INFORMATION LOSSLESS SPACE-TIME CODING FOR RAYLEIGH FADED MULTIPLE ACCESS SYSTEMS

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ABSTRACT

In this paper we consider multiple access systems affected by flat Rayleigh fading, where users and access point are equipped with multiple antennas. The exploitation of some of the MIMO potentials relies necessarily on space-time coding. However, such a processing may induce severe losses in terms of achievable information rates. In this paper we prove necessary and sufficient conditions ensuring that the space-time coding is information lossless, in the sense that it achieves the ergodic capacity region of the multiple access system. In particular, the information lossless property is guaranteed if each user makes use of a full-rate Trace-Orthogonal Design, that is a linear space-time code whose encoding matrices are orthogonal with respect to the trace inner product. Noteworthy, different users can also adopt the same set of encoding matrices.

1. INTRODUCTION

MIMO systems have attracted a lot of research in the recent years, since they yield an increase of spectral efficiency and diversity gain without sacrificing transmission power and/or bandwidth [1]. However, the exploitation of MIMO potentials, in particular diversity gain, relies on the use of space-time coding [2]. Unfortunately, such a processing may induce severe losses in terms of capacity [3]. Nevertheless, spectral efficiency is one of the major motivations for using MIMO systems. Hence, it is fundamental to discern which are the space-time coding properties essential to achieve the MIMO potentials without incurring capacity losses. In the single user scenario there are several studies in that direction, see, e.g., [4], [5] and references therein. However, it seems that no equivalent systematic study has been carried out in the multiuser setting. The characterization of the space-time coding strategies allowing for lossless information transfer, in the case of multiple access systems, was considered in [6], where the problem of invariance for the sum-rate was addressed, and [7], where the invariance with respect to the instantaneous capacity region was considered. In particular, [7] considers non ergodic channels and gives the necessary and sufficient condition on the space-time coding scheme ensuring that certain regions of achievable rates are not affected by the coding. The result is derived on a per-channel realization basis and thus it does not rely on the statistics of the channels. In this work, we take into account the channel statistics explicitly and consider multiple access systems affected by flat Raylegh fading with i.i.d.[1] channel coefficients. Assuming ergodic channels with long-term delay constraint, the set of achievable rates is described by the ergodic capacity region [10] of the multiple access system. We aim at characterizing the class of information lossless space-time coding schemes for such systems. In particular, we prove that the necessary and sufficient condition to achieve the ergodic capacity region of the multiple access system is that every user employs a full-rate Trace-Orthogonal Design, that is a linear space-time code whose encoding matrices are orthogonal with respect to the trace inner product. The work is organized as follows. The system model is outlined in Section 2 where in addition the ergodic capacity regions of interest are defined. In Section 3 we derive the necessary and sufficient conditions identifying the class of information lossless linear space-time codes. Section 4 follows with conclusive remarks. Throughout the paper, we use the following notation: $\mathbb{C}^{n\times p}$ denotes the space of $n \times p$ matrices with complex entries; matrices are denoted by bold uppercase letters and vectors by bold lowercase ones; $\boldsymbol{I}_n$ denotes the $n \times n$ identity matrix; $\boldsymbol{I}$ and $\boldsymbol{0}$ denote respectively an identity matrix and a null matrix with suitable dimensions.

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[1]Independent identically distributed.

2. SYSTEM MODEL

Consider a multiple access system composed of N users, each with n_T transmit antennas, and an access point (AP), with n_R receive antennas. Let us assume that the k-th user encodes its own n_s complex symbols $s_k(j), j = 1,\ldots,n_s$, through the following space-time linear encoder

$$\boldsymbol{X}_k = \sum_{j=1}^{n_s} \boldsymbol{A}_k(j) s_k(j) \tag{1}$$

where $\{\boldsymbol{A}_k(j), j = 1,\ldots,n_s\}$ is the set of $n_T \times Q$ complex matrices assigned to the k-th user.

A space-time encoder is a *Trace-Orthogonal Design* (TOD), if the corresponding matrices $\boldsymbol{A}_k(1),\cdots,\boldsymbol{A}_k(n_s)$ are orthonormal with respect to the trace inner product, that is they satisfy

$$\mathrm{tr}\left(\boldsymbol{A}_k^H(j)\boldsymbol{A}_k(m)\right) = \delta_{jm}, \tag{2}$$

where δ_{jm} is the Kronecker delta.

Applying the vec($\cdot$) operator to (1), we get

$$\boldsymbol{x}_k = \mathrm{vec}(\boldsymbol{X}_k) = \sum_{j=1}^{n_s} \mathrm{vec}\left(\boldsymbol{A}_k(j)\right) s_k(j) = \boldsymbol{F}_k \boldsymbol{s}_k, \tag{3}$$

where $\boldsymbol{F}_k$ is the $Qn_T \times n_s$ matrix whose j-th column is $\mathrm{vec}(\boldsymbol{A}_k(j))$ and $\boldsymbol{s}_k = [s_k(1)\cdots s_k(n_s)]^T$ is the vector of transmitted symbols for the k-th user.

