ONE METHOD FOR IIR FILTER DESIGN BASED ON COMPLEX ALLPASS FILTERS

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ABSTRACT

This paper presents a new direct method for the lowpass and highpass IIR filter design. The method is based on the design of complex allpass filters. In contrast to the existing direct methods for the IIR filter design, the method presented here uses the same specifications, namely the passband and stopband frequencies, and passband droop and stopband attenuation, as the specifications used in the analog filter based design of IIR filters. The resulting filter is a Butterworth-type filter.

1. INTRODUCTION

It is well known that the traditional IIR filter design is based on the corresponding analog filter design [1]. Recently, different direct methods have been proposed for designing IIR filters, such as [2–8]. In this paper we propose a method for a direct IIR filter design based on complex allpass filters. We use the result presented in [9] for constructing complex allpass filters. The design of a digital filter based on allpass filters has certain advantages, such as lower sensitivity to filter quantization [1].

The design parameters in this approach are the same as the ones in the traditional IIR filter design. These are shown in Fig. 1(a) and are passband frequency ω_p, stopband frequency ω_s, passband droop A_p, and stopband attenuation A_s.

An IIR lowpass filter $H(z)$ of an even order N can be expressed in terms of two complex allpass filters $A_0(z)$ and $A_1(z)$ [10]

$$H(z) = \frac{1}{2}\left[A_0(z) + A_1(z)\right]. \tag{1}$$

We rewrite this relation in the following form,

$$H(z) = \frac{A_0(z)}{2}\left[1 + A(z)\right], \tag{2}$$

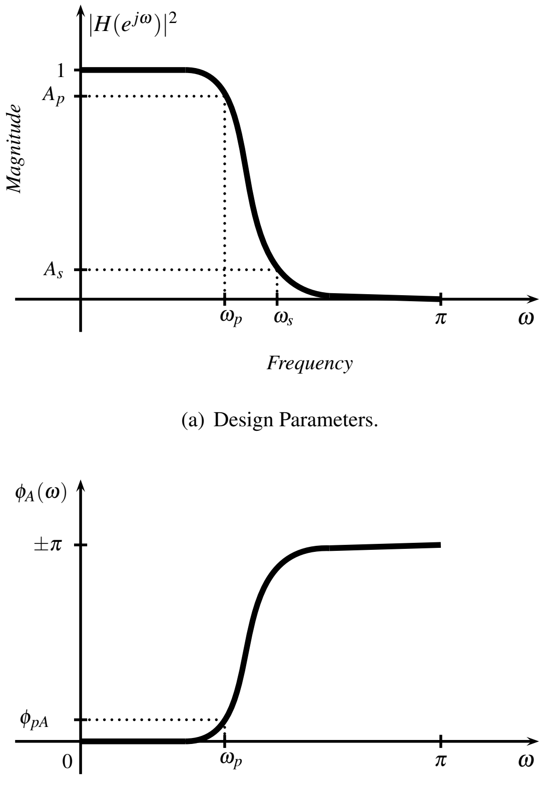
where $A(z) = A_1(z)/A_0(z)$.

From Eq.(2) we have

$$\begin{aligned} |H(e^{j\omega})|^2 &= \frac{1}{4}\left|1 + A(e^{j\omega})\right|^2 \\ &= \cos^2\left(\frac{\phi_A(\omega)}{2}\right), \end{aligned} \tag{3}$$

where $\phi_A(\omega)$ is the phase response of $A(z)$.

The relation (3) shows that the square magnitude response of $H(z)$ depends only on the phase of $A(z)$. In order to design a lowpass filter, the phase values of $\phi_A(\omega)$ at $\omega = 0$ and $\omega = \pi$ must be 0 and π, respectively. The phase value at $\omega = \omega_p$ has a certain value ϕ_{pA} as shown in Fig. 1(b). The



(a) Design Parameters.

(b) Phase response of $A(z)$.

