A RECURSIVE ERRORS-IN-VARIABLES METHOD FOR TRACKING TIME VARYING AUTOREGRESSIVE PARAMETERS FROM NOISY OBSERVATIONS

Julien Petitjean[1], Eric Grivel[1], Roberto Diversi[2] and Roberto Guidorzi[2]

[1] IMS, ENSEIRB-MATMECA – Université Bordeaux 1, 351 cours de la libération, 33405, TALENCE, France.
[2]DEIS – University of Bologna, Viale del Risorgimento 2, 40136 BOLOGNA, Italy.

[1]julien.petitjean@ims-bordeaux.fr, eric.grivel@ims-bordeaux.fr,
[2]roberto.diversi@unibo.it, roberto.guidorzi@unibo.it

ABSTRACT

Time Varying Autoregressive (TVAR) models play a key role in various applications such as radar processing, aeronautics and speech processing. Nevertheless, tracking TVAR parameters may be difficult, especially when the process is disturbed by an additive white noise. In this paper, we suggest the use of a recursive Errors-In-Variables method to estimate the variances of the driving process and the additive noise and to track TVAR parameters. This method is based on a Newton-Raphson algorithm. A comparative study with EKF, UKF and CDKF is also proposed.

1. INTRODUCTION

Autoregressive (AR) models and Multivariate Autoregressive (M-AR) models are used in a wide range of applications from speech processing to biomedical signal analysis, from radar processing to mobile communication systems, etc.

However, when the observations are disturbed by an additive noise, the Least-Squares (LS) estimates of the AR parameters, the estimates based on standard Yule-Walker equations and LS on-line methods, are biased.

To compensate the influence of the additive noise, various solutions have been proposed in the literature. When the additive noise is white[1], on-line noise-compensated methods such as the ρ-LMS can be considered. Mutually interactive approaches combining Kalman or H_∞ filtering have been also proposed. Some of them can be seen as recursive instrumental variable techniques and hence provide consistent estimates of the parameters. Their relevance has been studied for AR and M-AR parameters for speech enhancement, channel estimation for CDMA or OFDM systems, etc.

If off-line noise-compensated approaches are considered, it is possible to use modified Yule-Walker (MYW) equations that can be seen as an instrumental variable technique. Alternative solutions have been proposed by Davila [5] who maps this estimation issue into a quadratic eigenvalue problem. Iterative approaches based on the Noise-Compensated Yule Walker equations have been suggested by Zheng in [16]. This latter approach has been extended to the M-AR process case in [9] and to the case of additive colored noise in [15] and [10]. For TVAR tracking from noisy observations, one idea would be to derive an Expectation-Maximization (EM) algorithm to estimate the TVAR process itself (i.e. the complete data) from the noisy observations and the coefficients of the expansion of the TVAR parameters into the basis sequences. Nevertheless, the selection of the basis and of its size has to be done ; in addition, the number of parameters to be estimated increases much.

As an alternative, we have suggested using Errors-in-Variables (EIV) approaches that have the advantage of estimating the AR parameters, the variance of the driving noise and that of the additive noise directly from the noisy observations. This approach is based on the search of the kernel of a specific sample covariance matrix obtained from noisy data. It has been first studied with reference to a noisy scalar AR process and applied in the field of speech enhancement using a single microphone [4], channel estimation in mobile communication systems, and for radar sea clutter rejection [11]. EIV techniques have then been extended to M-AR processes [12]. These approaches provide significant results for signal-to-noise ratios (SNR) higher than 5 dB also when a limited number of samples (few hundreds) is available.

In various applications such as aeronautics [8], radar processing [1] and EEG analysis [3], time varying AR (TVAR) models are often used to design parametric approaches for non-stationary signals. For more details about TVAR processes the reader may refer to

[1] For more details, the reader is referred to [4].

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various pioneering works by Grenier in the field of speech analysis, transmission and recognition, like [7].

Usually, the TVAR parameters are approximated by a weighted combination of a small number of known functions such as Legendre polynomials, etc.

To our knowledge, only few papers deal with the direct estimation of the TVAR parameters from noisy observations. Possible approaches concern extended Kalman filtering, sigma point Kalman filters (SPKF) [14], that include the unscented Kalman filter, and the central difference Kalman filter. Nevertheless, they require the *a priori* knowledge of the variances of the noises. In [13] and [6], particle filtering is considered but the computational cost may be particularly high.