To guarantee symbol recovery[2] for each user, the matrices $\boldsymbol{F}_k$ must be full column rank, i.e., $\mathrm{rank}(\boldsymbol{F}_k) = n_s$. This means that the following inequality must be satisfied

$$n_s \leq Q \cdot n_T. \tag{4}$$

We will refer to codes for which $\boldsymbol{F}_k$ has full column rank, as *nonsingular* codes. Moreover, when (4) holds with equality, we will refer to the corresponding code as *full-rate*. In this case, in fact, the code rate, defined as $R = n_s/Q$, is equal to n_T.

[2]Symbol recovery is guaranteed if mapping (3) is injective.

As will be clear later, matrices $\boldsymbol{F}_k$ $(k = 1, \ldots, N)$ play a fundamental role in characterizing the properties of the codes. In particular, for full-rate Trace-Orthogonal Designs the following property holds

Property 1. *The matrix $\boldsymbol{F}_k$ of a full-rate Trace-Orthogonal Design is unitary.*

Proof. By definition

$$\boldsymbol{F}_k = [\mathrm{vec}(\boldsymbol{A}_k(1)) \ldots \mathrm{vec}(\boldsymbol{A}_k(n_s))] \,, \quad (5)$$

where $n_s = Q \cdot n_T$ since the code is full-rate. As a consequence, $\boldsymbol{F}_k$ is a square matrix and the generic element of the product $\boldsymbol{F}_k^H \boldsymbol{F}_k$, denoted by $\left\{\boldsymbol{F}_k^H \boldsymbol{F}_k\right\}_{i,j}$, can be written as

$$\begin{aligned}\left\{\boldsymbol{F}_k^H \boldsymbol{F}_k\right\}_{i,j} &= \mathrm{vec}(\boldsymbol{A}_k(i))^H \,\mathrm{vec}(\boldsymbol{A}_k(j)) \\ &= \mathrm{tr}\left(\boldsymbol{A}_k^H(i)\boldsymbol{A}_k(j)\right) = \delta_{ij}\,, \end{aligned} \quad (6)$$

where property $\mathrm{vec}(\boldsymbol{A})^H \mathrm{vec}(\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}^H\boldsymbol{B})$ has been exploited in the second equality, and (6) comes from (2). So, unitarity of $\boldsymbol{F}_k$ follows immediately. □

Conversely, any unitary matrix can be thought as the matrix $\boldsymbol{F}_k$ of a full-rate TOD. Note that, within the class of non-singular codes, for a full-rate TOD $\boldsymbol{F}_k^H\boldsymbol{F}_k = \boldsymbol{F}_k\boldsymbol{F}_k^H = \boldsymbol{I}_{n_s}$, as Property 1 states, whereas for a non full-rate $(n_s < Qn_T)$ TOD we have $\boldsymbol{F}_k^H\boldsymbol{F}_k = \boldsymbol{I}_{n_s}$, but $\boldsymbol{F}_k\boldsymbol{F}_k^H \neq \boldsymbol{I}_{Qn_T}$.

Now, let us consider the multiple access channel. Denoting by $\boldsymbol{H}_k \in \mathbb{C}^{n_R \times n_T}$ the channel matrix characterizing the link between the k-th user and the AP, and by $\tilde{\boldsymbol{x}}_k$ the corresponding vector of transmitted symbols, the received vector is

$$\tilde{\boldsymbol{y}} = \sum_{k=1}^{N} \boldsymbol{H}_k \tilde{\boldsymbol{x}}_k + \tilde{\boldsymbol{v}}, \quad (7)$$

where $\tilde{\boldsymbol{v}}$ is the noise vector, assumed to be zero mean, circularly symmetric complex Gaussian, with covariance matrix $\sigma_v^2\boldsymbol{I}$. We will refer to (7) as the *uncoded system.*

Consider now the system with space-time encoding, where the channels $\boldsymbol{H}_k$ are assumed to be constant over Q successive channel uses (block fading model). If each user transmits the matrix $\boldsymbol{X}_k$, built as in (1), the received matrix is

$$\boldsymbol{Y} = \sum_{k=1}^{N} \boldsymbol{H}_k \boldsymbol{X}_k + \boldsymbol{V}\,, \quad (8)$$

where $\boldsymbol{V}$ is the $n_R \times Q$ received noise matrix. Applying the $\mathrm{vec}(\cdot)$ operator[3] to (8) and using (1) and (3), we get

$$\boldsymbol{y} = \sum_{k=1}^{N} (\boldsymbol{I}_Q \otimes \boldsymbol{H}_k)\boldsymbol{F}_k \boldsymbol{s}_k + \boldsymbol{v}, \quad (9)$$

where $\boldsymbol{v} = \mathrm{vec}(\boldsymbol{V})$. We will refer to (9) or (8) as the *coded system.*