Figure 1: Design parameters and phase response of $A(z)$.

proposed design of an IIR lowpass filter $H(z)$ is based on the design of one complex allpass filter with the prescribed phase values, and Section 2 presents the design of such a complex allpass filter with desired phase characteristics. In Section 3 we propose the algorithm for the design of a lowpass filter while in Section 4 we present a way to design its highpass counterpart.

2. DESIGN OF COMPLEX ALLPASS FILTERS

A complex allpass filter of order N (N is even) is given by,

$$A(z) = z^{-N}\frac{\alpha\widetilde{F}(z)}{\alpha^* F(z)}, \tag{4}$$

where α is a complex constant with the unit magnitude, and z is a complex variable. The symbols $*$ and $\widetilde{}$ denote conjugate and paraconjugate, respectively. The complex polynomial $F(z)$ is

$$F(z) = 1 + \sum_{n=1}^{N} f_n z^{-n}. \tag{5}$$

where f_n are complex coefficients.

Using Eq.(4) the relationship between the phases of $A(z)$ and $F(z)$ can be expressed by

$$\phi_A(\omega) = -\omega N + 2\phi_\alpha - 2\phi_F(\omega), \tag{6}$$

where ϕ_α is the phase of α. Since the value of $\phi_A(0)$ is 0 it follows that $\phi_\alpha = \phi_F(0)$.

Considering the value of A_p in dB, the desired phase value $\phi_A(\omega)$ at ω_p is, (see Eq.(3))

$$\phi_{pA} = \phi_A(\omega_p) = 2\arccos\left(10^{-A_p/20}\right). \tag{7}$$

Using Eq.(6), the corresponding group delay $\tau_A(\omega)$ of the allpass filter is given as,

$$\tau_A(\omega) = N + \tau_F(\omega), \tag{8}$$

where $\tau_F(\omega)$ is the group delay of $F(z)$.

The designed filter $H(z)$ has flatness at $\omega = 0$ and $\omega = \pi$ (see Fig. 1(a)) i.e. it is a Butterworth-type filter. The degree of flatness at these frequency points is equal to $N-2$. Group delays of the allpass filter at $\omega = 0$ and $\omega = \pi$, are equal to zero, $\tau_A(0) = \tau_A(\pi) = 0$, as shown in Fig. 1(b). Therefore, from Eq.(8) the corresponding group delays of $F(z)$ at these points are $N/2$. In a similar way, by denoting the phase value of $F(z)$ at $\omega = 0$ as $-\phi_0$, it follows that the phase at $\omega = \pi$ is $-\phi_0 - (N+1)\pi/2$ (see Eq.(6)). Using the method proposed in [9], the corresponding filter coefficientes of $F(z)$ in Eq. (5) are

$$f_n = \begin{cases} \binom{N}{n} & n \text{ even} \\ \binom{N}{n} e^{-j2\phi_0} & n \text{ odd} \end{cases}. \tag{9}$$

Using Eqs.(6) and (9) it follows

$$\alpha = e^{-j\phi_0}. \tag{10}$$

We use the result [9],

$$\sum_{n=0}^{N} \sin(\omega_p n - \phi_p - \phi_0 - \phi_n) r_n = 0, \tag{11}$$

where r_n and ϕ_n are the magnitude and phase of f_n and

$$\phi_p = \frac{\phi_{pA} + \omega_p N}{2}. \tag{12}$$

Solving Eq.(11) we get,

$$\phi_0 = \frac{1}{2j} \ln\left\{\frac{D}{D^*}\right\}, \tag{13}$$

where $D = e^{j\phi_p} B_0^* - e^{-j\phi_p} B_1$ and

$$B_0 = \sum_{m=0}^{\frac{N}{2}-1} \binom{N}{2m+1} e^{j(2m+1)\omega_p}, \tag{14}$$

$$B_1 = \sum_{m=0}^{\frac{N}{2}} \binom{N}{2m} e^{j2m\omega_p}. \tag{15}$$

Since f_n is a Type I symmetric sequence, $F(z)$ can be expressed as,

$$F(z) = \beta z^{-n_0} F_0(z^{-1}) F_0(z), \tag{16}$$

where $F_0(z)$ is the polynomial with all zeros inside the unit circle, and n_0 is

$$n_0 = N/2. \tag{17}$$

The complex constant β is given by,

$$\beta = \frac{(-1)^{n_0}}{\prod_{k=1}^{n_0} R_k}, \tag{18}$$

where R_k are the zeros of the polynomial $F_0(z)$.

