# APPROXIMATE OPTIMAL PERIODOGRAM SMOOTHING FOR CEPSTRUM ESTIMATION USING A PENALTY TERM

*Johan Sandberg and Maria Hansson-Sandsten*

Mathematical Statistics, Centre for Mathematical Sciences, Lund University.
Address: Box 118, SE-221 00 Lund, Sweden. Phone: + 46 46 22 285 50. Fax: +46 46 22 246 23.
Email: sandberg@maths.lth.se, sandsten@maths.lth.se. Web: www.maths.lth.se/matstat

## ABSTRACT

The cepstrum of a random process is useful in many applications. The cepstrum is usually estimated from the periodogram. To reduce the mean square error (MSE) of the estimator, the periodogram may be smoothed with a kernel function. We present an explicit expression for a kernel function which is approximatively MSE optimal for cepstrum estimation. A penalty term has to be added to the minimization problem, but we demonstrate how the weighting of the penalty term can be chosen. The performance of the estimator is evaluated on simulated processes. Since the MSE optimal smoothing kernel depends on the true covariance function, we give an example of a simple data driven method.

## 1. INTRODUCTION

The *cepstrum*, an anagram of the word “spectrum”, was invented by Bogart, Healy and Tukey, in the early sixties [1,2]. It is defined as the inverse Fourier transform of the log-spectrum of a stationary random process. The cepstrum is often used in classification or detection problems where the *cepstrum coefficients* (discrete cepstrum) may serve as a feature input to a pattern recognition system. Recently, it has been used for classification of events using neural activity [3], detection of rot fungi in trees [4], and in fault detection of a mechanical gear system [5]. The cepstrum is most notably used in audio related applications, such as in speech and speaker recognition, but also in music genre classification and in speech synthesis, [6, 7]. Typically, the cepstrum is estimated by the Fourier transform of the log-periodogram. The periodogram suffers from large variance, causing large estimation errors in the cepstrum coefficients. The variance of the periodogram can be reduced by convolution with a smoothing kernel function. In the context of spectrum estimation, there has been a lot of research on how to select the smoothing kernel function. Periodogram smoothing is not usually performed in the context of cepstrum estimation. One reason to this may be that the optimal selection of the kernel function appropriate for cepstrum estimation has not been considered before. Therefore, we will in this paper discuss how the smoothing kernel should be selected in order to minimize the mean square error (MSE = squared bias + variance) of the cepstrum estimator defined in Section 2. We believe that the MSE is a useful optimization criterion since it takes both bias and variance of the cepstrum coefficients into account. In Section 3, for the first time, an approximate solution to this optimization problem is derived. The approximation is significantly improved by introducing a penalty term as described in Section 3.1. The weight of the penalty term is not very crucial. This is illustrated on a speech-like simulated process in Section 3.2, where we also compare with the MSE of the two most common estimators which are the Fourier transform of the log-periodogram and the Fourier transform of the Hanning-windowed log-periodogram. These estimators are also used as comparisons in Section 4, where we demonstrate how the performance depends on the pole location in an AR(2) process. Since the optimal smoothing depends on the true covariance function, which is unknown, one can in practical applications not expect to select the most optimal kernel function. Using the sample covariance function computed from observed data, instead of the true covariance function, we can still yield a better result compared to no smoothing at all, as shown in Section 5. Section 6 concludes the paper.

This work was supported by the Swedish Research Council.

## 2. CEPSTRUM ESTIMATION BY PERIODOGRAM SMOOTHING

Let $\{x(t), t = 1, \ldots, n\}$ denote a real-valued stationary random process with zero mean and finite moments. The covariance function $r : \{-n+1, \ldots, n-1\} \mapsto \mathbb{R}$ is defined by

$$r(\tau) = \mathsf{E}\left[x(t)x(t+\tau)\right].$$

The spectral density $S : \{-n+1, \ldots, n-1\} \mapsto \mathbb{R}^+$ at frequencies $p/N$, $p = -n+1, \ldots, n-1$, $N = 2n-1$, is defined by

$$S(p) = \sum_{\tau=-n+1}^{n-1} r(\tau) \mathrm{e}^{-i2\pi \frac{p}{N} \tau}.$$

Assuming that the spectrum is strictly positive, $S(p) > 0$, for every $p$, the cepstrum $c : \{-n+1, \ldots, n-1\} \mapsto \mathbb{R}$ of the process is defined as the inverse Fourier transform of the log-spectrum:

$$c(q) = \frac{1}{N} \sum_{p=-n+1}^{n-1} \log(S(p)) \mathrm{e}^{i2\pi \frac{p}{N} q}.$$

