

# AN UNBIASED PISARENKO HARMONIC DECOMPOSITION ESTIMATOR FOR SINGLE-TONE FREQUENCY

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## ABSTRACT

In this paper, closed-form frequency estimation of a single real tone in white noise is addressed. Based on the alternative derivation of Pisarenko's frequency estimate using the standard sample covariance of the noisy sinusoid with lags 1 and 2, we have devised novel sample covariance expressions, which are inspired from the modified covariance (MC) method, to achieve unbiased frequency estimation. The variance of the proposed Pisarenko harmonic decomposition (PHD) variant is also derived. Computer simulations are included to corroborate the theoretical development and to demonstrate its superiority over the MC and original PHD algorithms.

## 1. INTRODUCTION

Estimating the frequency of a sinusoid in noise has been an important research topic [1]-[5] because of its wide applicability in control theory, signal processing, digital communications, biomedical engineering as well as instrumentation and measurement. The discrete-time signal model for single sinusoidal frequency estimation is

$$x(n) = s(n) + q(n), \quad s(n) = \alpha \cos(\omega n + \phi), \\ n = 1, 2, \dots, N \quad (1)$$

where  $\alpha$ ,  $\omega \in (0, \pi)$  and  $\phi \in [0, 2\pi)$  are unknown but deterministic constants which represent the tone amplitude, frequency and phase, respectively, while the noise  $q(n)$  is assumed to be a zero-mean white process with unknown variance  $\sigma^2$ . The task is to find  $\omega$  from the  $N$  samples of  $\{x(n)\}$ .

Under Gaussian noise assumption, the maximum likelihood estimate of frequency [6] is obtained by maximizing a highly nonlinear and multimodal cost function and thus extensive computations are involved. For applications where real-time estimation is required, computationally efficient but suboptimal frequency estimators [3] such as notch filtering, Capon methods, linear prediction, Yule-Walker methods and subspace based approaches are widely used choices.

In this work, we focus on fast frequency estimation of a real-valued tone in white noise. The rest of the paper is organized as follows. In Section 2, we first review that the Pisarenko harmonic decomposition (PHD) method [7], which exploits the eigenstructure of the sample covariance matrix, can be derived [8] in an alternative and simpler manner with the sample covariance of  $x(n)$  with lags 1 and 2. Inspired by the modified covariance (MC) frequency estimator [2], we devise an unbiased variant of the PHD estimate. The frequency variance of the unbiased PHD method is also analyzed. Numerical examples are presented in Section 3 to

corroborate the theoretical development and to evaluate the performance of the proposed algorithm by comparing with the original PHD and MC methods. Finally, conclusion is drawn in Section 4.

## 2. UNBIASED PISARENKO'S ESTIMATOR

In this Section, we first review that the PHD solution for single tone frequency estimation can be obtained alternatively [8]. An unbiased variant of the PHD method is then proposed and analyzed. Denote the standard sample covariance of  $x(n)$  with lag  $k$  by  $r_k$ , which is expressed as

$$r_k = \frac{1}{N-k} \sum_{n=1}^{N-k} x(n)x(n+k) \quad (2)$$

For a sufficiently large  $N$ , it is easy to show that  $r_1$  and  $r_2$  can be approximated as

$$r_1 \approx \alpha^2 \cos(\omega)/2 \quad (3)$$

and

$$r_2 \approx \alpha^2 \cos(2\omega)/2 = \alpha^2(\cos^2(\omega) - 1/2) \quad (4)$$

Cross-multiplying (3) and (4), we obtain the following approximate equation:

$$2r_1 \cos^2(\hat{\omega}) - r_2 \cos(\hat{\omega}) - r_1 = 0 \quad (5)$$

where  $\hat{\omega}$  denotes an estimate of  $\omega$ . Only one root of (5) corresponds to the actual frequency and it is easy to verify that  $\hat{\omega}$  has the form of:

$$\hat{\omega} = \cos^{-1} \left( \frac{r_2 + \sqrt{r_2^2 + 8r_1^2}}{4r_1} \right) \quad (6)$$

