eta}{\Delta}\\\frac{\eta}{\Delta}\\0\end{pmatrix} \\ \frac{\eta}{\Delta}\mathbf{A}-\mathbf{I}_{|\mathcal{V}|}\end{pmatrix} +$$
$$+\frac{\eta}{\Delta}\begin{pmatrix}\frac{\Delta}{\eta}\left(\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top-\mathbf{I}\right)\mathbf{A}\otimes\begin{pmatrix}\frac{\eta}{\Delta}\\\frac{\eta}{\Delta}\\0\end{pmatrix} \\ \frac{\Delta}{\eta}\left(\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top-\mathbf{I}\right)\frac{\eta}{\Delta}\mathbf{A}\end{pmatrix} =$$
$$= \begin{pmatrix}\mathbf{I}_{|\mathcal{V}|}\otimes\begin{pmatrix}0\\0\\1\end{pmatrix} \\ \mathbf{0}_{|\mathcal{V}|}\end{pmatrix} + \begin{pmatrix}\frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\mathbf{A}\otimes\begin{pmatrix}\frac{\eta}{\Delta}\\\frac{\eta}{\Delta}\\0\end{pmatrix} \\ \frac{1}{|\mathcal{V}|}\mathbf{1}\mathbf{1}^\top\frac{\eta}{\Delta}\mathbf{A}\end{pmatrix}. \tag{32}$$

Thus, Theorem 2 is proved completely. ■

It must be noted that comparing Eq. (13) with Eq. (19), $\overline{\boldsymbol{\zeta}}$ corresponds to μ_ζ, and the term (32) replaces the non-invertible term $(\mathbf{I}-\mathbf{P})^{-1}\mathbf{R}$ for Censi's algorithm.

As mentioned in [8], term $(\mathbf{I}-\mathbf{P})$ is not invertible in this case, nonetheless, we had made an assumption that the matrix $\mathbf{R}$ acts as a stabilizing term, thus ensuring convergence. Taking the result of Theorem 2 we can conclude that this assumption was correct.

4. BOUNDS ON THE DRIFT FROM THE MEAN

In (19) we assumed that the noise $\boldsymbol{\zeta}(k)$ converges to some value $\overline{\boldsymbol{\zeta}}$. However, as mentioned before, in case of deterministic quantization noise this value depends on the step size η, initial states, quantization scheme and also topology. Therefore it is not easy to estimate it before-hand (see Tab. 2). However, bounds on drift from the mean can be set straightforwardly.

In Tab. 1 we recall the bounds on $\overline{\boldsymbol{\zeta}}$ for few simplest quantization schemes.

quantization scheme	*lower bound*	*upper bound*
round to nearest	-0.5	0.5
round up	0	1
round down	-1	0

Table 1: Lower and upper bounds on $\overline{\boldsymbol{\zeta}}$ for few simple quantization schemes.

Topology	*quantization scheme*	$\lvert\mathcal{V}\rvert$	$\overline{\zeta}_{ave}$
complete	rounding	10	0.0128 ± 0.46
		30	-0.0077 ± 0.459
	ceiling	10	0.49 ± 0.468
		30	0.502 ± 0.459
star	rounding	10	0.0034 ± 0.49
		30	0.0106 ± 0.48
	ceiling	10	0.4922 ± 0.4932
		30	0.504 ± 0.396
ring	rounding	10	-0.0093 ± 0.468
		30	-0.004 ± 0.46
	ceiling	10	0.5139 ± 0.455
		30	0.5005 ± 0.403
geometric[4]	rounding	9	0.0012 ± 0.47
		30	-0.000462 ± 0.45
	ceiling	9	0.4988 ± 0.4721
		30	0.4925 ± 0.4612

Table 2: Average $\overline{\boldsymbol{\zeta}}$ after 1000 randomly initialized runs, for few topologies and 2 different quantization schemes.

When considering the Censi's bound, we consider the bound as defined in Censi [3], i.e. $\eta\beta$ where $|q(x)-x|<\beta$, for any quantization function $q(\cdot)$.

4.1 A-priori bounds on the drift from the mean

For several typical network topologies, we can generate an adjacency matrix $\mathbf{A}_{\mathcal{G}}$ and provide a-priori bounds on the drift from the mean if the sent data is rounded to the nearest integer (see Tab. 3) and if the data is rounded up (operation ceiling) (see Tab. 4). The values are for $\mathbf{x}_1$, i.e. for the stored data in the nodes. Similarily, we can provide bounds on $\mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{y}$.