The cepstrum is symmetric, $c(q) = c(-q)$, $q = -n+1, \ldots, n-1$, and we will from now on only consider $q \geq 0$. Note that all the three functions $r$, $S$, and $c$, are defined in $n$ unique points. The above defined spectrum $S$ is thus up-sampled in comparison to the ordinary periodogram of $x$ which is usually only computed in $n/2$ points. This is necessary in order to guaranty that $c(q) \approx c_{\text{cont}}(q/N)$, $q = -n+1, \ldots, n-1$, where $c_{\text{cont}}$ is the continuous cepstrum, defined as the continuous Fourier transform of the continuous spectral density. This fact is also recognized in [8].

In order to estimate the cepstrum from an observed realization of the random process, one may first estimate the spectrum by the periodogram, and then apply the logarithm and the inverse Fourier transform to achieve a cepstrum estimate. We denote this estimator by

$$\hat{c}_{\mathrm{per}}(q) = \frac{1}{N}\sum_{p=-n+1}^{n-1} \log(\hat{S}_{\mathrm{per}}(p)) \mathrm{e}^{i2\pi\frac{p}{N}q}, \tag{1}$$

where $\hat{S}_{\mathrm{per}}$ is the periodogram:

$$\begin{aligned}\hat{S}_{\mathrm{per}}(p) &= \sum_{\tau=-n+1}^{n-1} \frac{1}{n} \sum_{t=1}^{n-|\tau|} x(t)x(t+|\tau|)\mathrm{e}^{-i2\pi\tau\frac{p}{N}} \\ &= \frac{1}{n}\left|\sum_{t=1}^{n} x(t)\mathrm{e}^{-i2\pi t\frac{p}{N}}\right|^2.\end{aligned}$$

The periodogram is an inconsistent estimator of the spectrum, since the variance does not approach zero as the data length, $n$, goes to infinity. This problem is naturally addressed by smoothing the periodogram with a kernel function. We denote and define this estimator by

$$\hat{S}_W(p) = \sum_{p'=-n+1}^{n-1} W(p')\hat{S}_{\mathrm{per}}(p-p'),$$

where $W : \{-n+1,\ldots,n-1\} \mapsto \mathbb{R}$ is a *smoothing kernel*. Let us define and denote the cepstrum estimator based on periodogram smoothing by

$$\hat{c}_W(q) = \frac{1}{N}\sum_{p=-n+1}^{n-1} \log\left(\hat{S}_W(p)\right) \mathrm{e}^{i2\pi\frac{p}{N}q}. \tag{2}$$

Choosing an appropriate smoothing kernel $W$ is a difficult matter. Our aim is to find the MSE optimal smoothing kernel for estimation of the $q$:th cepstrum coefficient:

$$W_{q-\mathrm{opt}} = \underset{W:\{-n+1,\ldots,n-1\}\mapsto\mathbb{R}}{\arg\min} F(W), \tag{3}$$

where

$$F(W) \triangleq \mathsf{E}\left[(c(q)-\hat{c}_W(q))^2\right]. \tag{4}$$

Note that the smoothing kernel is allowed to depend on which cepstrum coefficient $q$ we wish to estimate, but we exclude the zeroth coefficient, $q = 0$, which is equal to the mean of the log-spectrum, since in most applications this is not interesting. The variance and bias of the cepstrum estimated from a tapered periodogram has been derived in [9] (see also [8]), but such expressions lead to a difficult optimization problem. In the following section, we will demonstrate that it is possible to compute an approximative solution to (3) which will turn out to be accurate only after the addition of a penalty term as in Section 3.1.