Now, assume that the channels are affected by uncorrelated Rayleigh fading, i.e., $\boldsymbol{H}_k$ for $k = 1, \ldots, N$ is composed of i.i.d. zero mean circularly symmetric complex Gaussian random variables; P_k is the constraint on the transmit power for the k-th user, i.e., with reference to (7),

$$\mathrm{tr}\left(\mathbb{E}\left\{\tilde{\boldsymbol{x}}_k \tilde{\boldsymbol{x}}_k^H\right\}\right) \le P_k\,; \quad (10)$$

[3] In deriving (9) we have used $\mathrm{vec}(\boldsymbol{AXB}) = (\boldsymbol{B}^T \otimes \boldsymbol{A})\,\mathrm{vec}(\boldsymbol{X})$.

no channel state information (CSI) is available at the transmitters and the receiver at the AP has perfect CSI. Under these hypotheses, when channels are ergodic with long-term delay constraint, the multiple access system can be characterized in terms of the ergodic capacity region. For the uncoded system in (7), the region is a polytope given by [9], [10]

$$\boldsymbol{\mathcal{C}}^{\,\mathrm{unc}} = \Big\{(R_1, \ldots, R_N) \in \mathbb{R}_+^N \Big| \sum_{k\in\mathcal{S}} R_k \le C^{\,\mathrm{unc}}(\mathcal{S})\,, \quad (11)$$
$$\forall\mathcal{S} \subseteq \{1, \ldots, N\} \Big\},$$

where

$$C^{\,\mathrm{unc}}(\mathcal{S}) = \mathbb{E}\left\{\log\left|\boldsymbol{I}_{n_R} + \frac{1}{n_T\sigma_v^2}\sum_{k\in\mathcal{S}} P_k \boldsymbol{H}_k\boldsymbol{H}_k^H\right|\right\}\,, \quad (12)$$

and $\mathcal{S}$ is any nonempty subset of $\{1, \ldots, N\}$.

To determine the corresponding region for the coded system in (9), we can assume, without loss of generality, that each user transmits independent symbols, since any correlation can be incorporated into the matrix $\boldsymbol{F}_k$. In particular, assuming that the symbol variance for the k-th user is P_k/n_T, the power constraint for the coded system becomes

$$\mathrm{tr}\left(\boldsymbol{F}_k\boldsymbol{F}_k^H\right) \le n_T Q\,, \quad (13)$$

since the symbols are transmitted in Q channel uses. Thus, the ergodic capacity region for the coded system in (9) is [10]

$$\boldsymbol{\mathcal{C}}^{\,\mathrm{cod}} = \bigcup_{\substack{\mathrm{tr}\left(\boldsymbol{F}_k\boldsymbol{F}_k^H\right)\le n_T Q \\ \forall k\in\{1,2,\ldots,N\}}} \Big\{(R_1, \ldots, R_N) \in \mathbb{R}_+^N \Big| \quad (14)$$
$$\sum_{k\in\mathcal{S}} R_k \le C^{\,\mathrm{cod}}(\mathcal{S})\,, \forall\mathcal{S} \subseteq \{1, \ldots, N\}\Big\},$$

where

$$C^{\,\mathrm{cod}}(\mathcal{S}) = \frac{1}{Q}\,\mathbb{E}\left\{\log\left|\boldsymbol{I} + \right.\right. \quad (15)$$
$$\left.\left.\frac{1}{n_T\sigma_v^2}\sum_{k\in\mathcal{S}} P_k(\boldsymbol{I}_Q \otimes \boldsymbol{H}_k)\boldsymbol{F}_k\boldsymbol{F}_k^H(\boldsymbol{I}_Q \otimes \boldsymbol{H}_k^H)\right|\right\}$$

in which the factor $1/Q$ accounts for the Q channel uses.

3. INFORMATION LOSSLESS SPACE-TIME CODING

The objective of this section is to provide both *sufficient* and *necessary and sufficient* conditions on the encoding matrices for each user so that lossless information transfer is guaranteed. In general, the coded system (8) can experience rate reductions depending on the particular choice of the space-time encoder for each user. For example, in the single user scenario, it is well known [3] that orthogonal space-time block coding incurs severe loss in terms of capacity. In a multiuser scenario the rate loss is experienced in terms of modification and/or reduction of the corresponding region of achievable rates. We are interested in space-time coding strategies that are able to achieve the ergodic capacity region of the multiple access system. In this regard, the next theorem gives a sufficient condition.

Theorem 1. *Consider a MIMO multiple access system with channels affected by flat Rayleigh fading with i.i.d. channel coefficients, where perfect CSIR and no CSIT are assumed. The k-th user is constrained in its total power to P_k. Then, a sufficient condition to achieve the ergodic capacity region of the system is that every user employs a full-rate Trace-Orthogonal Design.*

