A NON-SEPARABLE 2D COMPLEX MODULATED LAPPED TRANSFORM AND ITS APPLICATIONS TO SEISMIC DATA FILTERING

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ABSTRACT

Oversampled transforms are useful tools for data analysis, since redundancy increases freedom in the choice of the processing. We propose here a framework for oversampled lapped transform of images. More specifically, we establish conditions for perfect reconstruction of 2D data using non-separable windows. We also provide an example of a transform which relies on this approach. We also show the benefit of this technique in directional filtering applications encountered in the field of seismic data processing.

1. INTRODUCTION

Filter banks have proven very efficient tools for signal and image processing. Lapped transforms (LT) are a particular kind of filter banks (FB), which were primarily aimed at reducing blocking artifacts in audio or image processing [1]. They have been developed under various flavours and names, with an emphasis on their local properties and custom design. Since seismic information on the subsurface is generally gathered in huge two- or tri-dimensional datasets, the locality of the LT motivated their use in geophysical applications.

Block processing is an efficient way for dealing with long signals. It is sometimes considered as unsuitable since it induces annoying artificial high frequencies. While natural images may stand block-by-block independent processing for compression (JPEG), the block boundaries generally hamper the quality of local image processing. One option consists in considering an analysis on overlapping blocks (similar to the short-term Fourier transform), while allowing invertibility of the transform (in the absence of intermediate processing of the transformed coefficients). LT may qualify several examples of such tools, for instance local trigonometric functions, windowed basis functions, cosine modulated or generalized DFT filter banks. They have been widely used in 1D signal processing, especially for audio coding and related applications. Redundancy offers increased noise immunity as well as increased design degrees of freedom. Several theoretical studies [2, 3] and design improvements have been proposed, including the introduction of complex transforms [4] to reduce aliasing effects. Recent works have proposed a direct FB design in a two-fold oversampled case where inverses are not unique [5]. For images, the tensor product extension of LT is straightforward. But the product of two 1D envelopes yields 2D separable windows which take relatively restricted forms. For this reason, non-separable transforms have been proposed, for instance with separable windows on non-separable continuous-space bases [6], or relying on non-separable sampling.

Seismic data features differ noticeably from those of natural images. Sensors, regularly located along lines on the ground surface, record one-dimensional signals resulting from propagating waves, reflected or refracted by the different interfaces between geological strata. Signals are then assembled in images, each sensor contributing to one image column. Numerous non-linear processing steps are then necessary to produce a representation of underground structures, generally stratified as apparent from two zooms in Figures 3(a-b). Steps include filtering, warping, deconvolution, corrections from different raypaths related delays; we refer to [7] for a detailed account on seismic signal processing. In most cases, the resulting images are tainted by different kinds of noise, including processing noise, requiring filtering to help geophysical interpretation. The very structure of the layers naturally induces local frequency processing in order to enhance the underlying structure. Since seismic data are highly anisotropic by nature, we propose a new design for 2D modulated non-separable windows; motivated by locally oriented analysis, we use a support basis reminiscent of [8] where a complex LT is proposed for motion estimation applications.

We first provide some notations for LT and recall expressions for a separable transform in Section 2. We then study more closely the general case of modulated LTs and give perfect reconstruction conditions for non-separable 2D windowing setting. Those results are illustrated in Section 3 with an example of a 2D complex transform derived from this framework. We demonstrate its usefulness in seismics with an application to directional filtering.

2. DESIGN OF A 2D NON-SEPARABLE MODULATED LT

2.1 From 1D LT to separable 2D LT

As a separable 2D FB is obtained by applying two 1D FBs (resp. on rows and columns), we first need to introduce some notations in the one-dimensional case. Figure 1 shows a 1D M-band FB with decimation factor N. In this paper, we are interested in the case where $(N,M) \in \mathbb{N}^2$ and $1 \leq N < M$, which corresponds to an oversampled FB.