In this paper, we derive a recursive EIV scheme to track TVAR parameters from noisy observations of a TVAR process. It does not require any *a priori* information about noise variances.

Given a generic algebraic process described by the variables $\{v_k\}_{k=1,\dots,K}$, the EIV estimation problem consists in determining, on the only basis of noisy observations $\{t_k = v_k + e_k\}_{k=1,\dots,K}$, the set of K-tuples $\{\lambda_k\}_{k=1,\dots,K}$ that satisfy the relation:

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_K v_K = 0. \tag{1}$$

By introducing R_x, R_t and R_b the autocorrelation matrices of $[v_1 \ \cdots \ v_K]$, $[t_1 \ \cdots \ t_K]$ and $[e_1 \ \cdots \ e_K]$ respectively, the above equation is equivalent to:

$$\begin{aligned}(R_t - R_e)[\lambda_1 \ \cdots \ \lambda_K]^T &= R_v[\lambda_1 \ \cdots \ \lambda_K]^T \\ &= \underline{0}\end{aligned} \tag{2}$$

where $\underline{0}$ is a zero row vector.

At that stage, the Frisch scheme [2] makes it possible to define the set of noise-compensating matrices such that $(R_t - R_e)$ is positive semidefinite. The kernel of $(R_t - R_e)$ hence corresponds to the set $\{\lambda_k\}_{k=1,\dots,K}$.

In our case, $\{\lambda_k\}_{k=1,\dots,K}$ represent the TVAR parameters.

The remainder of the paper is organized as follows: section 2 describes the problem statement. The Frisch scheme is then introduced to estimate the variances of the driving process and of the additive noise as well as the TVAR parameters. Section 3 describes a recursive EIV algorithm for parameter tracking while Section 4 reports a comparison of the proposed approach with EKF, UKF and SPKF which require the *a priori* knowledge of the noise variances.

2. PROBLEM STATEMENT

Let the TVAR process be defined as follows:

$$x(n) = -\sum_{l=1}^{p} a_l(n) x(n-l) + u(n) \tag{3}$$

where $\{a_l(n)\}_{l=1,\dots,p}$ denote the TVAR parameters[2] and $u(n)$ is a zero-mean white noise with variance σ_u^2. This process is considered as disturbed by an additive zero-mean white noise $b(n)$ with variance σ_b^2, uncorrelated with the driving process:

$$y(n) = x(n) + b(n). \tag{4}$$

Let now introduce the following vectors:

$$\bar{\varphi}_x(n) = [x(n) \ x(n-1) \cdots x(n-p)]^T = [x(n) \ \varphi_x^T(n)]^T$$

$$\bar{\varphi}_y(n) = [y(n) \ y(n-1) \cdots y(n-p)]^T = [y(n) \ \varphi_y^T(n)]^T$$

$$\bar{\varphi}_b(n) = [b(n) \ b(n-1) \cdots b(n-p)]^T = [b(n) \ \varphi_b^T(n)]^T$$

$$\bar{\varphi}_u(n) = \Big[u(n) \quad \underbrace{0 \ \cdots \ 0}_{p} \Big]^T$$

and the TVAR parameter vector:

$$\bar{\theta}(n) = [1 \ a_1(n) \cdots a_p(n)]^T = [1 \ \theta(n)^T]^T.$$

Then, equations (3) and (4) can be expressed in a matrix form as follows:

$$\left(\bar{\varphi}_x^T(n) - \bar{\varphi}_u^T(n)\right)\bar{\theta}(n) = 0 \tag{5}$$

and
$$\bar{\varphi}_y(n) = \bar{\varphi}_x(n) + \bar{\varphi}_b(n) \tag{6}$$

By pre-multiplying equation (5) by $[\bar{\varphi}_x(n) - \bar{\varphi}_u(n)]^*$ and by introducing $\bar{R}_x(n) = E\left[\bar{\varphi}_x(n)\bar{\varphi}_x^{T*}(n)\right]$, we obtain:

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[2] At time n: $H(z) = \dfrac{1}{\sum_{l=0}^{p} a_l(n) z^{-l}} = \dfrac{1}{\prod_{l=1}^{p}(1 - p_l(n) z^{-1})}$ where p_l are the poles.

$$\bar{R}^{*}_{x,u}(n)\bar{\theta}(n) = \left(\bar{R}^{*}_{x}(n) - diag\left[\sigma_u^2 \ \underbrace{0 \cdots 0}_{p} \right] \right) \bar{\theta}(n) = \underline{0} \quad (7)$$

The TVAR parameters could thus be obtained, should $\bar{R}^{*}_{x,u}(n)$ be available.