## 3. APPROXIMATIVE OPTIMIZATION

We will now solve the optimization problem in (3) using an approach where the MSE is rewritten on a form where the argument of the logarithm operator is likely to be close to 1, which allows us to apply the approximation $\log(z) \approx 1 - z^{-1}$, for $z \approx 1$. The MSE is invariant to scaling of the smoothing window, i.e. the MSE of $\hat{c}_W(q)$ equals the MSE of $\hat{c}_{W'}(q)$ if $W = \alpha W'$, $\alpha \neq 0$. Therefor, we may solve the minimization problem subject to $||W|| = 1$, where $||\cdot||$ denotes the $L^2$ norm, without imposing any restrictions to the final set of solutions. The function $F$, defined in (4), which we aim to minimize, can be written:

$$F(W) = \frac{1}{N^2}\mathsf{E}\left[\left(\sum_{p=-n+1}^{n-1} \log\left(\frac{S(p)}{\hat{S}_W(p)}\right) \mathrm{e}^{i2\pi\frac{p}{N}q}\right)^2\right].$$

Since $\hat{S}_W(p)$ is an estimate of $S(p)$, it seems reasonable to believe that the ratio between them is close to 1. Therefor, the approximation $\log\left(\frac{S(p)}{\hat{S}_W(p)}\right) \approx 1 - \frac{\hat{S}_W(p)}{S(p)}$ is justified and we approximate the function $F$ with the function $F_{\mathrm{approx}}$ defined by

$$\begin{aligned}F_{\mathrm{approx}}(W) &\triangleq \frac{1}{N^2}\mathsf{E}\left[\left(\sum_{p=-n+1}^{n-1} \left(1-\frac{\hat{S}_W(p)}{S(p)}\right) \mathrm{e}^{i2\pi\frac{p}{N}q}\right)^2\right] \\ &= \frac{1}{N^2}\mathsf{E}\left[\left(\sum_{p=-n+1}^{n-1} \frac{\hat{S}_W(p)}{S(p)} \mathrm{e}^{i2\pi\frac{p}{N}q}\right)^2\right],\end{aligned}$$

since $\sum \mathrm{e}^{i2\pi\frac{p}{N}q} = 0$, for $q \neq 0$. The requirement $||W|| = 1$ excludes the trivial solution $W \equiv 0$. To minimize $F_{\mathrm{approx}}$, we will now introduce the following vector notation:

$$\mathbf{A}_p = \begin{bmatrix} \sum_{t=1}^{n} x(t)^2 \\ 2\Re\left\{\sum_{t=1}^{n-1} x(t)x(t+1)\mathrm{e}^{-i2\pi\frac{p}{N}1}\right\} \\ 2\Re\left\{\sum_{t=1}^{n-2} x(t)x(t+2)\mathrm{e}^{-i2\pi\frac{p}{N}2}\right\} \\ \vdots \\ 2\Re\left\{\sum_{t=1}^{n-(n-1)} x(t)x(t+(n-1))\mathrm{e}^{-i2\pi\frac{p}{N}(n-1)}\right\} \end{bmatrix}$$

where $\Re$ denotes real-part and

$$\mathbf{w} = [w(0) \quad \ldots \quad w(n-1)]^T,$$

and where $w$ is defined by

$$w(\tau) = \frac{1}{N}\sum_{p=-n+1}^{n-1} W(p)\mathrm{e}^{i2\pi p\tau/N}.$$

With this notation, $\hat{S}_W(p) = \mathbf{A}_p^T\mathbf{w}$, and the function $F_{\mathrm{approx}}$ can be written:

$$F_{\mathrm{approx}}(W) = \frac{1}{N^2}\mathsf{E}\left[\left(\sum_{p=-n+1}^{n-1} \frac{1}{S(p)}\mathbf{A}_p^T\mathbf{w}\mathrm{e}^{i2\pi\frac{p}{N}q}\right)^2\right].$$