Proof. Under the assumption of flat Rayleigh fading with i.i.d. channel coefficients, the ergodic capacity region for the coded system is given by (14). Since every user employs a full-rate Trace-Orthogonal Design then, according to Property 1, the following identities hold

$$\boldsymbol{F}_k \boldsymbol{F}_k^H = \boldsymbol{I}_{Qn_T}\,, \qquad k = 1, \ldots, N\,, \tag{16}$$

which, substituted in (14), lead to

$$\boldsymbol{\mathcal{C}}^{\,\mathrm{cod}} = \Big\{ (R_1, \ldots, R_N) \in \mathbb{R}_+^N \Big| \sum_{k \in \mathcal{S}} R_k \leq C^{\,\mathrm{cod}}(\mathcal{S})\,, \tag{17}$$
$$\forall \mathcal{S} \subseteq \{1, \ldots, N\} \Big\},$$

with

$$C^{\,\mathrm{cod}}(\mathcal{S}) = \frac{1}{Q}\,\mathbb{E}\left\{ \log \left| \boldsymbol{I}_{Qn_R} + \frac{1}{n_T \sigma_v^2} \sum_{k \in \mathcal{S}} P_k (\boldsymbol{I}_Q \otimes \boldsymbol{H}_k \boldsymbol{H}_k^H) \right| \right\}$$
$$= \mathbb{E}\left\{ \log \left| \boldsymbol{I}_{n_R} + \frac{1}{n_T \sigma_v^2} \sum_{k \in \mathcal{S}} P_k \boldsymbol{H}_k \boldsymbol{H}_k^H \right| \right\}, \tag{18}$$

which is precisely the ergodic capacity region of the original (uncoded) multiple access system, as reported in (11) with (12). □

It is interesting to note that the use of a full-rate Trace-Orthogonal Design is also a necessary condition to achieve the ergodic capacity region, at least when the users have the same power budget. However, in order to prove this result, that will be the object of Theorem 2, we need some preliminary results from linear algebra and random matrix theory [11]:

Lemma 1. *If $\boldsymbol{A}$ is an $n \times n$ matrix with complex entries, then for any choice of n orthonormal vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ belonging to $\mathbb{C}^n$, the following identity holds true*

$$\mathrm{tr}(\boldsymbol{A}) = \sum_{k=1}^{n} \boldsymbol{u}_k^H \boldsymbol{A} \boldsymbol{u}_k\,. \tag{19}$$

Proof. Let us introduce the matrix $\boldsymbol{U} = [\boldsymbol{u}_1\ \boldsymbol{u}_2\ \cdots\ \boldsymbol{u}_n]$, which is unitary since its columns are orthonormal by hypothesis. Then,

$$\mathrm{tr}(\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{U}\boldsymbol{U}^H \boldsymbol{A}) = \mathrm{tr}(\boldsymbol{U}^H \boldsymbol{A} \boldsymbol{U}) = \sum_{k=1}^{n} \boldsymbol{u}_k^H \boldsymbol{A} \boldsymbol{u}_k$$

which proves the lemma. □

Definition 1. *A Hermitian random matrix $\boldsymbol{W}$ is said to be* unitarily invariant *if its distribution is invariant under the following transformation*

$$\boldsymbol{U}\boldsymbol{W}\boldsymbol{U}^H,$$

where $\boldsymbol{U}$ is any unitary matrix.

Lemma 2. *If $\boldsymbol{H}$ is a random matrix with i.i.d. entries which are circularly symmetric complex Gaussian random variables, then $\boldsymbol{H}^H\boldsymbol{H}$ is unitarily invariant.*

In particular, the channel matrix of a MIMO channel affected by flat Raylegh fading with i.i.d. channel coefficients satisfies Lemma 2.

Lemma 3. *If $\boldsymbol{W}$ is Hermitian unitarily invariant, then it admits the following spectral decomposition*[4]

$$\boldsymbol{W} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H\,, \tag{20}$$

where $\boldsymbol{U}$ and $\boldsymbol{\Lambda}$ are statistically independent. Moreover $\boldsymbol{U}$ is a Haar matrix, i.e., it is a random matrix uniformly distributed over the Stiefel manifold of unitary matrices.

We are now ready to prove the following result

Theorem 2. *Consider a MIMO multiple access system with channels affected by flat Rayleigh fading with i.i.d. channel coefficients, where perfect CSIR and no CSIT are assumed. Every user is constrained in its total power to P. Then, a space-time coding strategy based on nonsingular linear codes achieves the ergodic capacity region of the system* if and only if *every user employs a full-rate Trace-Orthogonal Design.*

Proof. The proof of the theorem will be carried out showing that the ergodic capacity region of the space-time coded system, as given in (14), can be achieved if and only if every user employs a full-rate TOD and then observing that such a region coincides with the ergodic capacity region of the original (uncoded) system, given in (11). Under the assumption of flat Rayleigh fading with i.i.d. channel coefficients, when every user is constrained in its total power to P, the ergodic capacity region for the coded system can be evaluated from (14) as

$$\boldsymbol{\mathcal{C}}^{\,\mathrm{cod}} = \bigcup_{\substack{\mathrm{tr}\left(\boldsymbol{F}_k \boldsymbol{F}_k^H\right) \leq n_T Q \\ \forall k \in \{1,2,\ldots,N\}}} \Big\{ (R_1, \ldots, R_N) \in \mathbb{R}_+^N \Big| \tag{21}$$
$$\sum_{k \in \mathcal{S}} R_k \leq C^{\,\mathrm{cod}}(\mathcal{S})\,, \forall \mathcal{S} \subseteq \{1, \ldots, N\} \Big\},$$

