REGULARIZATION OF MULTIVALUED IMAGES BY MEANS OF A WAVELET-BASED PARTIAL DIFFERENTIAL EQUATION

Aldo Maalouf, Philippe Carré, Bertrand Augereau, Christine Fernandez-Maloigne

University of Poitiers,
Signal-Image-Communication Laboratory
SP2MI-2 Bd Marie et Pierre Curie, PO Box 30179
86962 Futuroscope Chasseneuil, France
{maalouf, carre, augereau, fernandez}@sic.sp2mi.univ-poitiers.fr

ABSTRACT

In this work, a wavelet-based anisotropic diffusion partial differential equation (PDE) is developed. The new model makes use of a multiscale structure tensor as an extension of the single-scale structure tensor proposed by Di Zenzo [1]. The multiscale structure tensor allows for accumulating multiscale gradient information of local regions. Thus, averaging properties are maintained while preserving edge structure. This structure tensor is used in an anisotropic diffusion process of multispectral images, namely, in the Perona-Malik model [2]. Therefore, a more efficient and accurate formulation for edge-preserving diffusion is obtained.

1. INTRODUCTION

Since the formulation of anisotropic diffusion introduced by Perona and Malik [2], the use of partial differential equation (PDE) in image processing has become a raising research area. Some of these researches have been oriented toward developing stable equations [3] [4] [5], others toward extending and modifying anisotropic diffusion for fast implementations and modifying the diffusion equations for specific applications [6].

All of these approaches model the image in a continuous spatial domain so that it takes the advantages of effective treatments from PDE's theory and obtains high accuracy and stability of the processing with the help of numerical analysis. One of the most influential work in this aspect is the anisotropic diffusion introduced by Perona and Malik in 1990 [2]. Although based on a directional diffusion that preserves edges, the Perona and Malik model meets several serious practical and theoretical difficulties such as the sensitivity to noise and the existence of a local backward diffusion as discussed later. The motivation of this work is to introduce a new regularization in which the gradient of the image is adjusted by the wavelet coefficients. Therefore, providing a more efficient and accurate formulation for edge-preserving diffusion.

The paper is organized as follows. In section 2 a review of the multiscale edge representation by the wavelet transform is presented. In section 3, a description of the Perona-Malik formulation of the anisotropic diffusion is given. Section 4 is a summary of the single-scale structure tensor of DiZenzo and the extension of the perona-Malik diffusion equation to the multivalued images. In section 5 the new wavelet-based PDE is described. In section 6 some experimental results are presented. Section 7 presents some concluding remarks.

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2. MULTISCALE EDGE REPRESENTATION

We employed the multiscale edge representation described in [7] and [8]. In this approach, x and $y-$directional wavelets are given by the partial derivatives of a separable, nonorthogonal scaling function $\theta(x,y)$ as follows: $\left(\psi^1(x,y), \psi^2(x,y)\right) = \left(\partial\theta(x,y)/\partial x, \partial\theta(x,y)/\partial y\right)$. The associated two-dimensional wavelet coefficients of an image $I \in L^2\left(\mathbb{R}^2\right)$ at scale j are defined by:

$$\begin{pmatrix} W_j^1(x,y)I \\ W_j^2(x,y)I \end{pmatrix} = \begin{pmatrix} I * \psi_j^1(x,y) \\ I * \psi_j^2(x,y) \end{pmatrix} = \nabla\left(I * \theta\right) \quad (1)$$

Where $\psi_j^l\,(l=1,2)$ and θ_j represent the wavelet and the scaling function at scale j, respectively, defined by: $\psi_j^l(x,y) = \psi^l\left(x/2^j, y/2^j\right)\Big/\sqrt{2^j}$ and $\theta_j(x,y) = \theta\left(x/2^j, y/2^j\right)\Big/\sqrt{2^j}$. The symbol $*$ indicates the convolution operation. The direction of the gradient vector at a point (x_0, y_0) indicates the direcion along which the image I has the steepest slope. Therefore, a point x is regarded as an edge point at scale j if the magnitude of the wavelet coefficient attains a local maximum along the gradient direction.

This stipulates that the wavelet transform of an image consists of the components of the gradient of the image, smoothed by the dilated smoothing function θ_j.