With $\mathbf{Y} = \sum_{p=-n+1}^{n-1} \frac{1}{S(p)}\mathbf{A}_p \mathrm{e}^{i2\pi\frac{p}{N}q}$ ($\mathbf{Y}$ is a random element in $\mathbb{R}^n$):

$$F_{\mathrm{approx}}(W) = \frac{1}{N^2}\mathbf{w}\mathsf{E}\left[\mathbf{Y}\mathbf{Y}^T\right]\mathbf{w}.$$

That is, we need to minimize the following expression with respect to $\mathbf{w}$:

$$\mathbf{w}^T \mathbf{M} \mathbf{w},$$

where

$$\mathbf{M} = \mathsf{E}\left[\mathbf{Y}\mathbf{Y}^T\right]. \tag{5}$$

Under the constraint $||\mathbf{w}|| = 1$, the solution to this minimization problem is the eigenvector of $\mathbf{M}$ corresponding to its smallest eigenvalue. For Gaussian processes, the kernel function can be computed exactly as it only requires an eigenvector decomposition of $\mathbf{M}$ which can be expressed in terms of the covariance function as given in Appendix A. Simulation studies shows that $F_{\text{approx}}$ is minimized by a smoothing kernel function for which the approximation $F \approx F_{\text{approx}}$ unfortunately does not hold, due to that the ratio between $\hat{S}_W(p)$ and $S(p)$ is too far from 1. To this end, we will add a penalty term, as described next.

### 3.1 Approximative optimization with penalty term

We will now use the same approach as previously but we add a penalty term, which penalizes smoothing kernels for which the ratio between $\hat{S}_W(p)$ and $S(p)$ is too far from 1. Our aim is now to find the smoothing kernel function $W$ which minimizes:

$$\begin{aligned} G(W) \triangleq \mathsf{E}\Bigg[&(1-\rho)n\left(c(q)-\hat{c}_W(q)\right)^2 \\ &+\rho\frac{1}{N}\sum_{p=-n+1}^{n-1}\left(\frac{S(p)-\hat{S}_W(p)}{S(p)}\right)^2\Bigg] \end{aligned} \tag{6}$$

where $0 < \rho < 1$ is a constant which controls the influence of the penalty term. The factor $n$ in front of the squared cepstrum error is justified from the fact that $\mathsf{E}\left[(c(q)-\hat{c}_W(q))^2\right]$ asymptotically decays as $1/n$. Thus, a good choice of the weighting factor $\rho$ will not depend heavily on $n$. Due to the penalty term, the approximation $\log\left(\frac{S(p)}{\hat{S}_W(p)}\right) \approx 1 - \frac{\hat{S}_W(p)}{S(P)}$ is more accurate and we can approximate $G$ with $G_{\text{approx}}$:

$$\begin{aligned} G_{\text{approx}}(W) = \mathsf{E}\Bigg[&\frac{1-\rho}{N^2}n\left(\sum_{p=-n+1}^{n-1}\frac{\hat{S}_W(p)}{S(p)}\mathrm{e}^{i2\pi\frac{p}{N}q}\right)^2 \\ &+\rho\frac{1}{N}\sum_{p=-n+1}^{n-1}\left(\frac{S(p)-\hat{S}_W(p)}{S(p)}\right)^2\Bigg] \end{aligned} \tag{7}$$

The constraint $||W|| = 1$ is no longer needed in order to avoid trivial solutions. With the same vector notation as before we have:

$$\begin{aligned} G_{\text{approx}}(W) = \mathsf{E}\Bigg[&\frac{1-\rho}{N^2}n\left(\sum_{p=-n+1}^{n-1}\frac{\mathbf{A}_p^T\mathbf{w}}{S(p)}\mathrm{e}^{i2\pi\frac{p}{N}q}\right)^2 \\ &+\rho\frac{1}{N}\sum_{p=-n+1}^{n-1}\left(\frac{S(p)-\mathbf{A}_p^T\mathbf{w}}{S(p)}\right)^2\Bigg] \end{aligned}$$

with $\mathbf{M}$ as in (5):