Let $(y_i(n))_{n\in\mathbb{Z}}$ with $i \in \{1,\ldots,M\}$ be the i-th output of the analysis FB in Figure 1 when the input signal is $(x(n))_{n\in\mathbb{Z}}$. If $(H_i(n))_{n\in\mathbb{Z}}$ denotes the impulse response of the

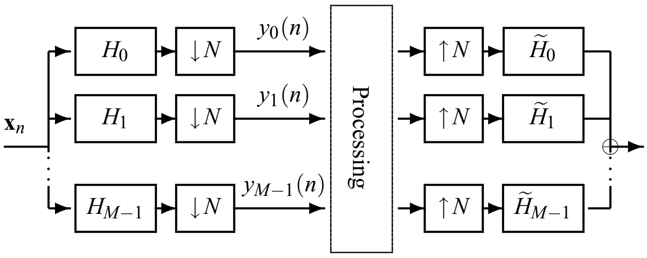

Figure 1: Diagram of an oversampled 1D LT.

filter in the i-th channel, we have for all $n \in \mathbb{Z}$,

$$\begin{aligned} y_{i+1}(n) &= \sum_{p} H_i(p)x(Nn-p) \\ &= \sum_{\ell} \sum_{j=0}^{N-1} H_i(N\ell+j)x(N(n-\ell)-j). \end{aligned} \tag{1}$$

Assuming that the considered impulse responses are causal with maximum length kN (with $k \in \mathbb{N}^*$), it is well-known that the computation of these subband coefficients can be viewed as the linear transform of a data block of length kN

$$\mathbf{x}_n = \big(x(nN), \ldots, x(nN-kN+1)\big)^\top$$

where $k-1$ is the number of overlapping blocks of size N. This can be expressed as

$$\mathbf{y}_n = \boldsymbol{P}\mathbf{x}_n \tag{2}$$

where $\mathbf{y}_n = (y_1(n), \ldots, y_M(n))^\top$, $\boldsymbol{P}$ is the block-matrix

$$\boldsymbol{P} = (\boldsymbol{P}_0 \quad \boldsymbol{P}_1 \quad \ldots \quad \boldsymbol{P}_{k-1}) \tag{3}$$

and, for $\ell \in \{0, \ldots, k-1\}$, $\boldsymbol{P}_\ell = (P_\ell(i,j))_{1 \le i \le M, 1 \le j \le N}$ with

$$P_\ell(i,j) = H_{i-1}(N\ell + j - 1).$$

Formally, the subband decomposition is a linear operator $\boldsymbol{T}$ transforming the infinite-dimensional vector $\bar{\mathbf{x}} = (\ldots\ \mathbf{x}_n^\top\ \mathbf{x}_{n-1}^\top\ \ldots)^\top$ into $\bar{\mathbf{y}} = (\ldots\ \mathbf{y}_n^\top\ \mathbf{y}_{n-1}^\top\ \ldots)^\top$. This transform corresponds to the infinite-dimensional block-Toeplitz matrix

$$\boldsymbol{T} = \begin{pmatrix} \ddots & & & & & 0 \\ & \boldsymbol{P}_0 & \boldsymbol{P}_1 & \cdots & \boldsymbol{P}_{k-1} & \boldsymbol{0}_{M\times N} \\ & \boldsymbol{0}_{M\times N} & \boldsymbol{P}_0 & \cdots & \boldsymbol{P}_{k-2} & \boldsymbol{P}_{k-1} \\ 0 & & & & & \ddots \end{pmatrix}.$$

An interesting case which is investigated in this work is when, for all $i \in \{0, \ldots, M-1\}$ and $n \in \{0, \ldots, kN-1\}$, $H_i(n) = E(i+1, n+1)h_a(n)$ where $(h_a(n))_{0 \le n \le kN-1}$ is a given analysis window and $\boldsymbol{E} = (E_a(i,n))_{1 \le i \le M, 1 \le n \le kN}$ is a complex semi-unitary matrix, i.e. $\boldsymbol{E}^*\boldsymbol{E} = \boldsymbol{I}_{kN\times kN}$ where $\boldsymbol{E}^*$ denotes the Hermitian adjoint of $\boldsymbol{E}$. Then, $\boldsymbol{P}$ can be written as: $\boldsymbol{P} = \boldsymbol{E}\boldsymbol{H}_a$ where $\boldsymbol{H}_a$ is the $kN \times kN$ diagonal matrix: $\boldsymbol{H}_a = \mathrm{Diag}(h_a(0), \ldots, h_a(kN-1))$. Such a family of FBs includes oversampled modulated FBs as well as redundant extensions of the Extended LT [1]. Notice that the semi-unitary condition implies that $kN = \mathrm{rank}(\boldsymbol{E}) \le M$. Therefor the redundancy factor M/N is here greater than or equal to k. Although this condition may appear restrictive, it provides degrees of freedom to build a selective directional analysis in image applications such as those considered in Section 3.1.