3. ANISOTROPIC DIFFUSION: PERONA-MALIK FORMULATION

Diffusion algorithms remove noise from an image by modifying the image via a PDE. For example, consider applying the heat equation given by $\partial I(x,y,t)/\partial t = div(\nabla I)$, using the original noisy image $I(x,y,0)$ as the initial condition, where ∇I is the image gradient. Modifying the image using this isotropic diffusion is equivalent to filtering the image with a gaussian filter. Perona and Malik [2] replaced the classical isotropic diffusion equation with

$$\frac{\partial I(x,y,t)}{\partial t} = div\left[g\left(\|\nabla I\|\right)\nabla I\right] \quad (2)$$

Where $\|\nabla I\|$ is the gradient magnitude, and $g\left(\|\nabla I\|\right)$ is an edge stopping function satisfying $g(x) \to 0$ when $x \to \infty$ so that the diffusion is "stopped" across edges.

However, as mentioned in the introduction, the Perona-Malik model meets several serious practical and theoretical difficulties. The first difficulty is that it is very sensitive to noise. Assume an image carries strong noise. The Perona-Malik model will conserve the noise in the processing. Another difficulty arises from the existence of the local backward diffusion in the area where $(c(\|\nabla I\|)\nabla I) < 0$. There is no existent theory supports the uniqueness of the solutions of equation (2). Examples show that (2) is unstable in the sense that very close images could produce divergent solutions [5].

4. STRUCTURE TENSOR OF DI ZENZO

Extending differential-based operations to color images is hindered by the multi-channel nature of color images. The derivatives in different channels can point in opposite directions, hence cancelation might occur by simple addition. The solution to this problem is given by the structure tensor for which opposing vectors reinforce each other.
In [1] Di Zenzo pointed out that the correct method to combine the first order derivative structure is by using a local tensor. Analysis of the shape of the tensor leads to an orientation and a gradient norm estimate. For a multichannel image $I = \left(I^1, I^2, \ldots\ldots, I^n\right)^T$ the structure tensor is given by

$$M = \begin{pmatrix} I_x^T I_x & I_x^T I_y \\ I_y^T I_x & I_y^T I_y \end{pmatrix} \tag{3}$$

The multichanel structure tensor describes the 2D first order differential structure at a certain point in the image.
The use of the Di Zenzo's structure tensor had permitted the extension of the Perona-Malik anisotropic diffusion for the case of multivalued images. The extended model for an m-valued image can be written as:

$$\begin{cases} \frac{\partial I_i}{\partial t} = div\left(g\left(\|M\|\right)\nabla I_i\right) \\ I_i(x,y,0) = I_{i_0}(x,y) \qquad for \qquad i = 1,2\ldots\ldots,m \\ \left.\frac{\partial I_i}{\partial n}\right|_{\partial\Omega} = 0 \end{cases} \tag{4}$$

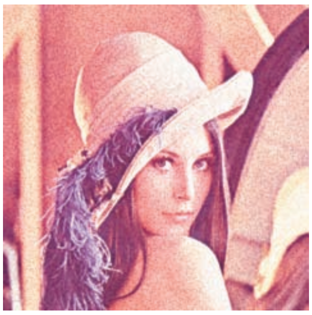
Where Ω is the image's domain.

5. WAVELET BASED ANISOTROPIC DIFFUSION

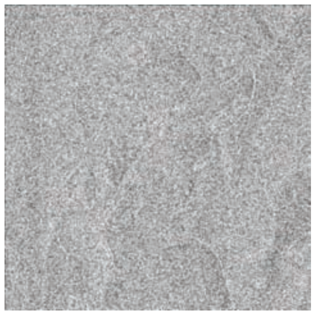
The Perona-Malik model tries to regularize ∇I to reduce the influence of noise. The effectiveness of a regularization depends on the type of noise on the image. For instance, if the noise does not obey Gaussian distribution, then the model does not provide a good regularization. The motivation of this work is to make the regularization of ∇I adjusted by the coefficients of the wavelet transform defined in (1). Based on the theory of the singularity detection by the wavelet transform proposed by Mallat et al. in [7] a multiscale multistructural diffusion tensor can be constructed. For an m-valued image, this structure tensor is defined at a scale j in terms of wavelet coefficients by:

$$G_w^j = \begin{pmatrix} \sum_{i=1}^{m} \sum_{n\in\Omega}\left(W_{n,j,i}^1\right)^2 & \sum_{i=1}^{m} \sum_{n\in\Omega} W_{n,j,i}^1 W_{n,j,i}^2 \\ \sum_{i=1}^{m} \sum_{n\in\Omega} W_{n,j,i}^1 W_{n,j,i}^2 & \sum_{i=1}^{m} \sum_{n\in\Omega}\left(W_{n,j,i}^2\right)^2 \end{pmatrix} \tag{5}$$