[4]Geometric graph is a randomly distorted rectangular grid where each node communicates with neighbours only in its predefined range.

Topology	$\lvert\mathcal{V}\rvert$	*worst-case steady-state value*	*bound Eq. (32)*	*Censi's bound*
complete	10	0.0459	± 0.05	0.05
	30	-0.0459	± 0.05	0.05
star	10	0.0077	± 0.01	0.05
	30	-0.003	± 0.0033	0.05
ring	10	0.045	± 0.05	0.05
	30	-0.045	± 0.05	0.05
geometric	9	0.0315	± 0.033	0.05
	30	0.0382	± 0.0408	0.05

Table 3: Bounds on the drift $d_{\mathbf{x}(k)}$, Eq. (19). Quantization scheme – rounding to the nearest integer, $\eta = 0.1$, worst case of true steady-state value after 1000 randomly initialized runs.

Topology	$\lvert\mathcal{V}\rvert$	*worst-case steady-state value (max/min)*	*bound Eq.(32) (max/min)*	*Censi's bound*
complete	10	0.0705/0.0188	0.1/0	0.1
	30	0.0961/0.0042	0.1/0	0.1
star	10	0.0188/0.0011	0.02/0	0.1
	30	0.0051/0.0015	0.00666/0	0.1
ring	10	0.0953/0.0045	0.1/0	0.1
	30	0.078/0.0236	0.1/0	0.1
geometric	9	0.0648/0.0019	0.066/0	0.1
	30	0.0778/0.0048	0.0817/0	0.1

Table 4: Bounds on the drift $d_{\mathbf{x}(k)}$, Eq. (19). Quantization scheme – ceiling, $\eta = 0.1$, worst case of true steady-state value after 1000 randomly initialized runs (maximum/minimum value).

In Figure 1 we observe a typical behaviour for the first 7 states in case of a geometric random network with 30 nodes, with the bounded asymptotic phase.

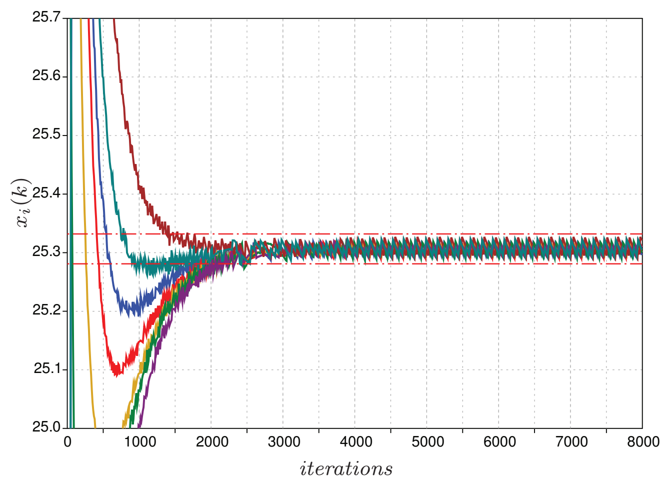

Figure 1: Example of convergence behaviour of $x_i(k)$ $(i = 1, 2, \ldots, 7)$ and bounds for a random geometric network; $\lvert\mathcal{V}\rvert = 30, \Delta = 6$.

5. CONCLUSION

We have proved and analyzed convergence of a quantized averaging algorithm with noise which is fed-back to system. We have proved that under simple requirements on the network and the noise, this type of algorithm always asymptotically converges to a consensus which can, however, be drifted from the true initial average. Moreover, in case of simple deterministic quantization schemes, the convergence is satisfied only in average. Nevertheless, also in this case, we can provide bounds on the drift from the steady-state. We must, however, emphasize that these bounds depend on the topology of network given by $\mathbf{A}_{\mathcal{G}}$, while Censi's bounds do not. As considered in one step of the proof in Censi [3, proof of *Proposition 1*], $x(k)$ can also be bounded by $\frac{1}{\lvert\mathcal{V}\rvert}\frac{\eta}{\Delta}Tr(\mathbf{D})\beta$, which gives the same qualitative bounds on $\mathbf{x}_1(k)$ like ours. Nevertheless, our approach using state-space description provides not only the bound on the drift from the mean for $\mathbf{x}_1(k)$ but also for average $\mathbf{y}(k)$, and last, but not least, provides a different insight to this problem.

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