This frequency estimate is in fact identical to the PHD estimate which is found from the eigenvector corresponding to the smallest eigenvalue of the following  $3 \times 3$  covariance matrix [7]:

$$\begin{bmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0 \end{bmatrix} \quad (7)$$

In our study, we propose to employ alternative expressions for  $r_1$  and  $r_2$ , which are inspired by the MC frequency estimate, denoted by  $\hat{\omega}_{MC}$  [2]:

$$\hat{\omega}_{MC} = \cos^{-1} \left( \frac{\sum_{n=3}^N x(n-1)(x(n) + x(n-2))}{2 \sum_{n=3}^N x^2(n-1)} \right) \quad (8)$$

where the numerator and denominator correspond to  $r_1 = \alpha^2 \cos(\omega)/2$  and  $r_0 = \alpha^2/2 + \sigma^2$ , respectively. Our proposed new  $r_1$  and  $r_2$  are:

$$r_1 = \sum_{n=4}^{N-1} x(n-1)(x(n) + x(n-2)) \quad (9)$$

and

$$r_2 = \sum_{n=5}^N x(n-2)(x(n) + x(n-4)) \quad (10)$$

The choice of (9) and (10) is to achieve unbiased frequency estimation and their computations require  $4(N-4)$  additions and  $(2N-4)$  multiplications. To illustrate the unbiasedness, we take the expected value of (9):

$$\begin{aligned} & E\{r_1\} \\ &= \sum_{n=4}^{N-1} E\{x(n-1)(x(n) + x(n-2))\} \\ &= \sum_{n=4}^{N-1} E\{x(n-1)(s(n) + s(n-2) + q(n) + q(n-2))\} \\ &= \sum_{n=4}^{N-1} E\{s(n-1)(2\cos(\omega)s(n-1) + q(n) + q(n-2))\} \\ &= 2\cos(\omega) \sum_{n=5}^N s^2(n-2) \end{aligned} \quad (11)$$

where  $E$  denotes the expectation operation. In a similar manner, we obtain:

$$E\{r_2\} = 2\cos(2\omega) \sum_{n=5}^N s^2(n-2) \quad (12)$$

From (3)–(5), we see that the new sample covariances of (9) and (10) are superior to the standard ones. As a result, our proposed frequency estimate is obtained by substituting (9) and (10) into (6).

To derive the variance of the modified PHD estimator, we follow the work of [9]. From (5), we define a quadratic function  $f(\rho)$ :

$$f(\rho) = 2r_1\rho^2 - r_2\rho - r_1 \quad (13)$$

where  $\rho = \cos(\hat{\omega})$  is one of its roots. For sufficiently large  $N$  and/or signal-to-noise ratio (SNR), this root will be located at a reasonable proximity of  $\cos(\omega)$ , and the variance of  $\hat{\omega}$ , denoted by  $\text{var}(\hat{\omega})$ , is evaluated as [9]:

$$\text{var}(\hat{\omega}) \approx \frac{E\{f^2(\rho)\}}{(E\{f'(\rho)\})^2} \Bigg|_{\rho=\cos(\omega)} \cdot \frac{1}{\sin^2(\omega)} \quad (14)$$

where  $f'(\rho)$  is the derivative of  $f(\rho)$  with respect to  $\rho$ . Assuming that  $q(n)$  is Gaussian distributed, we have shown that (See the Appendix)

$$\text{var}(\hat{\omega}) \approx$$

$$\begin{aligned} & \frac{1}{(N-4 + g(\omega, \phi, N-2, 0, 3))^2 (\cos(2\omega) + 2)^2} \times \\ & \left( \frac{2 + \mathcal{F}(\omega, \phi, N) + \mathcal{G}(\omega, \phi, N) + \mathcal{H}(\omega, \phi, N)}{2\text{SNR} \sin^2(\omega)} \right. \\ & \left. + \frac{\cos^2(\omega)(N-5) + \cos^2(2\omega)(N-4)}{\text{SNR}^2 \sin^2(\omega)} \right) \end{aligned} \quad (15)$$

where  $g(\omega, \phi, N, k, b)$ ,  $\mathcal{F}(\omega, \phi, N)$ ,  $\mathcal{G}(\omega, \phi, N)$  and  $\mathcal{H}(\omega, \phi, N)$  have been defined therein. For  $\text{SNR} \gg 1$  and sufficiently large  $N$ , the frequency variance can be simplified as (See the Appendix):

$$\text{var}(\hat{\omega}) \approx \frac{\text{SNR} + \cos^2(\omega)(N-5) + \cos^2(2\omega)(N-4)}{\text{SNR}^2 (N-4)^2 (\cos(2\omega) + 2)^2 \sin^2(\omega)} \quad (16)$$