Where $W_{n,j,i}^k$ is the undecimated wavelet coefficient computed at scale j and position n for the image channel i.
The norm of G_w^j is defined in terms of its eigenvalues, $\left|G_w^j\right| = \sqrt{\lambda_+ + \lambda_-}$, and it describes the total local derivative energy.
Figure 1 shows the norms of the multistructure tensor of Di Zenzo (Fig. 1(b)) and the norm of the multiscale multistructure tensor defined in (5) (Fig. 1(c)) of the noisy 'Lenna' image.
It is clear that the multiscale structure tensor provides a better characterization of the image edges. Recall that the noise distribution is singular everywhere, which can be characterized by negative Lipschitz orders [7]. Let $n(x,y)$ be a stationary, white noise random field of variance σ^2. Let $M_{\psi,s}n(x,y)$ be the modulus of the wavelet transform of $n(x,y)$ at a scale s ($s = 2^j$), and let $E(X)$ be the expected value of a random variable X. The author of [7] proved that:



(a) Noisy Lenna Image



(b) Norm of the Di Zenzo tensor

(c) Norm of the multiscale structure tensor

Figure 1: Norms of the Di Zenzo structure tensor and the multiscale structure tensor defined in (5)

$$E\left(\left(M_{\psi,s}n(x,y)\right)^2\right) = \frac{\sigma^2\left(\left\|\psi^1\right\|^2 + \left\|\psi^2\right\|^2\right)}{s} \tag{6}$$

Where ψ^1 and ψ^2 are the wavelets defined in section 2. Thus, we can discriminate the image singularity (which occurs at edge) from the noise singularity by their wavelet transform modulus across scales: as the scale s increases, the wavelet transform modulus of edge points increase while the modulus created by noise decrease. It is that behavior of the wavelet transform coefficients that pioneered the edge detection capability of the multiscale structure tensor.
The wavelet-based regularization is therefore defined by:

$$\begin{cases} \frac{\partial I_i}{\partial t} = div\left(g\left(\left\|G_w^j\right\|\right)\nabla I_i\right) \\ I_i(x,y,0) = I_{i_0}(x,y) \qquad for \qquad i = 1,2\ldots\ldots,m \\ \left.\frac{\partial I_i}{\partial n}\right|_{\partial\Omega} = 0 \end{cases} \tag{7}$$

That is, at each PDE scale, the multivalued structure tensor is computed using equations (1) and (5). The multivalued

structure tensor affords a good edge characterization despite of the presence of noise. This edge characterization will, in turn, provide a good orientation for the image gradient during the regularization process. Therefore, a better edge preserving anisotropic diffusion is obtained.
It is to be noted that the multiscale structure tensor should be computed at an optimized scale where the noise singularity is well distinguished from edges singularity as explained above.
The edge stopping function g (also called diffusivity function) always ensures that region boundaries are less blurred than flat regions. Common diffusivity functions are proposed by Perona [2], German and Reynolds [13], Aubert et al. [14] and Saint-Marc et al [15]. For example, the one proposed by Perona and Malik is defined by [2]:

$$\exp\left(-\left(\frac{\nabla I}{k}\right)^2\right)$$

The k parameters in these functions, also called edge threshold parameter, controls the shape of the diffusivity function, balancing the degrees of inter-region smoothing and edge enhancement in the diffusion process. Perona and Malik proposed to compute the histogram and then let k equals to 90% of the integral of the histogram. In our scheme, the k parameter is computed according to the noise level.
It can be shown that:

$$\sigma^2_{\sqrt{\lambda_j^+ + \lambda_j^-}} \approx N\sigma_j^2 \tag{8}$$

with N the number of pixels in the image and σ_j^2 the noise variance at scale j. σ_j^2 is given by:

$$\sigma_j^2 = \left\|\psi_j\right\|^2 \sigma^2 \tag{9}$$

where σ^2 is the noise variance of the image, and $\left\|\psi_j\right\|^2$ is the norm of the wavelet function. The k parameter is made proportional to σ_j:

$$k = c\sqrt{N}\sigma_j \tag{10}$$

Where c is a constant.