A 2D separable extension of the LT of interest reads, for all $(n_1, n_2) \in \mathbb{Z}^2$,

$$\begin{aligned} Y_{i_1,i_2}(n_1,n_2) = \sum_{p_1,p_2} & E(i_1+1, p_1+1)E(i_2+1, p_2+1) \\ & W_a(p_1,p_2)X(Nn_1-p_1, Nn_2-p_2) \end{aligned} \tag{4}$$

where $W_a(p_1,p_2) = h_a(p_1)h_a(p_2)$, $(X(n_1,n_2))_{n_1,n_2}$ is the input image and $(Y_{i_1,i_2}(n_1,n_2))_{n_1,n_2}$ is the resulting coefficient field in subband $(i_1,i_2) \in \{0, \ldots, M-1\}^2$. The design of such a decomposition straightforwardly follows from the 1D framework. However, the use of a separable window may not be the most appropriate for image analysis tasks.

2.2 Proposed 2D non-separable transform

In the above equation, $W_a(p_1,p_2)$ is the product $h_a(p_1)h_a(p_2)$ resulting in a separable window of rank 1. To increase design flexibility, we replace it by a "true" 2D generally non-separable window $W_a(p_1,p_2)$. The Perfect Reconstruction (PR) property is derived by rewriting the 2D decomposition in a matrix form.

Let $\mathbf{Y}_{n_1,n_2} = (Y_{n_1,n_2}(i))_{1 \le i \le M^2}$ be the vector obtained by using the column stacking operation:

$$Y_{n_1,n_2}(i_1 + Mi_2 + 1) = Y_{i_1,i_2}(n_1,n_2), \quad (i_1,i_2) \in \{0, \ldots, M-1\}^2.$$

At the same time, define the column vector $\mathbf{X}_{n_1,n_2} = (X_{n_1,n_2}(p))_{1 \le p \le (kN)^2}$ as

$$\begin{aligned} & X_{n_1,n_2}((q_1k+q_2)N^2 + r_2N + r_1 + 1) = \\ & \qquad X(n_1N - (q_1N + r_1), n_2N - (q_2N + r_2)), \\ & \qquad (q_1,q_2) \in \{0, \ldots, k-1\}^2, (r_1,r_2) \in \{0, \ldots, N-1\}^2. \end{aligned}$$

Similarly, we introduce the diagonal matrix $\boldsymbol{W}_a = \mathrm{Diag}\big(\mathscr{W}_a(0), \ldots, \mathscr{W}_a((kN)^2-1)\big)$ such that for $(q_1,q_2) \in \{0, \ldots, k-1\}^2$ and $(r_1,r_2) \in \{0, \ldots, N-1\}^2$

$$\mathscr{W}_a((q_1k+q_2)N^2 + r_2N + r_1) = W_a(q_1N + r_1, q_2N + r_2)$$

Then, Equation (4) can be obviously rewritten as

$$Y_{n_1,n_2}(i) = \sum_{p=0}^{(kN)^2-1} F(i, p+1)\mathscr{W}_a(p)X_{n_1,n_2}(p+1)$$

where $\boldsymbol{F} = (F(i,p))_{1 \le i \le M^2, 1 \le p \le (kN)^2} = \boldsymbol{E} \otimes \boldsymbol{E}$, the Kronecker product being denoted by $\otimes$. So, our 2D transform takes a form similar to the one in Eq. (2):

$$\mathbf{Y}_{n_1,n_2} = \boldsymbol{Q}\mathbf{X}_{n_1,n_2}, \quad \boldsymbol{Q} = \boldsymbol{F}\boldsymbol{W}_a.$$

Due to the basic properties of the Kronecker product of matrices, the semi-unitarity of $\boldsymbol{E}$ entails that $\boldsymbol{F}$ is a semi-unitary matrix. We are in a framework very similar to the one described in Section 2.1. Thus, an infinite-dimensional matrix $\boldsymbol{S}$ exists which is associated with this transform. We will now see how to perform the reconstruction of X by studying $\boldsymbol{S}$.