### 3. SIMULATION RESULTS

Computer simulations had been carried out to evaluate the proposed frequency estimator performance for a single real sinusoid in white Gaussian noise. We compared its mean square frequency error (MSFE) with those of the original PHD and MC methods as well as Cramér-Rao lower bound (CRLB) [3] for frequency estimation. The tone amplitude was assigned to  $\sqrt{2}$  and  $\phi = 0$  was used, while different SNRs were obtained by proper scaling the noise variance  $\sigma^2$ . All simulation results provided were averages of 1000 independent runs.

Figure 1 shows the MSFEs of the three estimators as well the CRLB versus  $\omega$  at  $\text{SNR} = 20$  dB and  $N = 10$ . The expressions of (15) and (16) are accompanied to check the validity of theoretical performance of the unbiased PHD method. It can be seen that the measured MSFEs of the proposed estimator agreed very well with the variance formula of (15) while they were close to (16) although its derivation assumes a large value of  $N$ . We observe that the proposed scheme outperformed the PHD solution in spite of the similarity in their algorithms. Moreover, it was superior to the MC method and had performance close to CRLB except when  $\omega \in (0.4\pi, 0.6\pi)$ . The mean frequency errors of the three methods, which were obtained by subtracting  $\omega$  from the corresponding mean frequency estimates, are shown in Figure 2. It is seen that the biases of the modified Pisarenko's estimator were much smaller than those of the PHD and MC method for the whole frequency range, which demonstrates the unbiasedness of the former. The above test was repeated for  $N = 100$  and the MSFE results are plotted in Figure 3. We see that the proposed estimator was the best among the three algorithms for all frequency values, although it had a larger degradation from the CRLB. Furthermore, the validity of (15) and (16) was confirmed.

Figures 4 plots the MSFEs versus SNR at  $N = 10$  and  $\omega = 0.265\pi$ . It is observed that the measured MSFEs of the proposed scheme, which again conformed to (15) as well as (16) except when  $\text{SNR} \leq 4$  dB, outperformed the remaining two estimators for higher SNRs. In addition, the unbiased Pisarenko's estimator approached the CRLB for  $\text{SNR} \geq 10$  dB.

#### 4. CONCLUSION

Inspired by the modified covariance (MC) method, we devise alternative expressions for the sample covariance to produce an unbiased variant of Pisarenko's frequency estimator. The performance of the proposed scheme is theoretically analyzed and evaluated via computer simulations which demonstrate its superiority over the MC and original Pisarenko's methods.

#### 5. APPENDIX

The derivations of (15) and (16) are given as follows. From (13), it is easy to show that

$$E\{f^2(\rho)\}|_{\rho=\cos(\omega)} = \cos^2(2\omega)E\{r_1^2\} - 2\cos(\omega)\cos(2\omega)E\{r_1r_2\} + \cos^2(\omega)E\{r_2^2\} \quad (\text{A.1})$$

The required terms, namely,  $E\{r_1^2\}$ ,  $E\{r_1r_2\}$  and  $E\{r_2^2\}$  are computed as

$$E\{r_1^2\} = \alpha^4 \cos^2(\omega) \{(N-4) + g(\omega, \phi, N-2, 0, 3)\}^2 + \alpha^2 \sigma^2 \{4N - 18 + (4N - 20)\cos(2\omega)\} + 2(2N - 8)\sigma^4 + \alpha^2 \sigma^2 \{(1 + 2\cos^2(\omega))g(\omega, \phi, N-2, 0, 3) + g(\omega, \phi, N-2, -2, 5) \times 4\cos(\omega)g(\omega, \phi, N-2, -1, 4)\} \quad (\text{A.2})$$