$$\begin{aligned} G_{\text{approx}}(W) &= \mathbf{w}^T\frac{1-\rho}{N^2}n\mathbf{M}\mathbf{w} \\ &\quad+\mathsf{E}\left[\rho\frac{1}{N}\sum_{p=-n+1}^{n-1}1-\frac{2\mathbf{A}_p^T\mathbf{w}}{S(p)}+\frac{(\mathbf{A}_p^T\mathbf{w})^2}{S(p)^2}\right] \\ &= \mathbf{w}^T\left(\frac{1-\rho}{N^2}n\mathbf{M}+\rho\frac{1}{N}\sum_{p=-n+1}^{n-1}\frac{\mathsf{E}\left[\mathbf{A}_p\mathbf{A}_p^T\right]}{S(p)^2}\right)\mathbf{w} \\ &\quad-2\rho\frac{1}{N}\sum_{p=-n+1}^{n-1}\frac{\mathsf{E}\left[\mathbf{A}_p^T\right]}{S(p)}\mathbf{w}+\rho \end{aligned}$$

With

$$\mathbf{\Psi} \triangleq \frac{1-\rho}{N^2}n\mathbf{M}+\rho\frac{1}{N}\sum_{p=-n+1}^{n-1}\frac{\mathsf{E}\left[\mathbf{A}_p\mathbf{A}_p^T\right]}{S(p)^2},$$

and

$$\mathbf{\Phi} \triangleq \rho\frac{1}{N}\sum_{p=-n+1}^{n-1}\frac{\mathsf{E}\left[\mathbf{A}_p^T\right]}{S(p)},$$

the minimization problem is written on the form:

$$\text{Minimize:} \quad \mathbf{w}^T\mathbf{\Psi}\mathbf{w}-2\mathbf{\Phi}^T\mathbf{w} \quad \text{w.r.t. } \mathbf{w}, \tag{8}$$

where $\mathbf{\Psi}$ is a symmetric matrix. The solution is

$$\mathbf{w}_{q-\text{opt}}^{\rho-\text{approx}} = \mathbf{\Psi}^{-1}\mathbf{\Phi}. \tag{9}$$

And the optimal smoothing kernel function is thus

$$W_{q-\text{opt}}^{\rho-\text{approx}}(p) = \sum_{\tau=-n+1}^{n-1} w_{q-\text{opt}}^{\rho-\text{approx}}(|\tau|)\mathrm{e}^{-i2\pi p\tau/N}. \tag{10}$$

### 3.2 Calibration of the weight of the penalty term

In order to investigate the effect of the penalty term, we choose an AR-process ($n = 240$) with parameters estimated from a recorded speech signal (sampling rate = 8 kHz, model order given by the Akaike final prediction error). We Monte Carlo compute (2000 simulations) the MSE of $\hat{c}_W(q)$, where $W$ is computed as in (10) for different values of the constant $\rho$, which regulates the influence of the penalty term. Fig. 1 shows the result as a function of $\rho$ for cepstrum coefficient $q = 3$. The MSE of $\hat{c}_{\text{per}}$, see (1), and the MSE of $\hat{c}_{\text{han}}$ defined by

$$\hat{c}_{\text{han}}(q) = \frac{1}{N}\sum_{p=-n+1}^{n-1}\log\left(\hat{S}_{\text{han}}(p)\right)\mathrm{e}^{i2\pi\frac{p}{N}q}, \tag{11}$$

where

$$\hat{S}_{\text{han}}(p) = \frac{1}{n}\left|\sum_{t=1}^{n}h(t)x(t)\mathrm{e}^{-i2\pi t\frac{p}{N}}\right|^2,$$

where $h$ is a Hanning window, is also shown as a comparison. Fig. 1 also shows the MSE summed up over all cepstrum coefficients $q = 1, \ldots, 15$ (these are the coefficients