where

$$C^{\,\mathrm{cod}}(\mathcal{S}) = \frac{1}{Q}\,\mathbb{E}\Big\{ \log \Big| \boldsymbol{I} + \tag{22}$$
$$\frac{P}{n_T \sigma_v^2} \sum_{k \in \mathcal{S}} (\boldsymbol{I}_Q \otimes \boldsymbol{H}_k) \boldsymbol{F}_k \boldsymbol{F}_k^H (\boldsymbol{I}_Q \otimes \boldsymbol{H}_k^H) \Big| \Big\}.$$

Since the matrix $\boldsymbol{F}_k$ associated to a full-rate Trace-Orthogonal Design is unitary (see Property 1), the first step is to prove that the boundaries $C^{\,\mathrm{cod}}(\mathcal{S})$ in (22), which define the ergodic capacity region (21), are jointly maximized, $\forall \mathcal{S} \subseteq \{1, \ldots, N\}$, if and only if

$$\boldsymbol{F}_1 \boldsymbol{F}_1^H = \cdots = \boldsymbol{F}_N \boldsymbol{F}_N^H = \boldsymbol{I}\,, \tag{23}$$

where every $\boldsymbol{F}_k$ is square and, as a consequence, unitary. The next step is to recognize that the ergodic capacity region in (21), with the optimal choice (23), reduces to the ergodic capacity region of the original (uncoded) system as given in (11) with $P_k = P$.

Following the line of reasoning outlined above, we will consider (22) with the objective of upper bounding it. So, let us take a closer look at the determinant inside (22). Denote by $\mathcal{S} = \{i_1, \cdots, i_K\}$ the generic subset of $\{1, 2, \ldots, N\}$ having K elements ($1 \leq K \leq N$). Then, let us introduce the block matrix

$$\boldsymbol{H} = [\boldsymbol{H}_{i_1}\ \boldsymbol{H}_{i_2} \cdots\ \boldsymbol{H}_{i_K}]\,, \tag{24}$$

[4] $\boldsymbol{U}$ is the unitary matrix of eigenvectors and $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues.

the block diagonal matrix

$$\boldsymbol{F} = \mathrm{diag}\left\{\boldsymbol{F}_{i_1}, \boldsymbol{F}_{i_2}, \cdots, \boldsymbol{F}_{i_K}\right\}, \tag{25}$$

and the following permutation matrix

$$\boldsymbol{\Pi} = [\boldsymbol{I}_Q \otimes \boldsymbol{P}_{i_1} \quad \boldsymbol{I}_Q \otimes \boldsymbol{P}_{i_2} \quad \cdots \quad \boldsymbol{I}_Q \otimes \boldsymbol{P}_{i_K}], \tag{26}$$

with $\boldsymbol{P}_k$ defined as

$$\boldsymbol{P}_k = \boldsymbol{u}_k \otimes \boldsymbol{I}_{n_T}, \tag{27}$$

where $\boldsymbol{u}_k$ is the k-th unit column vector in $\mathbb{C}^K$. Note that $\boldsymbol{H}$, $\boldsymbol{F}$, and $\boldsymbol{\Pi}$ depend on the subset $\mathcal{S}$, but this is not explicitly indicated in order to keep the notation simple.

Using positions (24)-(27), after some matrix algebra, formula (22) can be rewritten as

$$C^{\,\mathrm{cod}}(\mathcal{S}) = \frac{1}{Q}\,\mathbb{E}\left\{\log\left|\boldsymbol{I} + \frac{P}{n_T\sigma_v^2}(\boldsymbol{I}_Q \otimes \boldsymbol{H})\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H(\boldsymbol{I}_Q \otimes \boldsymbol{H}^H)\right|\right\}. \tag{28}$$

Now, the product $\boldsymbol{H}^H\boldsymbol{H}$ is a Hermitian positive semidefinite matrix and can be written as $\boldsymbol{H}^H\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H$, where $\boldsymbol{U}$ is unitary and $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_{Kn_T}\}$. Moreover $\boldsymbol{U}$ and $\boldsymbol{\Lambda}$ satisfy Lemma 3, since $\boldsymbol{H}^H\boldsymbol{H}$ is unitarily invariant according to Lemma 2. So, for the determinant within (28) the following chain of identities holds true

$$\begin{aligned}
&\left|\boldsymbol{I} + \frac{P}{n_T\sigma_v^2}(\boldsymbol{I}_Q \otimes \boldsymbol{H})\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H(\boldsymbol{I}_Q \otimes \boldsymbol{H}^H)\right| \\
&= \left|\boldsymbol{I} + \frac{P}{n_T\sigma_v^2}(\boldsymbol{I}_Q \otimes \boldsymbol{H}^H\boldsymbol{H})\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H\right| \\
&= \left|\boldsymbol{I} + \frac{P}{n_T\sigma_v^2}(\boldsymbol{I}_Q \otimes \boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^H)\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H(\boldsymbol{I}_Q \otimes \boldsymbol{U}\boldsymbol{\Lambda}^{1/2})\right|,
\end{aligned} \tag{29}$$

where $\boldsymbol{\Lambda}^{1/2}$ is the square root of $\boldsymbol{\Lambda}$ [8].