$$E\{r_1r_2\} = \alpha^4 \cos(\omega)\cos(2\omega) \{(N-4) + g(\omega, \phi, N-2, 0, 3)\}^2 + \alpha^2 \sigma^2 \{(4N - 20)\cos(\omega) + (4N - 22)\cos(3\omega)\} + \alpha^2 \sigma^2 \{(2\cos(2\omega) + 1)g(\omega, \phi, N-2, -1, 4) + 2\cos(\omega)\cos(2\omega)g(\omega, \phi, N-2, 0, 3) + 2\cos(\omega)g(\omega, \phi, N-2, -2, 5) + g(\omega, \phi, N-2, -3, 6)\} \quad (\text{A.3})$$

$$E\{r_2^2\} = \alpha^4 \cos^2(2\omega) \{(N-4) + g(\omega, \phi, N-2, 0, 3)\}^2 + \alpha^2 \sigma^2 \{4N - 20 + (4N - 24)\cos(4\omega)\} + 2(2N - 10)\sigma^4 + \alpha^2 \sigma^2 \{4\cos(2\omega)g(\omega, \phi, N-2, -2, 5) + (2\cos^2(2\omega) + 1)g(\omega, \phi, N-2, 0, 3) + g(\omega, \phi, N-2, -4, 7)\} \quad (\text{A.4})$$

where

$$g(\omega, \phi, N, k, b) = \sum_{n=b}^N \cos((2n+k)\omega + 2\phi) = \frac{\sin((2N+k+1)\omega + 2\phi) - \sin((2b-1+k)\omega + 2\phi)}{2\sin(\omega)}$$

$$\mathcal{F}(\omega, \phi, N) = \cos^2(2\omega) \{(1 + 2\cos^2(\omega))g(\omega, \phi, N-2, 0, 3) + g(\omega, \phi, N-2, -2, 5) + 4\cos(\omega)g(\omega, \phi, N-2, -1, 4)\} \\ \mathcal{G}(\omega, \phi, N) = -2\cos(\omega)\cos(2\omega) \{(2\cos(2\omega) + 1)g(\omega, \phi, N-2, -1, 4) + 2\cos(\omega)\cos(2\omega)g(\omega, \phi, N-2, 0, 3) + 2\cos(\omega)g(\omega, \phi, N-2, -2, 5) + g(\omega, \phi, N-2, -3, 6)\}$$

and

$$\mathcal{H}(\omega, \phi, N) = \cos^2(\omega) \{4\cos(2\omega)g(\omega, \phi, N-2, -2, 5) + (2\cos^2(2\omega) + 1)g(\omega, \phi, N-2, 0, 3) + g(\omega, \phi, N-2, -4, 7)\}$$

In a similar manner, we get

$$E\{f'(\rho)\}|_{\rho=\cos(\omega)} = \alpha^2(N-4 + g(\omega, \phi, N-2, 0, 3))(\cos(2\omega) + 2) \quad (\text{A.5})$$

Substituting (A.1)-(A.5) into (13) with  $\text{SNR} = \alpha^2/(2\sigma^2)$  and after simplifications, we obtain (15). For sufficiently large SNR and  $N$ , the terms  $g(\omega, \phi, N, k, b)$ ,  $\mathcal{F}(\omega, \phi, N)$ ,  $\mathcal{G}(\omega, \phi, N)$  and  $\mathcal{H}(\omega, \phi, N)$  can be ignored. In so doing, the asymptotic expression of (16) is obtained.

