GAIN CONTROL FREE BLIND FREQUENCY OFFSET ESTIMATOR FOR GENERAL QAM COMMUNICATION

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ABSTRACT

In this paper we present a novel not data aided gain control free frequency offset estimator for general Quadrature Amplitude Modulated constellations along with its theoretical performance analysis. The estimator is based on applying a tentative frequency offset compensation by means of a nonlinear transformation of the received signal samples and on estimating an accumulation function in different angular windows. For perfect frequency offset compensation, the measurements are suitably clustered and their accumulation, named "Constellation Phase Signature" (CPS), is a function of the window orientation made up by a set of pulses whose locations depend on the constellation. If the constellation is known, the CPS is known, and the estimated frequency offset is the one such that the preliminary frequency compensation of the non linearly transformed signal samples provides the best match between the observed phase histogram and its expected value corresponding to zero frequency offset. The performance analysis is shown to match the numerical simulations for medium to high values of SNR.

Index Terms— Frequency estimation, Quadrature amplitude modulation.

1. INTRODUCTION

In general Quadrature Amplitude Modulated (QAM) transmission, preliminary carrier phase and frequency offset estimation needs to be performed at the output of the receiver, either in a trained or in blind fashion. Although many standard communication systems adopt trained transmission, great bandwidth savings are achieved when the estimation is performed using blind estimators. In [1], the authors introduce a family of blind feed-forward nonlinear least-squares estimators for joint carrier phase and frequency offset estimation, relying on a constellation dependent nonlinear estimator that, after automatic gain control, minimizes the asymptotic (large sample) error variance. In [2] nondata-aided carrier frequency offset estimation for non-circular modulations in unknown frequency-selective channels is described; the estimator exploits oversampling of the received waveform to induce unconjugated cyclostationary statistics on the samples, while in [3] nondata-aided frequency offset estimation is performed exploiting a suitable linear precoding. In [4], a frequency estimator is developed, and its performance analyzed based

on the unique conjugate cyclic frequency of the received signal, which is equal to twice the frequency offset. In [5], an estimator is derived by linearizing the Maximum-likelihood cost function, both for preamble-based and blind acquisition.

In [6, 7] a novel blind frequency offset estimator for cross QAM constellations has been introduced extending the phase offset estimator presented in [8] and [9]. This estimator is based on the observation that, when the frequency offset is perfectly removed by preliminary compensation, a suitable nonlinear transformation of the received signal samples exhibits a particular phase distribution, named Constellation Phase Signature (CPS). In ideal, noise-free QAM signalling, the CPS is constituted by a discrete number of pulses whose locations depend on the signal constellation, and retains a significant peakness also in presence of channel noise. Hence, the frequency offset can be estimated by searching the frequency compensation that maximizes the peakness of the CPS estimated on the compensated data. The resulting blind frequency estimator does not need neither gain/SNR nor constellation knowledge and performs well on cross constellations for medium to high values of SNR. Here we present a new constellation dependent estimator extending this latter. Under the hypothesis of known constellation, the phase distribution is known. Hence, we devise a frequency estimator such that the preliminary frequency compensation of the non linearly transformed signal samples provides the best match between the observed phase histogram and its expected value corresponding to zero frequency offset.

This paper is organized as follows: In Sect.2 we introduce the model of the received signal; in Sect.3 we present the frequency the CPS estimator while in Sect. 4 we present its performance analysis. Sect.5 shows results of both theoretical performance analysis and numerical simulations; for comparison sake, results of selected state-of-the-art estimators [1] are also reported.

2. DISCRETE-TIME SIGNAL MODEL

Let S[n] be the *n*-th transmitted symbol drawn from a power normalized *M*-ary QAM constellation $\mathcal{A} = \{S_0, ..., S_{M-1}\}$. Let us assume for the samples of the received signal extracted at symbol rate X[n] the following analytical model:

$$X[n] = G_C e^{j\theta + j2\pi f_e n} S[n] + W[n]$$
⁽¹⁾

where G_C is the unknown overall gain, θ and f_e are the unknown phase and frequency offset, and W[n] is a realization of a circularly complex Gaussian stationary noise process, statistically independent of X[n]. The signal-to-noise ratio (SNR) is defined as SNR $\stackrel{\text{def}}{=} G_c^2 / \sigma_w^2$, being $\sigma_w^2 \stackrel{\text{def}}{=} \operatorname{E} \{ |W[n]|^2 \}$ the noise variance. We address here the estimation of the carrier frequency offset f_e given a sample of N consecutive observations X[n], $n = 0, \dots, N-1$. Due to the quadrant constellation symmetry, without loss of generality, the frequency offset is limited to $|f_e| < 1/8$ [1].