In order to proceed, it is useful to rewrite the product $\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H$ in (29), as a partitioned matrix with $Q \times Q$ blocks, where each block has dimensions $Kn_T \times Kn_T$ and is denoted by $\boldsymbol{\mathcal{F}}_{ij}$,

$$\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H = \begin{bmatrix} \boldsymbol{\mathcal{F}}_{11} & \cdots & \boldsymbol{\mathcal{F}}_{1Q} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\mathcal{F}}_{Q1} & \cdots & \boldsymbol{\mathcal{F}}_{QQ} \end{bmatrix}. \tag{30}$$

With position (30), the second term in the determinant (29) becomes

$$\begin{aligned}
&(\boldsymbol{I}_Q \otimes \boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^H)\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H(\boldsymbol{I}_Q \otimes \boldsymbol{U}\boldsymbol{\Lambda}^{1/2}) \\
&= \begin{bmatrix} \boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^H\boldsymbol{\mathcal{F}}_{11}\boldsymbol{U}\boldsymbol{\Lambda}^{1/2} & \cdots & \boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^H\boldsymbol{\mathcal{F}}_{1Q}\boldsymbol{U}\boldsymbol{\Lambda}^{1/2} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^H\boldsymbol{\mathcal{F}}_{Q1}\boldsymbol{U}\boldsymbol{\Lambda}^{1/2} & \cdots & \boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^H\boldsymbol{\mathcal{F}}_{QQ}\boldsymbol{U}\boldsymbol{\Lambda}^{1/2} \end{bmatrix}.
\end{aligned} \tag{31}$$

Each diagonal block in (31) can be further simplified. In fact, denoting by $\boldsymbol{u}_j$ the j-th column of $\boldsymbol{U} = [\boldsymbol{u}_1 \cdots \boldsymbol{u}_{Kn_T}]$, we have

$$\begin{aligned}
\boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{U}\boldsymbol{\Lambda}^{1/2} &= \boldsymbol{\Lambda}^{1/2}\begin{bmatrix} \boldsymbol{u}_1^H \\ \vdots \\ \boldsymbol{u}_{Kn_T}^H \end{bmatrix}\boldsymbol{\mathcal{F}}_{jj}\left[\boldsymbol{u}_1 \cdots \boldsymbol{u}_{Kn_T}\right]\boldsymbol{\Lambda}^{1/2} \\
&= \boldsymbol{\Lambda}^{1/2}\begin{bmatrix} \boldsymbol{u}_1^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_1 & \cdots & \boldsymbol{u}_1^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_{Kn_T} \\ \vdots & \ddots & \vdots \\ \boldsymbol{u}_{Kn_T}^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_1 & \cdots & \boldsymbol{u}_{Kn_T}^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_{Kn_T} \end{bmatrix}\boldsymbol{\Lambda}^{1/2}.
\end{aligned} \tag{32}$$

Now, applying Hadamard's inequality [8] to (29), taking into account (31) and (32), we arrive at the following inequality

$$\begin{aligned}
&\left|\boldsymbol{I} + \frac{P}{n_T\sigma_v^2}(\boldsymbol{I}_Q \otimes \boldsymbol{H})\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H(\boldsymbol{I}_Q \otimes \boldsymbol{H}^H)\right| \\
&\qquad \leq \prod_{j=1}^{Q}\prod_{k=1}^{Kn_T}\left(1 + \frac{P\lambda_k}{n_T\sigma_v^2}\boldsymbol{u}_k^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_k\right),
\end{aligned} \tag{33}$$

where equality holds, when $\boldsymbol{H} \neq \boldsymbol{0}$, if and only if $\boldsymbol{U}^H\boldsymbol{\mathcal{F}}_{ij}\boldsymbol{U} = \delta_{ij}\boldsymbol{\mathcal{D}}_j$, where $\boldsymbol{\mathcal{D}}_j$ is a *diagonal* matrix.