Using Eq.(16) we express $A(z)$ in the following form,

$$A(z) = \frac{A_1(z)}{A_0(z)} = z^{-N} \frac{\alpha\gamma^* z^{n_0} \widetilde{F}_0(z^{-1}) \widetilde{F}_0(z)}{\alpha^* \gamma z^{-n_0} F_0(z^{-1}) F_0(z)}, \tag{19}$$

where

$$\gamma = \frac{\beta}{|\beta|}, \tag{20}$$

$$A_0(z) = \left(\frac{\alpha}{\gamma}\right)^* z^{-n_0} \frac{F_0(z^{-1})}{\widetilde{F}_0(z^{-1})}, \tag{21}$$

$$A_1(z) = \left(\frac{\alpha}{\gamma}\right) z^{-n_0} \frac{\widetilde{F}_0(z)}{F_0(z)}. \tag{22}$$

Applying the results (21) and (22), we now rewrite Eq.(1) for the desired IIR filter as,

$$H(z) = \frac{1}{2}\left[\left(\frac{\alpha}{\gamma}\right)^* z^{-n_0} \frac{F_0(z^{-1})}{\widetilde{F}_0(z^{-1})} + \left(\frac{\alpha}{\gamma}\right) z^{-n_0} \frac{\widetilde{F}_0(z)}{F_0(z)}\right]. \tag{23}$$

In the next section we propose an algorithm for the design of an IIR filter based on Eq.(23).

3. ALGORITHM FOR LOWPASS FILTER DESIGN

The algorithm for the filter design is described in the following steps.

1. Estimate the order N of the filter using the well known relation for the traditional digital Butterworth filter design [1],

$$N = \left\lceil \frac{\log\left(\frac{10^{A_s/10}-1}{10^{A_p/10}-1}\right)}{2\log\left(\frac{\omega_s'}{\omega_p'}\right)} \right\rceil, \tag{24}$$

   where the symbol $\lceil\cdot\rceil$ denotes the ceiling function, and

$$\omega_p' = \tan\left(\frac{\omega_p}{2}\right), \tag{25}$$

$$\omega_s' = \tan\left(\frac{\omega_s}{2}\right). \tag{26}$$

   If the estimated value N is odd we increase it by one.
2. Calculate the phase value at ω_p of the allpass filter using Eq. (7). Based on this result and the estimated value of N, compute the value ϕ_p using (12) and the phase ϕ_0 of the filter $1/F(z)$ based on Eq. (13).

3. Calculate the coefficients f_n given by Eq.(9). Compute the minimum and maximum phase subfilters $F_0(z)$ and $z^{-n_0}F_0(z^{-1})$, and their corresponding paraconjugates $z^{-n_0}\widetilde{F}_0(z)$ and $\widetilde{F}_0(z^{-1})$.
4. Finally, using the values of α and γ calculated from Eqs. (10) and (20), compute the coefficients of the desired lowpass IIR filter $H(z)$ using the relation (23).

We illustrate the proposed algorithm with one example.

Example 1. We design an IIR lowpass filter $H(z)$ with the following specifications: passband frequency $\omega_p = 0.3\pi$, stopband frequency $\omega_s = 0.6\pi$, passband droop, $A_p = 1$ dB, and stopband attenuation, $A_s = 40$ dB.

Our calculation procedure is as follows.