2.3 PR Conditions

The PR property holds if and only if a left inverse $\widetilde{\boldsymbol{S}}$ of $\boldsymbol{S}$ exists. The invertibility of $\boldsymbol{S}$ is also equivalent to the invertibility of $\boldsymbol{S}^*\boldsymbol{S}$. Let us now determine the form of this operator.

As was done for the matrix $\boldsymbol{P}$ in Eq. (3), we use a block decomposition of the matrix $\boldsymbol{Q}$, the only differences being that the index ℓ now varies in $\{0,...,k^2-1\}$ and the size of each block $\boldsymbol{Q}_\ell$ is $M^2 \times N^2$. Then, we have:

$$\begin{cases} \sum_{\ell=0}^{k^2-1} \boldsymbol{Q}_\ell^* \boldsymbol{Q}_\ell = \sum_{\ell=1}^{k^2} \boldsymbol{W}_a(\ell)^* \boldsymbol{W}_a(\ell) \\ \sum_{\ell=d}^{k^2-1} \boldsymbol{Q}_\ell^* \boldsymbol{Q}_{\ell-d} = \boldsymbol{0}, \qquad \forall d \in \{1,\dots,k^2-1\} \end{cases}$$

where $\boldsymbol{W}_a$ has been decomposed into a block-diagonal form as $\mathrm{Diag}(\boldsymbol{W}_a(1),\dots,\boldsymbol{W}_a(k^2))$. To find these expressions we have used the fact that $\boldsymbol{F}$ is a semi-unitary matrix. From these relations, it is readily checked that $\boldsymbol{S}^*\boldsymbol{S}$ is an infinite-dimensional diagonal matrix with blocks $\boldsymbol{D} = \sum_{\ell=1}^{k^2} \boldsymbol{W}_a(\ell)^* \boldsymbol{W}_a(\ell)$ on the diagonal. Consequently, $\boldsymbol{S}^*\boldsymbol{S}$ is invertible if and only if the diagonal matrix $\boldsymbol{D}$ is invertible. As, for all $j \in \{0,\dots,N^2-1\}$, the j-th diagonal element of $\boldsymbol{D}$ is equal to

$$D(j) = \sum_{\ell=0}^{k^2-1} |\mathscr{W}_a(N^2\ell + j)|^2$$

we infer that a necessary and sufficient condition for $\boldsymbol{S}$ to be left-invertible is

$$\forall j \in \{0,\dots,N^2-1\}, \qquad \sum_{\ell=0}^{k^2-1} |\mathscr{W}_a(N^2\ell + j)|^2 \neq 0.$$

Coming back to the 2D indexation, the PR condition reads: for all $(j_1, j_2) \in \{0,\dots,N-1\}^2$,

$$\sum_{\ell_1=0}^{k-1}\sum_{\ell_2=0}^{k-1} |W_a(N\ell_1 + j_1, N\ell_2 + j_2)|^2 \neq 0.$$

2.4 Optimal reconstruction

When the previous PR condition is satisfied, due to the redundancy in the considered transform, there does not exist a *unique* inverse $\widetilde{\boldsymbol{S}}$ such that $\widetilde{\boldsymbol{S}}\boldsymbol{S} = \boldsymbol{I}$. A choice for $\widetilde{\boldsymbol{S}}$ possessing good reconstruction properties is the pseudo-inverse operator $\boldsymbol{S}^\sharp = (\boldsymbol{S}^*\boldsymbol{S})^{-1}\boldsymbol{S}^*$. Uppon reconstruction, $\boldsymbol{S}^\sharp$ allows to cancel the effects of the perturbations of the decomposition coefficients which do not belong to $\mathrm{Im}(\boldsymbol{S})$.

With the same approach as in Section 2.3, it is easy to see that $\boldsymbol{S}^\sharp$ corresponds to an infinite-dimensional matrix built from the blocks

$$\mathrm{Diag}(\underbrace{\boldsymbol{D}^{-1},\dots,\boldsymbol{D}^{-1}}_{k\text{ times}})\boldsymbol{Q}^* = \boldsymbol{W}_s\boldsymbol{F}^*$$

where $\boldsymbol{W}_s = \mathrm{Diag}(\boldsymbol{D}^{-1},\dots,\boldsymbol{D}^{-1})\boldsymbol{W}_a^*$.