Inequality (33) is an upper bound to the determinant within (28). Taking into account the expected value, and considering that $\mathbb{E}_{\boldsymbol{H}}\{\cdots\} = \mathbb{E}_{\boldsymbol{\Lambda}}\{\mathbb{E}_{\boldsymbol{U}}\{\cdots\}\}$ due to Lemma 3, we are able to state the following chain of equalities/inequalities for (28), or equivalently (22), that is,

$$C^{\,\mathrm{cod}}(\mathcal{S}) = \frac{1}{Q}\,\mathbb{E}_{\boldsymbol{\Lambda}}\left\{\mathbb{E}_{\boldsymbol{U}}\left\{\log\left|\boldsymbol{I} + \frac{P}{n_T\sigma_v^2}(\boldsymbol{I}_Q \otimes \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H)\boldsymbol{\Pi}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{\Pi}^H\right|\right\}\right\} \tag{34}$$

can be upper bounded exploiting (33) as

$$\leq \frac{1}{Q}\,\mathbb{E}_{\boldsymbol{\Lambda}}\left\{\mathbb{E}_{\boldsymbol{U}}\left\{\sum_{j=1}^{Q}\sum_{k=1}^{Kn_T}\log\left(1 + \frac{P\lambda_k}{n_T\sigma_v^2}\boldsymbol{u}_k^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_k\right)\right\}\right\} \tag{35}$$

where equality holds if and only if $\boldsymbol{U}^H\boldsymbol{\mathcal{F}}_{ij}\boldsymbol{U} = \delta_{ij}\boldsymbol{\mathcal{D}}_j$. Then, an application of Jensen's inequality to the inner expectation leads to

$$\leq \frac{1}{Q}\,\mathbb{E}_{\boldsymbol{\Lambda}}\left\{\sum_{j=1}^{Q}\sum_{k=1}^{Kn_T}\log\left(1 + \frac{P\lambda_k}{n_T\sigma_v^2}\,\mathbb{E}_{\boldsymbol{U}}\left\{\boldsymbol{u}_k^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_k\right\}\right)\right\} \tag{36}$$

where equality holds if and only if $\boldsymbol{u}_k^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_k = \mathbb{E}_{\boldsymbol{U}}\left\{\boldsymbol{u}_k^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_k\right\}$ with probability one. Now taking into account that the eigenvectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{Kn_T}$ have the same marginal distribution, since $\boldsymbol{U}$ is a Haar matrix [11], we can write

$$= \frac{1}{Q}\,\mathbb{E}_{\boldsymbol{\Lambda}}\left\{\sum_{j=1}^{Q}\sum_{k=1}^{Kn_T}\log\left(1 + \frac{P\lambda_k}{n_T\sigma_v^2}\frac{1}{Kn_T}\,\mathbb{E}_{\boldsymbol{U}}\left\{\sum_{h=1}^{Kn_T}\boldsymbol{u}_h^H\boldsymbol{\mathcal{F}}_{jj}\boldsymbol{u}_h\right\}\right)\right\} \tag{37}$$

and applying Lemma 1 to the inner sum

$$= \frac{1}{Q}\,\mathbb{E}_{\boldsymbol{\Lambda}}\left\{\sum_{j=1}^{Q}\sum_{k=1}^{Kn_T}\log\left(1 + \frac{P\lambda_k}{n_T\sigma_v^2}\frac{1}{Kn_T}\,\mathbb{E}_{\boldsymbol{U}}\left\{\mathrm{tr}\left(\boldsymbol{\mathcal{F}}_{jj}\right)\right\}\right)\right\} \tag{38}$$

since $\mathrm{tr}\left(\boldsymbol{\mathcal{F}}_{jj}\right)$ is not random, we have, equivalenty

$$= \frac{1}{Q}\,\mathbb{E}_{\boldsymbol{\Lambda}}\left\{\sum_{j=1}^{Q}\sum_{k=1}^{Kn_T}\log\left(1 + \frac{P\lambda_k}{n_T\sigma_v^2}\frac{1}{Kn_T}\,\mathrm{tr}\left(\boldsymbol{\mathcal{F}}_{jj}\right)\right)\right\} \tag{39}$$

applying Jensen's inequality to the average running over the index j, we get

$$\leq \mathbb{E}_{\boldsymbol{\Lambda}}\left\{\sum_{k=1}^{Kn_T}\log\left(1 + \frac{P\lambda_k}{n_T\sigma_v^2}\frac{1}{Kn_TQ}\sum_{j=1}^{Q}\mathrm{tr}\left(\boldsymbol{\mathcal{F}}_{jj}\right)\right)\right\} \tag{40}$$

where equality holds if and only if $\mathrm{tr}(\boldsymbol{\mathcal{F}}_{11}) = \cdots = \mathrm{tr}(\boldsymbol{\mathcal{F}}_{QQ})$. Taking into account that $\boldsymbol{\mathcal{F}}_{jj}$ is the j-th diagonal block of $\boldsymbol{F}\boldsymbol{F}^H$, the inner sum can be written as

$$= \mathbb{E}_{\boldsymbol{\Lambda}} \left\{ \sum_{k=1}^{Kn_T} \log \left(1 + \frac{P\lambda_k}{n_T\sigma_v^2} \frac{1}{Kn_TQ} \mathrm{tr}\left(\boldsymbol{F}\boldsymbol{F}^H\right)\right)\right\} \tag{41}$$

that achieves the maximum when (and only when) $\mathrm{tr}(\boldsymbol{F}_{i_j}\boldsymbol{F}_{i_j}^H) = Q \cdot n_T$ (see (25)), for $j=1,\ldots,K$, leading to

$$\leq \mathbb{E}_{\boldsymbol{\Lambda}} \left\{ \sum_{k=1}^{Kn_T} \log \left(1 + \frac{P\lambda_k}{n_T\sigma_v^2}\right)\right\} \tag{42}$$

$$= \mathbb{E}_{\boldsymbol{H}} \left\{ \log \left| \boldsymbol{I}_{Kn_T} + \frac{P}{n_T\sigma_v^2}\boldsymbol{H}^H\boldsymbol{H}\right|\right\} \tag{43}$$

$$= \mathbb{E}_{\boldsymbol{H}} \left\{ \log \left| \boldsymbol{I}_{n_R} + \frac{P}{n_T\sigma_v^2}\sum_{k\in\mathcal{S}}\boldsymbol{H}_k\boldsymbol{H}_k^H\right|\right\}. \tag{44}$$