However, in all real cases, only the observation sample covariance matrix $\bar{R}^{*}_{y}(n)$ can be computed.

Let us now focus our attention on a recursive approach to estimate the correlation of the noisy observations and the variances of the additive noise and of the driving process. It should be noted that for every estimation of the additive-noise variance σ_b^2, one can easily deduce $\theta(n)$ and σ_u^2. Given (6) and (7), the extended vector $\bar{\theta}(n)$ satisfies the condition:

$$\left(\bar{R}^{*}_{y}(n) - diag\left[\sigma_u^2 + \sigma_b^2 \ \sigma_b^2 \cdots \sigma_b^2 \right] \right) \bar{\theta}(n)$$
$$= \left(\begin{bmatrix} \sigma_y^2(n) & r^{T*}(n) \\ r(n) & R^{*}_{y}(n) \end{bmatrix} - diag\left[\underbrace{\sigma_u^2 + \sigma_b^2}_{1} \ \underbrace{\sigma_b^2 \cdots \sigma_b^2}_{p} \right] \right) \bar{\theta} \,. \quad (8)$$
$$= \underline{0}$$

Given $\bar{\theta}(n)$, equation (8) can be split into the following equalities:

$$\begin{cases} \sigma_y^2 - \sigma_u^2 - \sigma_b^2 + r^{T*}(n)\theta(n) = 0 & (9) \\ r(n) + (R^{*}_{y}(n) - \sigma_b^2 I_p)\theta(n) = \underline{0} & (10) \end{cases}$$

Thus, given an estimation of σ_b^2, the TVAR parameters $\theta(n)$ can be estimated by using (10); the variance σ_u^2 can then be deduced from equation (9).

In the following the TVAR parameter vector is denoted as a function of σ_b^2, i.e. $\bar{\theta}(n,\sigma_b^2) = [1\ \theta(n,\sigma_b^2)^T]^T$ and a recursive algorithm, based on the solution of higher-order Yule-Walker equations, is proposed to estimate the additive-noise variance σ_b^2.

For this purpose, let us consider two column vectors of size q with $q \geq p$:

$$\varphi_x^h(n) = \left[x(n-p-1)\ x(n-p-2) \cdots x(n-p-q) \right]^T$$

and

$$\varphi_y^h(n) = \left[y(n-p-1)\ y(n-p-2) \cdots y(n-p-q) \right]^T .$$

Since $\quad R_y^h(n) = E\left[\varphi_y^h(n) \bar{\varphi}_y^{T*}(n) \right] = R_x^h(n)$,

$\bar{\theta}(n,\sigma_b^2)$ satisfies the Yule-Walker equation :

$$(R_y^h(n))^{*} \bar{\theta}(n,\sigma_b^2) = \underline{0} . \quad (11)$$

We thus suggest estimating σ_b^2 by minimizing the following cost function:

$$\begin{aligned} J(\sigma_b^2) &= \left\| (R_y^h(n))^{*} \bar{\theta}(n,\sigma_b^2) \right\|_2^2 \\ &= \bar{\theta}(n,\sigma_b^2)^{T*} (R_y^h(n))^T (R_y^h(n))^{*} \bar{\theta}(n,\sigma_b^2) \\ &= \bar{\theta}(n,\sigma_b^2)^{T*} \Sigma^h(n) \bar{\theta}(n,\sigma_b^2) \\ &= \bar{\theta}(n,\sigma_b^2)^{T*} \begin{bmatrix} \sigma(n) & \rho^{T*}(n) \\ \rho(n) & \Sigma(n) \end{bmatrix} \bar{\theta}(n,\sigma_b^2) \end{aligned} \quad (12)$$

where $0 \leq \sigma_b^2 \leq \sigma_{b,\max}^2$ and $\sigma_{b,\max}^2$ is the lowest eigenvalue of $\bar{R}_y(n)$.

In [2] and [4], off-line methods have been proposed to estimate the noise variances. In [11], a recursive approach has been derived for time-invariant AR processes.

Here, our purpose is to use a Newton-Raphson algorithm to track the TVAR parameters.