1. From Eq. (24) it follows that $N = 6$.
2. Using Eqs.(7), (12) and (13), we calculate the following phase values, $\phi_A(\omega_p) = 0.9414$, $\phi_p = 3.298134$ and $\phi_0 = 1.536422$.
3. The coefficients f_n computed using Eq.(9) are given in Table 1.

n		f_n	n	f_{0n}
0	6	1	0	1
1	5	$-5.9858 - j0.4122$	1	$-1.0111 + j0.4232$
2	4	15	2	$0.5562 - j0.2307$
3		$-19.9528 - j1.3739$	3	$-0.1012 + j0.0634$

Table 1: Filter coefficients of $A(z)$ and $F_0(z)$ in Example 1.

Using Eq.(5) and the values of f_n we compute the coefficients of the subfilter $F_0(z)$. The filter coefficients f_{0n} are shown in Table 1.

4. We calculate α and γ using Eqs.(10) and (20) to be $\alpha = 0.0344 - j0.9994$ and $\gamma = -0.8475 - j0.5308$. Finally, the coefficients of the filter $H(z)$ are obtained from Eq. (23) and they are shown in Table 2.

n		Numerator	n	Denominator		
0	6	0.0041	0	1	4	0.6208
1	5	0.0246	1	-2.0222	5	-0.1418
2	4	0.0615	2	2.3138	6	0.0142
3		0.0820	3	-1.5223		

Table 2: Filter coefficients of $H(z)$ in Example 1.

The magnitude response of the designed filter is plotted in Fig. 2. The passband and stopband details, given in Fig. 2(b), demonstrate that the specifications are satisfied.

4. DESIGN OF HIGHPASS FILTERS

A relation useful for the highpass filter design that involves two complex allpass filters is [10],

$$G(z) = \frac{1}{2j}\left[A_0(z) - A_1(z)\right]. \tag{27}$$

Using Eqs.(19)–(22) we now have,

$$G(z) = \frac{1}{2j}\left[\left(\frac{\alpha}{\gamma}\right)^{*} z^{-n_0}\frac{F_0(z^{-1})}{\widetilde{F}_0(z^{-1})} - \left(\frac{\alpha}{\gamma}\right) z^{-n_0}\frac{\widetilde{F}_0(z)}{F_0(z)}\right]. \tag{28}$$

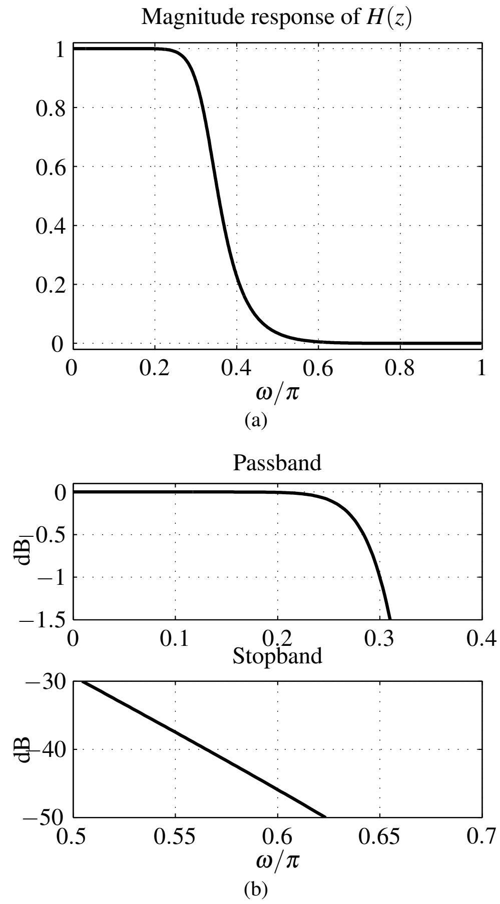


Figure 2: Example 1.

In this case the phase value ϕ_{pA} is given as,

$$\phi_{Ap} = 2\arcsin\left(10^{-A_p/20}\right). \tag{29}$$

The procedure for the design of a highpass filter $G(z)$ is similar to the one described in the previous section. The only difference is that the frequencies in Eqs.(25) and (26) are now

$$\omega_p' = \tan\left(\frac{\pi - \omega_p}{2}\right), \tag{30}$$

$$\omega_s' = \tan\left(\frac{\pi - \omega_s}{2}\right), \tag{31}$$

and the corresponding relations given by (7) and (23) are now as given by Eqs.(29) and (28), respectively.