3. CPS BASED FREQUENCY OFFSET ESTIMATION Let us consider the following nonlinear function [1, 8, 6] of the received signal samples X[n] comprising a frequency compensation of f_0 :

$$Y^{(f_0)}[n] = |X[n]| \cdot e^{j4 \cdot \arg\{X[n]\}} e^{-j2\pi 4f_0 n}$$
(2)

In the noise free case, and for equiprobable constellation symbols, the pdf of the random variable $Y^{(f_0)}[n]$ in polar form $(Y^{(f_0)}[n] = r_n e^{j\varphi_n})$ can be written as:

$$p_{R,\Phi}(r_n,\varphi_n;f_0) = \frac{1}{M} \sum_{m=0}^{M-1} \delta(r_n - G_C |S_m|) \\ \cdot \delta(\varphi_n - 4\theta_n - 4\arg S_m)$$
(3)

where $\theta_n = \theta + 2\pi (f_e - f_0)n$ is the time-variant phase-offset

due to the residual frequency offset $f_e - f_0$. For perfect frequency compensation $f_0 = f_e$, the noise-free pdf of the random variable $Y^{(f_0)}[n]$ becomes:

$$p_{R,\Phi}(r_n,\varphi_n;f_e) = \frac{1}{M} \sum_{m=0}^{M-1} \delta(r_n - G_C |S_m|) \delta(\varphi_n - 4\theta - 4\arg S_m)$$
(4)

From (3), (4), we see that for $f_0 \neq f_e$ the pdf of the random variable $Y^{(f_0)}[n]$ is cyclically shifted of $4\theta_n$ with respect to the variable φ , and $p_{R,\Phi}(r_n,\varphi_n;f_0) = p_{R,\Phi}(r_n,\varphi_n - t_0)$ $2\pi 4(f_e - f_0)n; f_e).$

In presence of additive noise, the Dirac pulses appearing in (3) become wider pulses whose shape depends on the SNR and the noise pdf; however, we can still observe the pdf cyclic shift by the time varying phase-offset θ_n due to the residual frequency offset $f_e - f_0$.

Now let us consider the Magnitude Weighed Tomographic Projection (MWTP)¹ of the probability density function (pdf) $p_{B \Phi}(r_n, \varphi_n; f_e)$ under the hypothesis of perfect frequency compensation, namely:

$$g_{\Phi}^{(\mathcal{A},\theta)}\left(\varphi_{n}\right) \stackrel{\text{def}}{=} \int_{0}^{+\infty} r_{n} \cdot p_{R,\Phi}\left(r_{n}\varphi_{n}; f_{e}\right) dr_{n} \qquad (5)$$

The MWTP $g_{\Phi}^{(\mathcal{A},\theta)}\left(\varphi_{n}\right)$ behaves as an ordinary pdf; hence it can be estimated by subdividing the phase interval $[0, 2\pi)$ in

K intervals and evaluating the normalized area of $g^{(\mathcal{A},\theta)}_{\Phi}\left(\varphi_{n}\right)$ in the k-th phase interval. At this aim, let us define

$$f^{(\mathcal{A},\theta)}(\psi) \stackrel{\text{def}}{=} \frac{K}{2\pi} \cdot \int_{\psi}^{\psi+2\pi/K} g_{\Phi}^{(\mathcal{A},\theta)}(\varphi_n) \, d\varphi_n \qquad (6)$$

In the limit $K \to \infty$, $f^{(\mathcal{A},\theta)}(\psi_k)$ tends to $g_{\Phi}^{(\mathcal{A},\theta)}(\psi_k)$ The CPS $f^{(\mathcal{A},\theta)}(\psi)$ depends on the constellation \mathcal{A} and is typically made up by a finite set of pulses whose locations and widths depend on the signal constellation and signal-to-noise power ratio, respectively, and it is exploited in [8, 9] to develop a blind phase offset estimator for general QAM signals². For perfect compensation $f_e = f_0$ the CPS can be estimated as:

$$\hat{f}^{(\mathcal{A},\theta,f_{e})}(\psi_{k}) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=0}^{N-1} |Y^{(f_{e})}[n]| d_{K}^{k} \left(Y^{(f_{e})}[n]\right) d_{K}^{k} \left(Y\right) = \frac{K}{2\pi} \operatorname{rect}_{2\pi/K}(\arg(Y) - 2\pi(k+1/2)/K) k = 0, \cdots, K-1$$
(7)

where $\psi_k \stackrel{\text{def}}{=} 2\pi k/K$ denotes the reference phase of the kth phase interval. For perfect frequency compensation, the accumulation function $\hat{f}^{(\mathcal{A},\theta,f_e)}(\psi_k)$ is an unbiased estimator of the CPS, in fact:

For $f_0 \neq f_e$, instead, we have

Hence, the expected value of the function $\hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k)$ is a temporal average of N suitably shifted versions of the per-fectly compensated CPS, *i.e.* $f^{(\mathcal{A},\theta)}(\psi_k)$. Based on this observation, we look for a frequency estimator \hat{f}_{CPS} that best focalizes the sample function $\hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k)$ towards the perfectly compensated CPS $f^{(\mathcal{A},\theta)}(\psi_k)$.

¹The MWTP is first introduced in [9]where a closed form calculation of the MWTP of a generic QAM signal can also be found.

²We remark that both the MWTP and the CPS are cyclically shifted by 4θ due to the residual phase offset θ , that is $g_{\Phi}^{(\mathcal{A},\theta)}(\varphi_n) = g_{\Phi}^{(\mathcal{A},0)}(\varphi_n - 4\theta)$ and $f^{(\mathcal{A},\theta)}(\psi_k) = f^{(\mathcal{A},0)}(\psi_k - 4\theta)$ [9].

If the constellation is known, the perfectly compensated CPS $f^{(\mathcal{A},\theta)}(\psi_k)$ is known, unless the unknown cyclic phase shift due to the phase offset. Then, for every tentative compensation frequency, the observed $\hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k)$ is matched to the zero phase offset CPS. To properly take into account the unknown phase offset, which is seen as a shift of the observed $\hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k)$, the matching is performed by means of a cross-correlation procedure. The maximum output of the cross-correlation is evaluated according to

$$\hat{\mathcal{C}}(f_0) \stackrel{\text{def}}{=} \max_{\psi_k} \hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k) \otimes f^{(\mathcal{A},0)}(\psi_k) \tag{8}$$

and the frequency offset estimator is written as 3 :

$$\hat{f}_{CPS}^{KC} = \arg\max_{f_0} \{\hat{\mathcal{C}}(f_0)\}$$

The estimator \hat{f}_{CPS}^{KC} doesn't need any gain control but only a coarse SNR estimate to evaluate the CPS, as deeply discussed in [9].

Following the approach in [10], the maximization of the gain function $\hat{C}(f_0)$ of the sample function $\hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k)$ can be performed in two steps, *i.e.* by first scanning the admissible range of f_0 , *i.e.* [-1/8, 1/8], with step Δf between candidate frequencies to evaluate an intermediate coarse estimate \hat{f}_c , and then by interpolating the estimate \hat{f}_c around the maximum to obtain the fine estimate \hat{f}_{fine} . Here, we adopt a parabolic approximation for the gain function around its maximum, yielding the following formula for the fine estimate \hat{f}_{fine} :

$$\hat{f}_{fine} = \hat{f}_c + \frac{1}{8}\Delta f_0 \frac{\hat{\mathcal{C}}(\hat{f}_c + \Delta f) - \hat{\mathcal{C}}(\hat{f}_c - \Delta f)}{\hat{\mathcal{C}}(\hat{f}_c + \Delta f) + \hat{\mathcal{C}}(\hat{f}_c - \Delta f) - 2\hat{\mathcal{C}}(\hat{f}_c)}$$

The parabolic approximation captures the local variations of the gain function and allows the estimator performance analysis to be carried out in closed form [10].

4. ANALYTICAL PERFORMANCE EVALUATION

In this Section we provide performance analysis for the CPS based frequency offset estimator.

In evaluating the accuracy of the estimator (3), we observe that two error components appear. The first component occurs when the coarse estimate \hat{f}_c is not correct, in the sense that it does not maximize the expected value of the gain function $\hat{C}(k\Delta f)$ over the index k. The second error component is due to the sample peakness estimation error and to the misfit of the parabolic approximation around its maximum, and definitely limits the estimator accuracy. Numerical simulations show that, for a large range of SNR values, the coarse estimate is correct, and the first error component is zero. Hence, following the approach indicated in [10], the bias and the variance of \hat{f}_{fine} can be analytically

evaluated as a function of the gain function mean, variance and covariances.