3. A NEWTON-RAPHSON ALGORITHM FOR PARAMETER TRACKING

The criterion defined in (12) can be expressed as:

$$\begin{aligned} J(\sigma_b^2) &= \theta^{T*}(n,\sigma_b^2)\Sigma\theta(n,\sigma_b^2) \\ &\quad + \rho^{T*}(n)\theta(n,\sigma_b^2) + \theta^{T*}(n,\sigma_b^2)\rho(n) + \sigma(n) \\ &= f(\theta) = f(g(\sigma_b^2)) \end{aligned}$$

where $g(n,\sigma_b^2) = -(R^{*}_{y}(n) - \sigma_b^2 I_p)^{-1} r(n)$.

By denoting $\dfrac{\partial^2 J(\sigma_b^2)}{\partial(\sigma_b^2)^2}$ as $J''(\sigma_b^2(n))$, the Newton-Raphson algorithm makes it possible to estimate the variance σ_b^2 iteratively. At time n, let the initial guess $\hat{\sigma}_{b,0}^2(n)$ of the variance be equal to the value estimated at time $n-1$. Since the function $J'(.)$ is wellbehaved, a better approximation of the variance can be iteratively obtained as follows:

$$\begin{aligned}\hat{\sigma}^2_{b,i+1}(n) &= \hat{\sigma}^2_{b,i}(n) - \frac{J'(\hat{\sigma}^2_{b,i}(n))}{J''(\hat{\sigma}^2_{b,i}(n))} \\ &= \hat{\sigma}^2_{b}(n) - \frac{g'(\hat{\sigma}^2_{b,i}(n))^T f'(\hat{\theta}_i(n))}{g'(\hat{\sigma}^2_{b,i}(n))^T f''(\hat{\theta}_i(n)) g'(\hat{\sigma}^2_{b,i}(n))}\end{aligned} \qquad (13)$$

where $\hat{\sigma}^2_{b,i}(n)$ is the i+1th estimate of the variance and $\hat{\theta}_i(n)$ the corresponding TVAR parameters. Moreover:

$$f'(\hat{\theta}_i) = \left.\frac{\partial f}{\partial \theta}\right|_{\theta=\hat{\theta}_i(n)} = 2(\hat{\Sigma}\hat{\theta}_i(n) + \hat{\rho}),$$

$$f''(\hat{\theta}) = \left.\frac{\partial^2 f}{\partial \theta^2}\right|_{\theta=\hat{\theta}_i(n)} = 2\hat{\Sigma},$$

$$\begin{aligned}g'(\hat{\sigma}^2_{b,i}) &= (\hat{R}^*_y(n) - \hat{\sigma}^2_{b,i} I_p)^{-1}\hat{\theta}_i(n) \\ &= (\hat{R}^*_{x,i}(n))^{-1}\hat{\theta}_i(n)\end{aligned}. \qquad (14)$$

By substituting (14) in (13) we obtain:

$$\begin{aligned}\hat{\sigma}^2_{b,i+1}(n) &= \hat{\sigma}^2_{b,i}(n) \\ &- \frac{((\hat{R}^*_{x,i})^{-1}(n)\hat{\theta}_i(n))^T(\hat{\Sigma}(n)\hat{\theta}_i(n) + \hat{\rho}(n))}{((\hat{R}^*_{x,i})^{-1}(n)\hat{\theta}_i(n))^T\hat{\Sigma}(n)(\hat{R}^*_{x,i})^{-1}(n)\hat{\theta}_i(n)}\end{aligned} \cdot \qquad (15)$$

The process is repeated until convergence that leads to the estimate of the additive-noise variance at time *n*, $\hat{\sigma}^2_b(n)$.

For TVAR tracking, after an initialization step requiring *Ntrame* samples, the autocorrelation matrix is updated as follows:

$$\begin{aligned}\hat{R}_y(n+1) &= \hat{R}_y(n) + \frac{\varphi_y(n+1)\varphi_y^{T*}(n+1)}{Ntrame} \\ &- \frac{\varphi_y(n-Ntrame+1)\varphi_y^{T*}(n-Ntrame+1)}{Ntrame}\end{aligned}. \qquad (16)$$

It should be noted that $\hat{r}(n+1)$ and $\hat{R}^h_y(n+1)$ are updated in a similar way. In addition, it holds:

$$\hat{\sigma}^2_y(n+1) = \hat{\sigma}^2_y(n) + \frac{\left|y(n+1)\right|^2}{Ntrame} - \frac{\left|y(n-Ntrame+1)\right|^2}{Ntrame}.$$