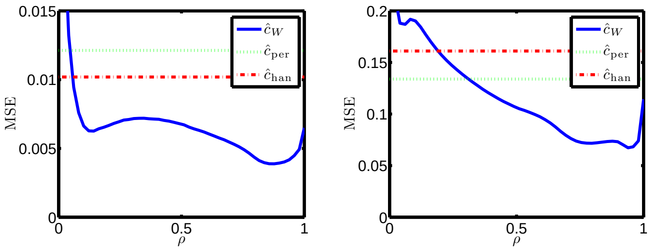


Figure 1: Left: MSE of the three estimators: $\hat{c}_W(q)$, $\hat{c}_{\mathrm{per}}(q)$ and $\hat{c}_{\mathrm{han}}(q)$, for $q = 3$ on a speech like simulated process. The smoothing kernel function $W$ used in the estimator $\hat{c}_W(q)$ is computed as in (10), for different values on $\rho$. Right: The same, but summed over $q = 1, \ldots, 15$.

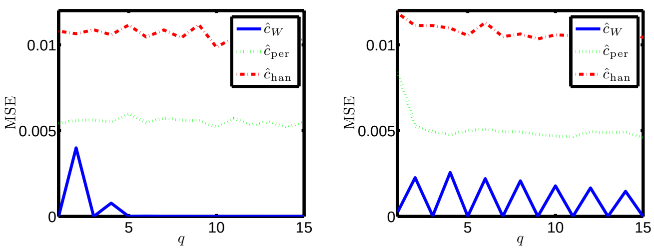


Figure 2: MSE of the estimators $\hat{c}_W(q)$ ($\rho = 0.8$), $\hat{c}_{\mathrm{per}}(q)$ and $\hat{c}_{\mathrm{han}}(q)$, as a function of cepstrum coefficient, $q$, for two different AR(2)-processes with poles in $0.5\mathrm{e}^{\pm i2\pi 0.25}$ (left) and $0.98\mathrm{e}^{\pm i2\pi 0.25}$ (right).

most often considered in audio applications). We have Monte Carlo computed the MSE as a function of $\rho$ on different AR-processes with parameters estimated from speech signals. Based on this, we can propose that $\rho$ should be chosen between 0.5 and 0.9, but the exact choice does not seem to be very important.

## 4. EVALUATION ON AR(2)-PROCESSES

We will now study how the cepstrum estimator $\hat{c}_W$, where $W$ is computed as in (10), behaves on a set of AR(2) processes with poles in $\alpha \mathrm{e}^{\pm i2\pi\nu}$. We choose $\rho = 0.8$ as weight for the penalty term. Fig. 2 shows Monte Carlo computed (2000 simulations) MSE of $\hat{c}_W(q)$ for $\nu = 0.25$ and for $\alpha = 0.5$ and $\alpha = 0.98$ for different cepstrum coefficients. The MSE of $\hat{c}_{\mathrm{per}}(q)$ and of $\hat{c}_{\mathrm{han}}(q)$ is also shown. The spectrum with a sharp peak, $\alpha = 0.98$, is more difficult to smooth than the slowly varying spectrum, $\alpha = 0.5$. In both cases, however, $\hat{c}_W$ has considerably lower MSE than the other estimators. Based on experiments where we have changed the parameter $\nu$, we can conclude that this parameter does not affect the MSE of the cepstrum estimate much. This is expected, since the periodogram is convolved with the smoothing kernel function, and thus, a shift of the spectrum should not have a large influence of the quality of the estimator.

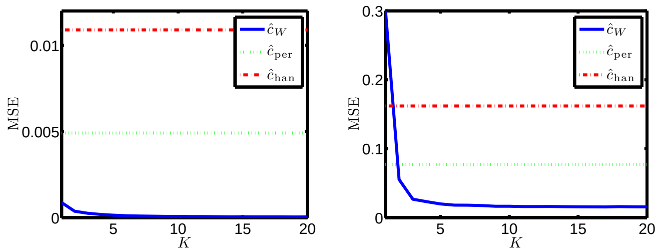


Figure 3: MSE as a function of the sample size $K$. Left: for cepstrum coefficient $q = 3$. Right: averaged over cepstrum coefficients $q = 1, \ldots 15$.