This inverse transform takes a very simple form: it is built from a synthesis window associated with the diagonal matrix $\boldsymbol{W}_s$ and the Hermitian adjoint of the orthogonal matrix $\boldsymbol{F}$ used in the direct transform. More precisely, similarly to the study for the analysis FB, it can be shown that the impulse responses of the synthesis FB are "anti-causal" sequences given by: for all $(i_1, i_2) \in \{0,\dots,M-1\}^2$ and $(p_1, p_2) \in \{0,\dots,kN-1\}^2$,

$$\widetilde{H}_{i_1,i_2}(-p_1,-p_2) = E(i_1+1, p_1+1)^* E(i_2+1, p_2+1)^* W_s(p_1, p_2)$$

where, for all $(\ell_1, \ell_2) \in \{0,\dots,k-1\}^2$ and $(j_1, j_2) \in \{0,\dots,N-1\}^2$,

$$W_s(N\ell_1 + j_1, N\ell_2 + j_2) = \frac{W_a(N\ell_1 + j_1, N\ell_2 + j_2)^*}{\sum_{q_1=0}^{k-1}\sum_{q_2=0}^{k-1} |W_a(Nq_1 + j_1, Nq_2 + j_2)|^2}. \tag{5}$$

2.5 Tight frame condition

When $\boldsymbol{S}^*\boldsymbol{S} = \alpha\boldsymbol{I}$ with $\alpha \in \mathbb{R}_+^*$, the overcomplete LT corresponds to a so-called discrete-time tight frame operator. Hence the energy of any image X is preserved after decomposition (up to a factor α):

$$\sum_{m_1,m_2} |X(m_1, m_2)|^2 = \alpha \sum_{i_1,i_2}\sum_{n_1,n_2} |Y_{i_1,i_2}(n_1, n_2)|^2.$$

From the results in Section 2.2, we deduce the following necessary and sufficient condition to obtain a tight frame decomposition: for all $(j_1, j_2) \in \{0,\dots,N-1\}^2$,

$$\sum_{\ell_1=0}^{k-1}\sum_{\ell_2=0}^{k-1} |W_a(N\ell_1 + j_1, N\ell_2 + j_2)|^2 = \alpha.$$

When this condition is fulfilled, Eq. (5) shows that the synthesis window takes the simpler form:

$$W_s(p_1, p_2) = \alpha^{-1} W_a(p_1, p_2)^*, \quad (p_1, p_2) \in \{0,\dots,kN-1\}^2.$$

3. APPLICATION TO SEISMIC DATA FILTERING

3.1 Non-separable 2D Complex Lapped Transform

In the previous section, we have derived a general framework allowing the use of any arbitrary semi-unitary matrix $\boldsymbol{E}$. In the considered application, the main processing step is to detect local directions. In addition, since features of interest often present an oscillatory behaviour, a frequency transform seems appropriate. 2D real transforms (such as DCT) exhibit symmetries in the frequency plane, which prevent them from separating oriented features (with angle θ) from features in the opposite direction (with angle $-\theta$). Thence, a complex-valued harmonic transform such as a DFT should be preferred in order to perform a directional analysis. More precisely with $M = kN$ we chose a matrix derived from the extended Complex Lapped Transform proposed in [8]: for all $(j, p) \in \{1,\dots,kN\}^2$,

$$E(j,p) = \frac{1}{\sqrt{kN}} e^{-\imath(j-\frac{Nk}{2}-\frac{1}{2})(p-\frac{Nk}{2}-\frac{1}{2})\frac{2\pi}{kN}}.$$

For this application we have used the following analysis window: for all $(i, j) \in \{1,...,kN\}^2$,

$$W_a(i,j) = \cos\left(\frac{\pi}{2}\left(\sqrt{a\,(2i-kN-1)^2 + b\,(2j-kN-1)^2} - R\right)\mathbf{1}_A(i,j)\right)$$