#### Acknowledgement

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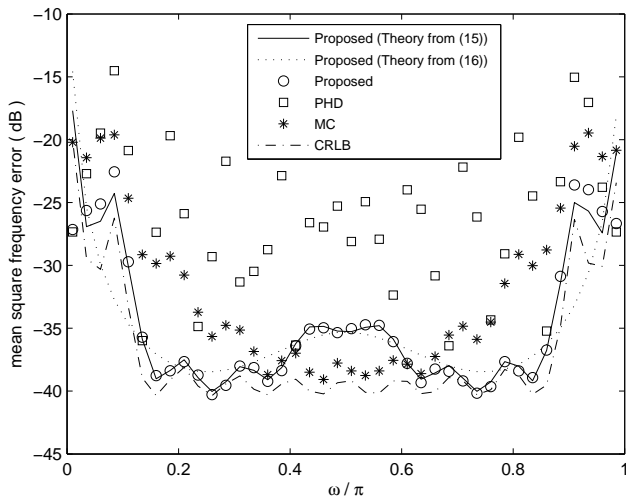


Figure 1: Mean square error versus  $\omega$  at SNR = 20dB and  $N = 10$

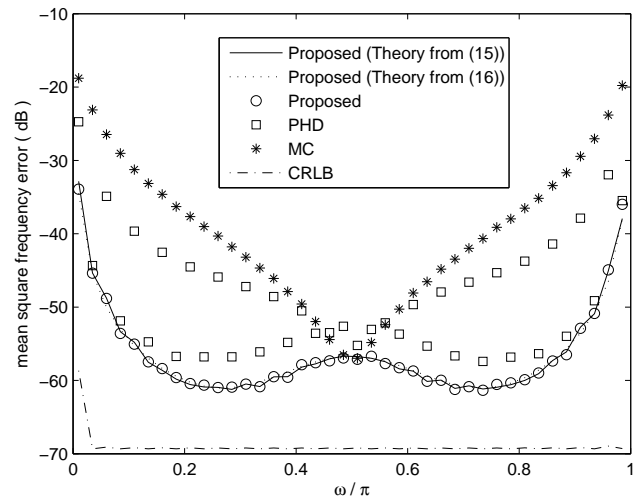


Figure 3: Mean square error versus  $\omega$  at SNR = 20dB and  $N = 100$

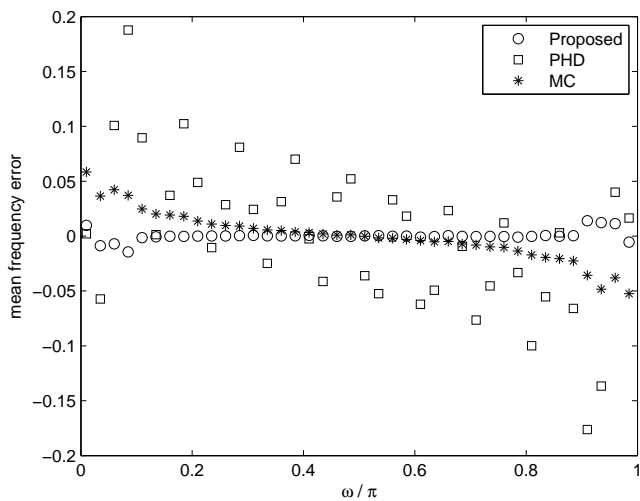


Figure 2: Mean frequency error versus  $\omega$  at SNR = 20dB and  $N = 10$

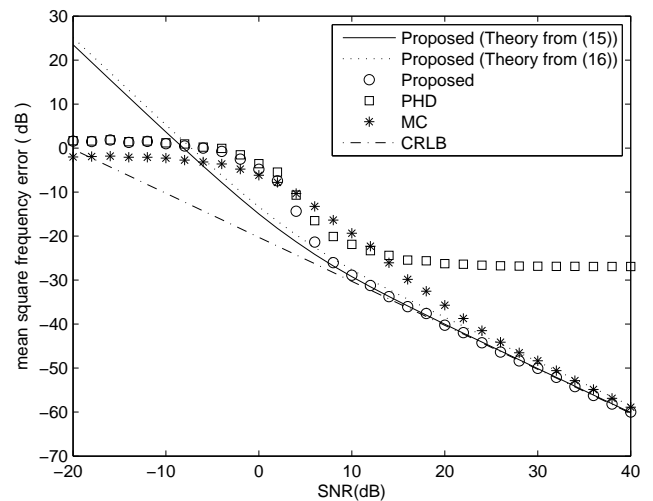


Figure 4: Mean square error versus SNR at  $\omega = 0.265\pi$  and  $N = 10$