Let us denote $E_{f_0} \stackrel{\text{def}}{=} \hat{\mathcal{C}}(f_0) - \mathrm{E}\left\{\hat{\mathcal{C}}(f_0)\right\}$ and let us pose

$$X = \mathbf{E} \left\{ \hat{\mathcal{C}}(f_c + \Delta f) \right\} \quad Y = \mathbf{E} \left\{ \hat{\mathcal{C}}(f_c - \Delta f) \right\}$$
$$Z = \mathbf{E} \left\{ \hat{\mathcal{C}}(f_c) \right\}$$

and c = X - Y d = X - 2Z + Y.

Then, resorting to the following first-order approximation of (3):

$$\hat{f}_c - \hat{f}_{fine} \approx \frac{\Delta f}{8} \left(\frac{c}{d} + \frac{d-c}{d^2} E_{f_c + \Delta f} - \frac{d+c}{d^2} E_{f_c - \Delta f} - \frac{2c}{d^2} E_{f_c} \right)$$

the following results hold [10]:

$$\begin{aligned} \operatorname{bias}(\widehat{f}_{fine}) &= -\frac{\Delta f}{2} \frac{2}{d^3} \cdot \left[(Y - Z) \operatorname{Var} \left(\widehat{\mathcal{C}}(f_c + \Delta f) \right) \right. \\ &+ (Z - X) \operatorname{Var} \left(\widehat{\mathcal{C}}(f_c - \Delta f) \right) - 2 \left(X - Y \right) \operatorname{Var} \left(\widehat{\mathcal{C}}(f_c) \right) \\ &- \left(Y + 2Z - 3X \right) \operatorname{Cov} \left(\widehat{\mathcal{C}}(f_c - \Delta f), \widehat{\mathcal{C}}(f_c) \right) \\ &+ \left(Y + 2Z - 3X \right) \operatorname{Cov} \left(\widehat{\mathcal{C}}(f_c), \widehat{\mathcal{C}}(f_c + \Delta f) \right) \\ &- \left(X - Y \right) \operatorname{Cov} \left(\widehat{\mathcal{C}}(f_c + \Delta f), \widehat{\mathcal{C}}(f_c - \Delta f) \right) \right] - b_{pm} \end{aligned}$$

where the term $b_{pm} = (f_e - f_c) + \Delta f/2 \cdot c/d$ accounts for the parabolic misfit [10]. As long as the variance of the estimator \hat{f}_{fine} is concerned, we have

$$\begin{aligned} \operatorname{aVar}(\hat{f}_{fine}) &\stackrel{\text{det}}{=} \lim_{N \to \infty} N \cdot \operatorname{Var}(\hat{f}_{fine}) \\ &= \frac{\Delta f^2}{64} \Big[\left(\frac{d-c}{d^2} \right)^2 \operatorname{Var}(\hat{\mathcal{C}}(f_c + \Delta f)) \\ &+ \left(\frac{d+c}{d^2} \right)^2 \operatorname{Var}(\hat{\mathcal{C}}(f_c - \Delta f)) + \left(\frac{2c}{d^2} \right)^2 \operatorname{Var}(\hat{\mathcal{C}}(f_c)) \\ &- \left(\frac{d^2 - c^2}{d^4} \right) \operatorname{Cov}(\hat{\mathcal{C}}(f_c + \Delta f), \hat{\mathcal{C}}(f_c - \Delta f)) \\ &+ \left(\frac{2dc + 2c^2}{d^4} \right) \operatorname{Cov}(\hat{\mathcal{C}}(f_c), \hat{\mathcal{C}}(f_c - \Delta f)) \\ &+ \left(\frac{2dc - 2c^2}{d^4} \right) \operatorname{Cov}(\hat{\mathcal{C}}(f_c + \Delta f), \hat{\mathcal{C}}(f_c)) \Big] \end{aligned}$$

Since the constellation the CPS $f^{(\mathcal{A},\theta)}(\psi_k)$ can be evaluated in close form, unless an unknown cyclic phase shift. Then, for every tentative compensation frequency, the gain function $\hat{\mathcal{C}}(f_0)$ is evaluated as

$$\hat{\mathcal{C}}(f_0) = \max_k \hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k) \otimes f^{(\mathcal{A},0)}(\psi_k)$$

$$= \sum_{k=0}^{K-1} \hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k - \psi_{k_{max}}) f^{(\mathcal{A},0)}(\psi_k)$$

$$\approx \sum_{k=0}^{K-1} \hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k - \theta) f^{(\mathcal{A},0)}(\psi_k)$$
(9)

³Interestingly enough, when $\hat{C}(f_0)$ is evaluated at the frequency \hat{f}_{CPS}^{KC} , the location of the cross-correlation maximum provides a coarse phase estimate as described in [9].