## 5. OPTIMAL SMOOTHING KERNEL BASED ON SAMPLE COVARIANCE

The approximative solution given in (10) of the MSE optimal periodogram smoothing for cepstrum estimation defined in (3) depends on the true covariance function, which in practical applications is unknown. Instead, one has to rely either on a priori knowledge about the application at hand, or on data driven methods. A simple example of a data driven approach is to first get a rough approximation of the covariance structure for the current type of data. The covariance structure can be plugged into (10) in order to compute a suitable smoothing kernel function. In the following example we have chosen an AR(2) process with poles in $0.98\mathrm{e}^{\pm i2\pi 0.25}$. We compute the average of $K$ sample covariance functions, $r_{\mathrm{samp}}(\tau) = \frac{1}{K}\sum_{i=1}^{K}\frac{1}{n}\sum_{t=1}^{n-|\tau|} x_i(t)x_i(t+|\tau|)$, where $x_i$, $i = 1, \ldots, K$, are independent realizations of this process. Using this sample covariance function, we compute the approximate optimal smoothing kernel function. On a new realization from the AR(2) process, we use the smoothing kernel function to estimate the cepstrum. In Fig. 3 the Monte Carlo computed ($100 \times 100$ simulations) MSE for $q = 3$ and the MSE summed over cepstrum coefficients $q = 1, \ldots, 15$ of such a procedure are shown as a function of $K$. The MSE of $\hat{c}_{\mathrm{per}}$ and of $\hat{c}_{\mathrm{han}}$ are also shown as a comparison. From only two realizations a smoothing kernel $W$ can be computed which will make $\hat{c}_W$ superior.

## 6. DISCUSSION

We have for the first time presented an approximative solution to the MSE optimal periodogram smoothing kernel function for cepstrum estimation. For the solution to be accurate a penalty term has to be included. On a speech like simulated process we have shown that the weighting of the penalty term is not very crucial. The optimal smoothing kernel depends on the true covariance, which in practical applications is unknown, and hence one has to rely either on a priori knowledge about the application at hand, or on data driven methods. A priori knowledge may be captured in a model. The model can be too restrictive for the application, but still good enough to provide input to (10), by which a good smoother is computed. As an example of a data driven method, we have shown how the sample covariance function, estimated from data, can be used to compute a good smoothing kernel function. The approximative solution that we present also provides a possibility in future research to gain general insights in how the optimal smoothing kernel depends on cer-

$$\mathbf{Y} = \sum_{p=-n+1}^{n-1} \frac{1}{S(p)} \mathbf{A}_p \mathrm{e}^{i2\pi \frac{p}{N} q}$$

$$= \begin{bmatrix} \left( \sum_{p=-n+1}^{n-1} \frac{\cos\left(2\pi \frac{p}{N} 0\right)}{S(p)} \mathrm{e}^{i2\pi \frac{p}{N} q} \right) \left( \sum_{t=1}^{n} x(t)^2 \right) \\ \left( \sum_{p=-n+1}^{n-1} \frac{\cos\left(2\pi \frac{p}{N} 1\right)}{S(p)} \mathrm{e}^{i2\pi \frac{p}{N} q} \right) \left( 2 \sum_{t=1}^{n-1} x(t)x(t+1) \right) \\ \vdots \\ \left( \sum_{p=-n+1}^{n-1} \frac{\cos\left(2\pi \frac{p}{N} (n-1)\right)}{S(p)} \mathrm{e}^{i2\pi \frac{p}{N} q} \right) \left( 2 \sum_{t=1}^{n-(n-1)} x(t)x(t+(n-1)) \right) \end{bmatrix} \tag{12}$$

$$\mathbf{M}_{ab} = i_{a,b} \left( \sum_{p=-n+1}^{n-1} \frac{\cos\left(2\pi \frac{p}{N} (a-1)\right)}{S(p)} \mathrm{e}^{i2\pi \frac{p}{N} q} \right) \left( \sum_{p=-n+1}^{n-1} \frac{\cos\left(2\pi \frac{p}{N} (b-1)\right)}{S(p)} \mathrm{e}^{i2\pi \frac{p}{N} q} \right)$$
$$\times \sum_{t_1=1}^{n-(a-1)} \sum_{t_2=1}^{n-(b-1)} \mathsf{E}\left[x(t_1)x(t_1+(a-1))x(t_2)x(t_2+(b-1))\right] \tag{13}$$

where $i_{a,b} = 1$ if $a = 1$ and $b = 1$, $i_{a,b} = 2$ if $a = 1$ and $b \neq 1$ or if $a \neq 1$ and $b = 1$, and $i_{a,b} = 4$ otherwise.