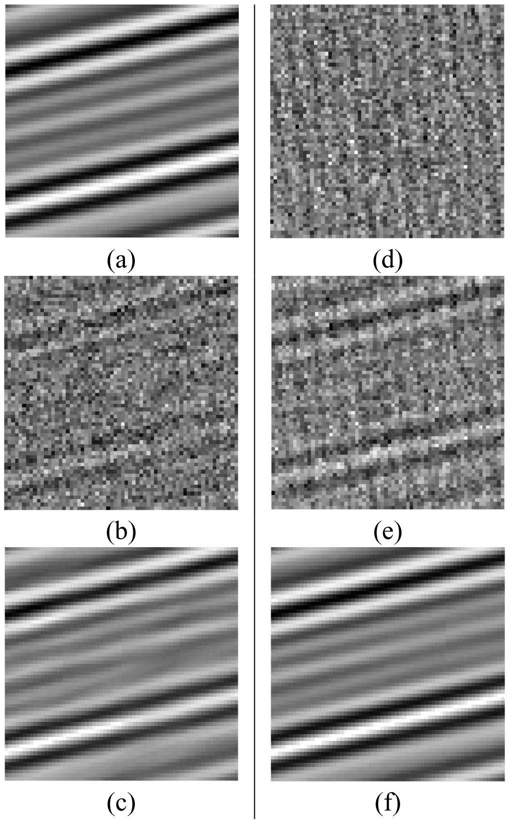

Figure 2: (a) Sample of original synthetic data (b) Noisy image (Gaussian noise) (c) Reconstructed image (Gaussian noise) (d) Directional noise used (e) Noisy image (directional noise) (f) Reconstructed image (directional noise)

where $\mathbf{1}_A$ is the characteristic function of the set $A = \{(u,v)\,|\,a(2u-kN-1)^2+b(2v-kN-1)^2 \geq R^2\}$.
We chose this window to offer a trade-off between good decay properties, in order to avoid boundary problems, and having a large area with no or little attenuation, to get fine enough analyses with small size data samples. The synthesis window is computed using Eq. (5).

3.2 Filtering and results

We propose the following empirical procedure to enhance the dominant structures in an image of the underground. First we detect the locally dominant orientation by finding the subband coefficient with highest magnitude at a given location. We then remove all the coefficients which do not correspond to this direction. Finally, a threshold cancels the small remaining coefficients.
Since locally seismic images are made of many parallel layers, we will approximate them by the sum $f(m,n) = L^{-1} \sum_{i=1}^{L} \sin(a_i m + b_i n + \phi_i)$ where the reals vectors $(a_i,b_i)_1 \leq i \leq L$ are collinear and the phases $(\phi_i)_{1\leq i \leq L}$ are randomly chosen in $[0,2\pi)$. In the following simulation $k=5$ and $N=16$. We first added a Gaussian white noise with $\sigma = 1$. Figures 2(b) and 2(c) show the noisy and reconstructed images. We see that the orientation was well detected and preserved. Note however that seismic data often exhibit *directional* noise that we wish to remove. We have generated a structured noise (Fig. 2(d)) and added it to the original image. The denoised image clearly shows that we are able to extract dominant structure from directional noise. On Figure 3(a) or 3(b), we observe on actual seismic data that the dominant horizontal structure is perturbed by many other directional interferences. Images 3(c) and 3(d) show how those perturbations are removed while keeping relevant information.

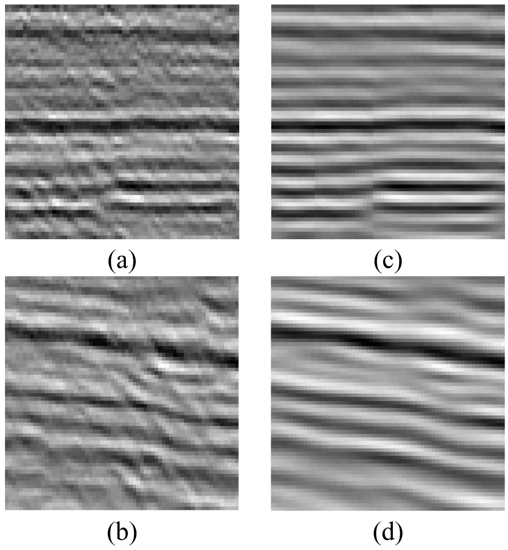

Figure 3: (a), (b) Samples of real seismic data (c), (d) Processed images

4. CONCLUSION

We proposed a simple framework for a 2D oversampled non-separable LT and obtained very promising results for directional filtering of seismic data. We still have to study thoroughly the design of the 2D windows which would be the most appropriate for different applications. We should also perform adaptive forms of processing in order to better retrieve areas around seismic faults.

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