Fig. 1. Frequency estimator MSE vs. SNR for 64-QAM (right) and 128 QAM (left) constellations; CPS-based estimator for known constellation (theoretical, solid line black, and numerical, circle black) and optimal NLS estimator in [1] (WA03) (theoretical, solid line gray, and numerical, diamond gray). The dashed line represents the MCRB [11].



Fig. 2. Frequency estimator MSE vs. SNR for 256-QAM (right) and 512 QAM (left) constellations; CPS-based estimator for known constellation (theoretical, solid line black, and numerical, circle black) and optimal NLS estimator in [1] (WA03) (theoretical, solid line gray, and numerical, diamond gray). The dashed line represents the MCRB [11].

where in (8) we disregard the influence of the coarse phase estimate error $\theta - \psi_{k_{max}}$ on the estimated gain function. To proceed, let us introduce the zero mean random process $\epsilon_k^{f_0}$:

$$\epsilon_k^{f_0} \stackrel{\text{def}}{=} \hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k) - \hat{F}^{(\mathcal{A},\theta,f_0)}(\psi_k)$$
$$\hat{F}^{(\mathcal{A},\theta,f_0)}(\psi_k) = E\left\{\hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k)\right\}$$
(10)

Based on this assumption the gain function in (9) can be written as:

$$\hat{\mathcal{C}}(f_0) = \sum_{k=0}^{K-1} (\hat{F}^{(\mathcal{A},\theta,f_0)}(\psi_k) + \epsilon_k^{f_0}) f^{(\mathcal{A},0)}(\psi_k)$$
(11)

With this position, the moments of $\hat{C}(f_0)$ can be expressed as a function of the expected value of $\hat{f}^{(\mathcal{A},\theta,f_0)}(\psi_k)$

and of the moments of the error process $\epsilon_k^{f_0}$. More specifically we have:

$$E\{\hat{\mathcal{C}}(f_0)\} = \frac{1}{K} \sum_{k=0}^{K-1} \hat{F}^{(\mathcal{A},\theta,f_0)}(\psi_k) f^{(\mathcal{A},0)}(\psi_k)$$

and

$$E\{\hat{\mathcal{C}}(f_0)^2\} = \frac{1}{K^2} \sum_{k_1}^{K-1} \sum_{k_2}^{K-1} \left(\hat{F}^{(\mathcal{A},\theta,f_0)}(\psi_{k_1})f^{(\mathcal{A},0)}(\psi_{k_1})\hat{F}^{(\mathcal{A},\theta,f_0)}(\psi_{k_2})f^{(\mathcal{A},0)}(\psi_{k_2}) + f^{(\mathcal{A},0)}(\psi_{k_1})f^{(\mathcal{A},0)}(\psi_{k_2})m_{f_0}^{(2)}[k_1,k_2]\right)$$

and

$$E\{\hat{\mathcal{C}}(f_0)\hat{\mathcal{C}}(f_1)\} = \frac{1}{K^2} \sum_{k_1}^{K-1} \sum_{k_2}^{K-1} \left(\hat{F}^{(\mathcal{A},\theta,f_0)}(\psi_{k_1})f^{(\mathcal{A},0)}(\psi_{k_1})\hat{F}^{(\mathcal{A},\theta,f_1)}(\psi_{k_2})f^{(\mathcal{A},0)}(\psi_{k_2}) + f^{(\mathcal{A},0)}(\psi_{k_1})f^{(\mathcal{A},0)}(\psi_{k_2})m^{(1,1)}_{f_0,f_1}[k_1,k_2]\right)$$

where we adopted the following notations:

$$m_{f_0}^{(r)}[k] \stackrel{\text{def}}{=} \mathbb{E}\left\{\left(\epsilon_k^{f_0}\right)^r\right\}$$

$$m_{f_0}^{(r,s)}[k_1, k_2] \stackrel{\text{def}}{=} \mathbb{E}\left\{\left(\epsilon_{k_1}^{f_0}\right)^r \left(\epsilon_{k_2}^{f_0}\right)^s\right\}$$

$$m_{f_0, f_1}^{(r,s)}[k_1, k_2] \stackrel{\text{def}}{=} \mathbb{E}\left\{\left(\epsilon_{k_1}^{f_0}\right)^r \left(\epsilon_{k_2}^{f_1}\right)^s\right\}$$
(12)