$$\sum_{t_1=1}^{n-(a-1)} \sum_{t_2=1}^{n-(b-1)} \mathsf{E}\left[x(t_1)x(t_1+(a-1))x(t_2)x(t_2+(b-1))\right] \tag{14}$$
$$= r(a-1)r(b-1)(n-a+1)(n-b+1) + \sum_{t_1=1}^{n-(a-1)} \sum_{t_2=1}^{n-(b-1)} r(t_2-t_1)r(t_2+b-t_1-a) + r(t_2+b-1-t_1)r(t_2-t_1-a+1).$$

tain properties of the random process.

## A. THE MATRIX $\mathbf{M}$

The random vector $\mathbf{Y}$ is expressed in terms of the random process $x$ in (12) and the $(a,b)$:th element of the matrix $\mathbf{M}$ is given in terms of $x$ in (13). For a Gaussian process with zero mean, the formula $\mathsf{E}[ABCD] = \mathsf{E}[AB]\mathsf{E}[CD] + \mathsf{E}[AC]\mathsf{E}[BD] + \mathsf{E}[AD]\mathsf{E}[BC]$ can be applied and the expectation inside the expression of $\mathbf{M}$ can be expressed as in (14).

## REFERENCES


[1] B. P. Bogert, M. J. R. Healy, and J. W. Tukey, "The quefrency alanysis of time series for echoes: Cepstrum, pseudo-autocovariance, cross-cepstrum and saphe cracking," in *Proc. Symp. Time Series Analysis*, M. Rosenblatt, Ed., chapter 15. John Wiley, 1963.

[2] A. V. Oppenheim and R. W. Schafer, "From frequency to quefrency: a history of the cepstrum," *IEEE Signal Process. Mag.*, vol. 21, no. 5, pp. 95–106, Sept. 2004.

[3] H. S. Bokil, B. Pesaran, R. A. Andersen, and P. P. Mitra, "A method for detection and classification of events in neural activity," *IEEE Trans. Biomed. Eng*, vol. 53, no. 8, pp. 1678–1687, Aug. 2006.

[4] M. Hansson-Sandsten and J. Axmon, "Multiple-window cepstrum analysis for estimation of periodicity," *IEEE Trans. Signal Process.*, vol. 55, no. 2, pp. 474–481, Feb. 2007.

[5] M. E. Badaoui, F. Guillet, and J. Danière, "New applications of the real cepstrum to gear signals, including definition of a robust fault indicator," *Mechanical Systems Signal Process.*, vol. 18, pp. 1031–1046, 2004.

[6] J. Sandberg, M. Hansson-Sandsten, T. Kinnunen, R. Saeidi, P. Flandrin, and P. Borgnat, "Multitaper estimation of frequency-warped cepstra with application to speaker verification," *IEEE Signal Proc. Lett.*, vol. 17, no. 4, pp. 343–346, April 2010.

[7] T. F. Quatieri, *Discrete-Time Speech Signal Processing*, Upper Saddle River, NJ: Prentice-Hall, 2002.

[8] Y. Ephraim and M. Rahim, "On second-order statistics and linear estimation of cepstral coefficients," *IEEE Trans. Speech Audio Process.*, vol. 7, no. 2, pp. 162–176, Mar. 1999.

[9] T. Gerkmann and R. Martin, "On the statistics of spectral amplitudes after variance reduction by temporal cepstrum smoothing and cepstral nulling," *IEEE Trans. Signal Process.*, vol. 57, no. 11, pp. 4165–4174, Nov. 2009.