6. EXPERIMENTAL RESULTS

This section is devoted to comparing the wavelet-based anisotropic diffusion scheme that is presented in this paper with previous work on image restoration.
To achieve this, the noisy image shown in figure 2(b) is processed by equation (7) as well as by the Perona-Malik scheme. The noisy image is obtained by adding a white gaussian noise to the image of figure 2(a) whose variance is equal to 0.2.
In figures 3(a) and 3(b), the results obtained by filtering the noisy images (fig. 2(b)) by the Perona-Malik approach (equation 2) and the wavelet-based regularization (equation 7) are shown respectively.
If both, figure 3(a) and figure 3(b) are compared, one can observe a better denoising performance, less blurring and better edge structures preservation.
We have also processed the noisy image of figure 2(b) with the edge enhancement diffusion [9], the coherence enhancement diffusion [10], the Tikhonov diffusion [12], the color total variation schemes [11] and the undecimated wavelet coefficients hard thresholding and soft thresholding. The results are shown in figure 4. The hard and soft thresholding schemes work in three steps: (1) compute an M-level undecimated wavelet transform. (2) Modify the detail coefficients by hard and soft thresholding and (3) compute the inverse wavelet transform. Both methods set the coefficients below the threshold T to zero. Soft thresholding additionally reduces the amplitude of the other coefficients by T, a procedure called shrinkage. The problem with wavelet coefficient thresholding is that setting coefficients to zero leads to smooth image (Fig. 4(f)) and destroy details which cause blur and artifacts (Fig. 4(e)).
Compared to all the other listed schemes, the wavelet-based anisotropic diffusion showed better details preserving, less blurring and better image restoration.
The quality of the filtered images is also evaluated using $CIEDE2000$ color difference equations [16] [17]. The $CIEDE2000$ evolved from traditional colorimetry and color difference calculations is tested using several psychophysical datasets. The color differences between the original image (Fig. 2(a)) and each of the filtered images obtained using different denoising schemes are shown in figure 5. The PDE wavelet based approach showed the lowest color difference and therefore it approaches the original image more than the other denoising techniques. Therefore, it respects the colorimetric characteristics of the original image.

(a) (b)

Figure 2: Original image and the noisy image (SNR=27.21dB) obtained by adding a white gaussian noise

(a) (b)

Figure 3: Filtered image obtained by: (a) The Perona-Malik approach (SNR=31.59dB), (b) the wavelet-based approach(SNR=33.61dB)