Closed form for the moments in (12) have been derived, based on the expectation of the accumulation function (7). Without loss of generality here we report only the final expressions: N-1

$$m_{f_0}^{(2)}[k] = \frac{1}{N^2} \sum_{n=0}^{N-1} \left(\hat{F}^{(\mathcal{A},f_0,2)} \left(\psi_k - \theta_n^{(f_0)} \right) - \hat{F}^{(\mathcal{A},\theta,f_0)} \left(\psi_k - \theta_n^{(f_0)} \right)^2 \right)$$
$$m_{f_0}^{(2)}[k,j] = \frac{1}{N^2} \sum_{n=0}^{N-1} \left(\hat{F}^{(\mathcal{A},f_0,2)} \left(\psi_k - \theta_n^{(f_0)} \right) \delta_{kj} - \hat{F}^{(\mathcal{A},\theta,f_0)} \left(\psi_k - \theta_n^{(f_0)} \right) \hat{F}^{(\mathcal{A},\theta,f_0)} \left(\psi_j - \theta_n^{(f_0)} \right) \right)$$
$$m_{f_0,f_1}^{(1,1)}[k,j] = \frac{1}{N^2} \sum_{n=0}^{N-1} \left(\hat{F}^{(\mathcal{A},f_0,2)} \left(\psi_k - \theta_n^{(f_0)} \right) \delta_{k(j-L(f_0-f_1)n)} - \hat{F}^{(\mathcal{A},\theta,f_0)} \left(\psi_k - \theta_n^{(f_0)} \right) \hat{F}^{(\mathcal{A},\theta,f_1)} \left(\psi_j - \theta_n^{(f_1)} \right) \right)$$

where δ_{kj} is the Kronecker delta and where we adopted the compact notations:

$$\hat{F}^{(\mathcal{A}, f_0, 2)} \Big(\psi_k - \theta_n^{(f_0)} \Big) \stackrel{\text{def}}{=} E \Big\{ |Y_n^{(f_0)}|^2 \ d_L^k \left(Y_n^{(f_0)} \right) \Big\}$$
$$\theta_n^{(f_0)} \stackrel{\text{def}}{=} 2\pi 4 (f_e - f_0) n$$

Finally, in the limit in which the sample error is asymptotically (large N) described by a normal distribution, we have:

$$m_{f_0}^{(2,2)}[k,j] = m_{f_0}^{(2)}[k] m_{f_0}^{(2)}[j] + 2\left(m_{f_0}^{(1,1)}[k,j]\right)^2 m_{f_0,f_1}^{(2,2)}[k,j] = m_{f_0}^{(2)}[k] m_{f_1}^{(2)}[j] + 2\left(m_{f_0,f_1}^{(1,1)}[k,j]\right)^2$$
(13)

Let us remark that we used the approximation $f^{(\mathcal{A},0)}(\psi_k) \approx g_{\Phi}^{(\mathcal{A},0)}(\psi_k)$ for the numerical evaluation of the above reported first and second order moments.

5. NUMERICAL EXPERIMENTS

Here, we present numerical results assessing the theoretical and experimental performance of the estimator introduced in

Sect.3. Each experiment consists of 1000 Monte Carlo trials, each one with sample size N = 2000 for cross constellations and N = 512 for square constellations; the value assumed for the frequency offset to be estimated is $f_e = 0.05 + \Delta f/4$ being $\Delta f = 1.4 \cdot 10^{-6}$. The experimental and theoretical Mean Square Error (MSE) of the frequency estimator \hat{f}_{CPS} introduced in Sect.3 versus SNR is shown in Figs.1 - 2 for 64-QAM, 128-QAM, 256-QAM and 512-QAM constellations. We observe a good agreement between the analytical and the numerical performance for medium to high SNR. For comparison sake, we report also the results obtained using the optimal NLS estimator [1]. We remark that both the CPS and the NLS estimator require the knowledge of the constellation and of the SNR. The results shown here assume perfect gain and SNR knowledge. The NLS estimator requires also the knowledge of the gain while the CPS estimator is gain control free. A performance degradation is expected for estimated gain. For reference sake we also report the Modified Cramer Rao lower bound (MCRB) [11], which well approximates the Cramer Rao Bound for high values of SNR.

All the figures shows a substantial performance gain for all the constellations at medium-high SNR values; the gain is more pronounced for complex constellations and it accounts for the facts that the CPS allows to jointly exploit measurements corresponding to all the constellation points.

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