Example 2. In this example we illustrate the algorithm for the IIR highpass filter design with the passband and stopband frequencies 0.8π and 0.6π, and passband droop and stopband attenuation 2 dB and 50 dB, respectively.

1. Using Eq.(24), the estimated value of N is 8.
2. From Eqs.(29), (12) and (13), compute $\phi_{pA} = 1.8358$, $\phi_p = 10.9710$ and $\phi_0 = -1.6243 \times 10^{-4}$.
3. The filter coefficients f_n computed using Eq.(9) are given in Table 3.

Using Eq.(5) arrive at the filter $F(z)$, and find its subfilter $F_0(z)$. The resulting coefficients of $F_0(z)$ are shown in Table 3.

n		f_n	n	f_{0n}
0	8	1	0	1
1	7	$7.9999 + j0.0026$	1	$2.3413 + j0.3032$
2	6	28	2	$2.2351 + j0.4750$
3	5	$55.9999 + j0.0182$	3	$0.9957 + j0.2814$
4		70	4	$0.1728 + j0.0589$

Table 3: Filter coefficients of $A(z)$ and $F_0(z)$ in Example 2.

4. The corresponding values of α and γ are $\alpha = 0.99999 + j0.00016$ and $\gamma = -0.9465 + j0.3227$. The calculated filter coefficients of $G(z)$ are shown in Table 4.

n		Numerator	n	Denominator		
0	8	2.965×10^{-5}	0	1	5	5.5634
1	7	-23.728×10^{-5}	1	4.6826	6	1.8992
2	6	83.047×10^{-5}	2	10.0438	7	0.3773
3	5	-166.095×10^{-5}	3	12.7455	8	0.0333
4		207.618×10^{-5}	4	10.4001		

Table 4: Filter coefficients of $G(z)$ in Example 2.

The magnitude response of $G(z)$ given in Fig. 3 verifies that the specifications are satisfied.

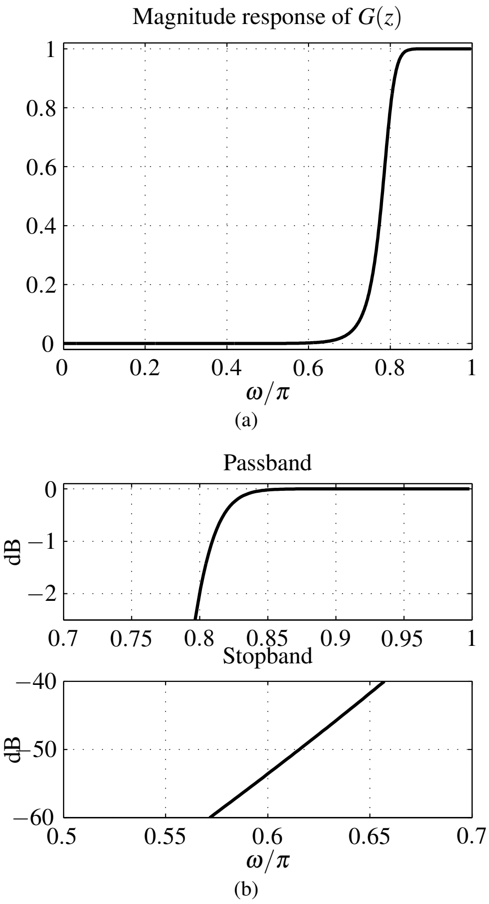


Figure 3: Example 2.

5. CONCLUSION

This paper presents a new direct method for the design of lowpass and highpass IIR filters based on complex allpass filters. The design parameters are the same as in the traditional IIR filter design, and these are the passband and stopband frequencies, passband droop and stopband attenuation. The resulting filter has a flat magnitude response in both passband and stopband. The proposed design is implemented in MATLAB where the inputs are the filter design parameters and the outputs are the IIR and complex allpass filter coefficients, computed according to Eqs.(23) and (28).

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