From (44), we can deduce that the boundaries of the ergodic capacity region are jointly maximized, whichever is the subset $\mathcal{S}$, if and only if the same relation holds true for every $\boldsymbol{F}_k$. In fact, let us analyze the condition under which equality is achieved in the upper bound (44). Towards this end we have to consider all the equality conditions[5] met through the upper bounding steps. Starting from the equality in (35), we get that the following identity must hold

$$\boldsymbol{U}^H\boldsymbol{\mathcal{F}}_{ij}\boldsymbol{U} = \delta_{ij}\boldsymbol{\mathcal{D}}_j\,, \tag{45}$$

where $\boldsymbol{\mathcal{D}}_j$ is a diagonal matrix. Equality in (36) implies that $\boldsymbol{\mathcal{D}}_j = c_j\boldsymbol{I}$, where c_j is a constant. So, taking into account that $\boldsymbol{U}$ is unitary, (45) becomes

$$\boldsymbol{\mathcal{F}}_{ij} = c_j\delta_{ij}\boldsymbol{I}\,, \tag{46}$$

with c_j constant. Equality in (40) is achieved if and only if $\mathrm{tr}(\boldsymbol{\mathcal{F}}_{11}) = \cdots = \mathrm{tr}(\boldsymbol{\mathcal{F}}_{QQ})$, which implies $c_j = c$ in (46), where c is the common constant. Finally equality in (42) holds if and only if $\mathrm{tr}(\boldsymbol{F}_{i_j}\boldsymbol{F}_{i_j}^H) = Q \cdot n_T$, which implies $c = 1$ in (46). This last condition can be easily derived taking the trace of both sides of (30) and exploiting the fact that Π is unitary (see (26)). So, equality in (44) is achieved if and only if $\boldsymbol{\mathcal{F}}_{ij} = \delta_{ij}\boldsymbol{I}$, or equivalently (see (30) and (25))

$$\boldsymbol{F}_1\boldsymbol{F}_1^H = \cdots = \boldsymbol{F}_N\boldsymbol{F}_N^H = \boldsymbol{I}\,. \tag{47}$$

Taking into account the hypothesis that coding is based on nonsingular codes, i.e., (4) holds, identities (47) imply that every $\boldsymbol{F}_k$ is unitary. From Property 1, this is equivalent to say that every user must employ a full-rate Trace-Orthogonal Design. The last step is to recognize that the ergodic capacity region of the coded system, as obtained assuming position (47), coincides with the corresponding region of the original (uncoded) system. This is easily accomplished exploiting Theorem 1, and the proof is complete. □

As a consequence of Theorem 2, it is interesting to remark that, at least when users have the same power budget constraint, the use of a full-rate Trace-Orthogonal Design is a necessary and sufficient condition to achieve the ergodic capacity region of multiple access systems affected by uncorrelated Rayleigh flat fading. From a practical point of view, the common constraint on the power budget is equivalent to admit the existence of a power control policy such that the received SNR, at the base station, is the same for all the users.

It is worthwhile noting that the technique used to prove Theorem 2 can be exploited for computing the ergodic capacity region (or the ergodic capacity) of a multiuser (single user) system when flat uncorrelated Rayleigh fading is assumed. This is possible setting $Q = 1$, i.e., only one channel use, and interpreting $\boldsymbol{F}_k\boldsymbol{F}_k^H$ as the optimal covariance matrix for the k-th user (or the only user in the single user case). In this regard it is a novel and alternative proof to Telatar's results [1].

[5]Note that all such conditions are necessary and sufficient to achieve equality.

4. CONCLUSION

In this work we have considered multiple access systems affected by flat Rayleigh fading, where users and access point are equipped with multiple antennas. To exploit some of the MIMO potentials we have to resort to space-time coding. We have studied the way to carry out such a coding strategy in order to avoid information losses. In particular, we have proved that, under a common power constraint, the necessary and sufficient condition for a space-time coding scheme to achieve the ergodic capacity region of the multiple access system, is that every user employs a full-rate Trace-Orthogonal Design. If the power constraints are different, the condition is only sufficient. Moreover, no other constraints are imposed on the choice of the encoding matrices apart from belonging to a Trace-Orthogonal Design. So, as far as the invariance of the achievable rates region is concerned, all users can share the same set of encoding matrices.

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