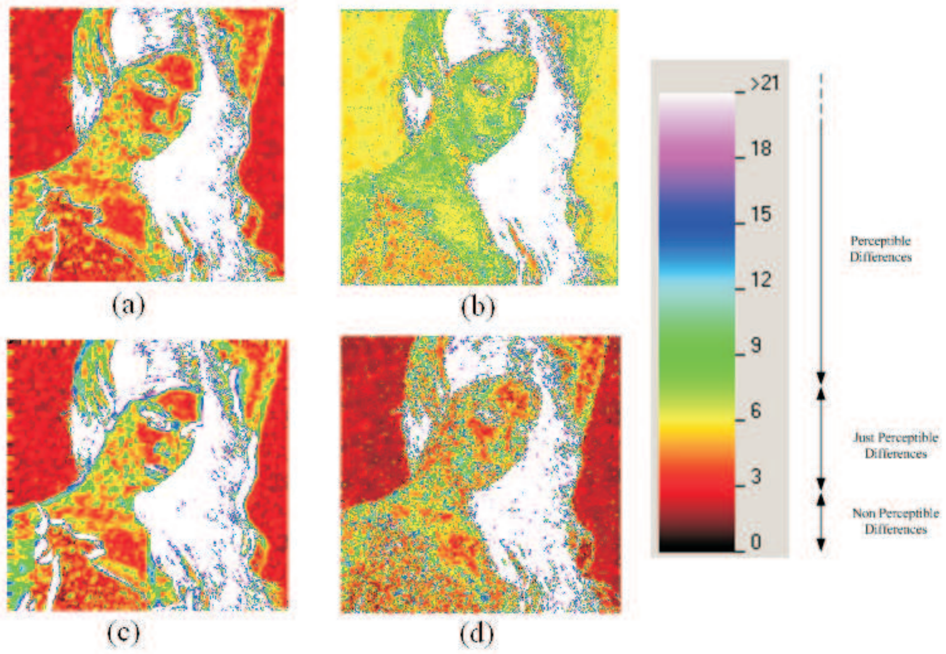


Figure 5: $CIEDE2000$ color difference between the original image and the filtered image obtained by: (a) wavelet based approach, (b) Wavelet hard thresholding, (c) Perona and Malik approach (d)Color Total Variation

7. CONCLUSION

In this paper, a wavelet-based anisotropic diffusion is described. It uses a multiscale structure tensor that is computed from the wavelet coefficients of the image being processed. Based on the theory of the singularity detection using the wavelet transform [7], the multiscale structure tensor provides a better edge characterization than the structure tensor of Di Zenzo. The use of the multiscale multistructure tensor in the anisotropic diffusion affords a better orientation of the gradient toward the maximal direction of intensity variation in the diffusion process in spite of the presence of noise. As a result, a better edge preserving and less blurring are obtained in the processed image. The proposed scheme is compared with other anisotropic diffusion schemes proposed in early work and with the wavelet thresholding techniques.

A future perspective could be to introduce an optimized scheme in which the scale of the wavelet transform used to compute the multiscale structure tensor is automatically computed in accordance with the noise level. Another perspective could be to study the efficiency of the proposed approach in denoising images contaminated with impulsive noise.

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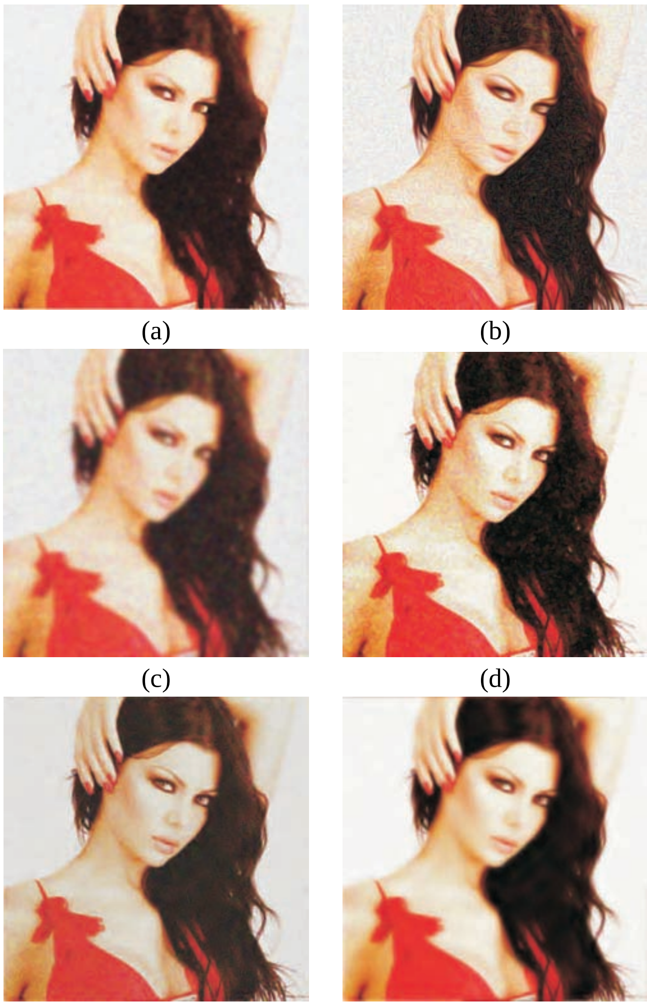

(a) (b) (c) (d) (e) (f)

Figure 4: Filtered image obtained by: (a) edge enhancement diffusion (SNR=29.058dB), (b) coherence enhancement diffusion (SNR=29.11dB), (c) Tikhonov diffusion(SNR=32.11dB) and (d) Color Total Variation (SNR=28.76dB), (e) Wavelet hard thresholding(SNR=38.95dB) and (f) wavelet soft thresholding(SNR=31.28dB)

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