TVAR tracking can thus be summarized as follows*:*

1. *Update of $\hat{\sigma}^2_b(n)$ by means of (15). It should be noted that several iterations of the Newton-Raphson algorithm may be required.*
2. *Update of $\hat{R}_y(n)$, $\hat{r}(n)$ and $\hat{\sigma}^2_y(n)$ using (16).*
3. *Computation of $(\hat{R}^*_x(n+1))^{-1} = (\hat{R}^*_y(n+1) - \hat{\sigma}^2_b(n+1)I_p)^{-1}$ and of the AR parameters at time n by means of the relation $\hat{\theta}(n+1) = -(\hat{R}^*_x(n+1))^{-1}\hat{r}(n+1)$.*
4. *Update of $\hat{\sigma}^2_u(n)$ by means of (9).*
5. *Update of $\hat{R}^h_y(n)$ and computation of $\hat{\Sigma}(n+1)$ and $\hat{\rho}(n+1)$.*

4. SIMULATION RESULTS

In this section, the performance of the approach proposed in this paper is compared with that of other on-line approaches such as standard Kalman filtering, EKF and SPKF (including UKF and CDKF). It should be noted that noise compensated approaches like the ρ-LMS cannot be used in TVAR parameter tracking.

For a simpler exposition, the order of the simulated TVAR process has been taken equal to 2; 2048 samples have been used. The AR parameters evolve in time according to the variation of the associated poles reported in fig. 3 and 4. The signal-to-Noise Ratio (SNR) is equal to 10 dB. The way the poles evolve in time is given in fig. 1 whereas the resulting spectrogram points out the non stationarity of the signal in fig. 2.

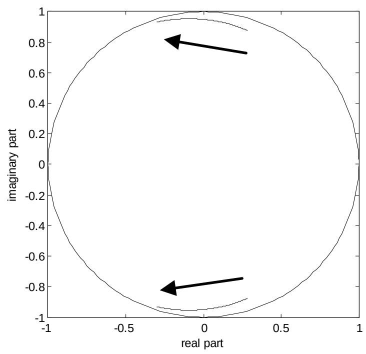

Figure 1: Evolution of the poles associated to the AR signal.

The simulations that have been carried out show that UKF and CDKF approaches provide similar results.

The number of iterations required by the Newton-Raphson algorithm used by the approach described in the paper has been low (typically equal to 2).

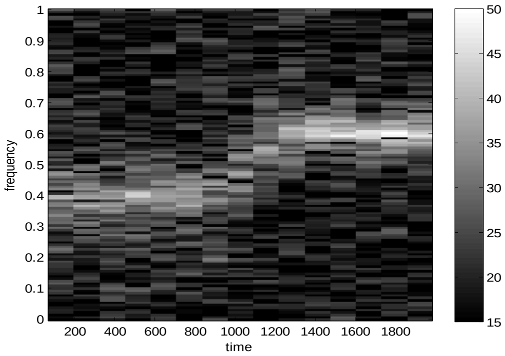

Figure 2: Signal spectrogram (dB).

5. CONCLUSION

It is thus possible to conclude that the approach described in the paper provides results quite similar to those of other methods like EKF and SPKF but with the advantage that it does not require any a priori information on the noise variances.

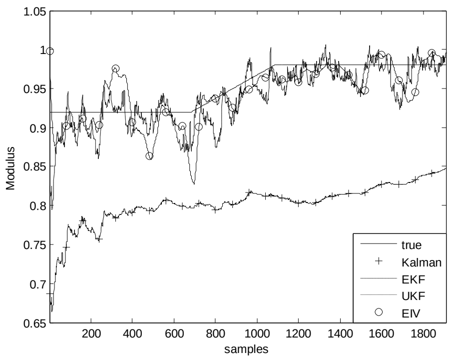

Figure 3: Estimation of the pole modulus

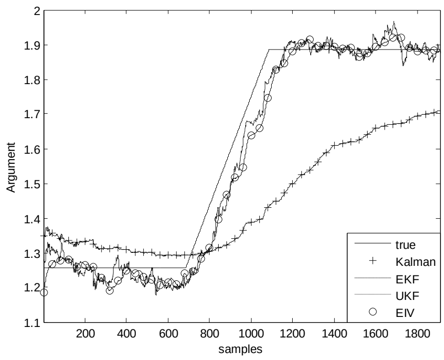

Figure 4: Estimation of the pole argument.

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