A FACTOR GRAPH APPROACH TO DESIGN CLOSE-TO-OPTIMAL RECEIVERS IN THE PRESENCE OF A TIMING UNCERTAINTY.

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ABSTRACT

This paper considers the design of close-to-optimal receivers in the presence of a timing uncertainty. The problem is placed into the factor-graph and the sum-product (SP) algorithm framework. A simplified version of the SP algorithm is considered and the expectation-maximization (EM) algorithm is used to implement it. The proposed approach, combining the SP and EM algorithms, is shown to outperform classical approaches while exhibiting a low complexity.

1. INTRODUCTION

During the last decade, a lot of efforts have been devoted to the design of receivers achieving close-to-optimum performance while exhibiting a reasonable complexity. A crucial step towards this kind of receiver was made with the discovery of the "turbo principle" by Berrou and Glavieux [1]. The authors showed that near-optimum decoding may be achieved by iteratively exchanging a (so-called) extrinsic information between two soft-in soft-out (SISO) decoders. The outstanding performance achieved by this approach has led to applying this principle to various receiver tasks: joint demodulation and decoding, joint equalization and decoding... Although leading to receivers with impressive performance, the turbo principle was applied for a long time without any mathematical justification of its efficiency. Recently, the factor graph representation and the associated sum-product (SP) algorithm [2] have provided both a justification of this principle and a rigorous framework for the design of iterative receivers.

Besides the transmitted symbols, the received observations also depend on some synchronization parameters. Unfortunately, the implementation of an optimal receiver which would be able to deal with the uncertainty relative to these parameters is totally intractable. In this paper, we propose therefore a suboptimal solution based on the SP-algorithm and the factor graph framework. The SP algorithm has already been considered in several papers [3]-[4] for the design of suboptimal receivers in the case of carrier phase uncertainty. In this paper, we will focus on the receiver design in the particular case of an uncertainty on the timing epoch. The sequel of the paper is organized as follows. In section 2 we set the model and the notations. The optimal receiver expressions as well as a common suboptimal approach are explained in section 3. Section 4 and 5 give two particular implementations of suboptimal receivers. In section 6 and 7, we place the receiver design in the presence of a timing uncertainty in the factor graph framework. Finally, in section 8 simulation results show the performance achieved by the proposed approach.

2. SYSTEM MODEL

We consider a burst transmission where a sequence of L information bits is encoded by a channel encoder with code rate R. The coded bits sequence is mapped to a signalling constellation Ω of size M, resulting in a symbol sequence $\mathbf{a}$, and shaped by a unit energy square-root raised-cosine pulse $u(t)$ with roll-off α. After propagation through an AWGN channel with delay τ, the received signal can be written as

$$r(t) = \sum_{k=0}^{K-1} a_k u(t - kT - \tau) + n(t), \tag{1}$$

where $a_k \in \Omega$ are the transmitted symbols, T is the symbol duration, K is the number of symbols in the burst and $w(t)$ is the complex envelope of an AWGN with passband two-sided power spectral density $N_0/2$. At the receiver, after anti-aliasing filtering, signal $r(t)$ is sampled at a rate of $1/T_s$ ($T_s < T/1+\alpha$) leading to samples

$$r(lT_s) = \sum_{k=0}^{K-1} a_k u(lT_s - kT - \tau) + n(lT_s). \tag{2}$$

Samples $r(lT_s)$ are sufficient statistics of the received signal and may therefore be used to design the optimal receiver.

3. OPTIMAL RECEIVER AND SUBOPTIMAL APPROACH

Let $\mathbf{r}$ denote the vector which contains the samples of the received signal $r(lT_s)$. The symbol-wise optimal receiver is the one which enables to minimize the symbol error probability or equivalently,

$$\hat{a}_k = \arg\max_{\tilde{a} \in \Omega} p(a_k = \tilde{a}|\mathbf{r}), \tag{3}$$

where $\tilde{a}$ is a trial value and $\hat{a}_k$ is the decision made on transmitted symbol a_k. Note that probability $p(a_k|\mathbf{r})$ may also be regarded as the marginal of a joint probability:

$$p(a_k|\mathbf{r}) = \sum_{\sim\{a_k\}} p(\mathbf{a}|\mathbf{r},\tau)\, p(\tau|\mathbf{r}), \tag{4}$$

where the notation $\sim \{a_k\}$ denotes that the summation is made over all the variables but a_k. The computation of probability (4) is unfortunately intrinsically too complex. A common approach to simplify the problem consists therefore in approximating probability density function $p(\tau|\mathbf{r})$ by a function $\tilde{p}(\tau|\mathbf{r})$ such as

$$\tilde{p}(\tau|\mathbf{r}) = \delta(\tau - \hat{\tau}) \tag{5}$$

i.e. probability density $p(\tau|\mathbf{r})$ is assumed to be concentrated in a neighborhood of a point $\hat{\tau}$. As long as approximation (5) holds, the efficiency of the receiver will then depend on the relevance of point $\hat{\tau}$ chosen[1] to characterize distribution $p(\tau|\mathbf{r})$. A common approach is to choose the point at which the probability $p(\tau|\mathbf{r})$ is maximized i.e.

$$\hat{\tau} = \arg\max_{\tilde{\tau}} p(\tilde{\tau}|\mathbf{r}). \tag{6}$$

Since we do not have any a priori knowledge about τ, its a priori distribution $p(\tau)$ may be considered uniform and maximization problem (6) reduces to

$$\hat{\tau} = \arg\max_{\tilde{\tau}} p(\mathbf{r}|\tilde{\tau}), \tag{7}$$

i.e. to a maximum-likelihood (ML) estimation problem. Approximation (5) together with (7) constitute the basis of most of the classical approaches used to approximate optimal solution (3). Note that the ML estimation problem defined in (7) is often itself an intrinsically complex problem. In sections 4 and 5, we will remind some existing approaches to deal with this issue. Then, in the remainder of the paper, we will expose a method to approximate the optimal receiver based on the SP algorithm.

4. CONVENTIONAL ML-BASED SYNCHRONIZERS

As mentioned in the previous section the ML estimation of the timing offset (7) is an intractable problem. Consequently, rather than computing the exact ML estimate conventional synchronizers proposed in the literature [5] are smart approximations of the true ML solution. In particular, non-data-aided (NDA) synchronizers enable to decrease the problem complexity by assuming that all the possible transmitted sequences are a priori equiprobable, although the transmission may be coded. As an example, Oerder and Meyr [5] have derived a closed-form expression for the NDA timing estimation:

$$\hat{\tau} = \frac{T_s}{2\pi} \arg\left\{\sum_m |y(mT_s)|^2 e^{-j2\pi\frac{mT_s}{T}}\right\}. \tag{8}$$

where $y(mT_s)$ is the matched filter output computed at time mT_s. This method exhibits a very low complexity and is therefore well-suited for practical implementation. However, as mentioned above, (8) does not deliver the actual solution of the maximization problem (7). In some systems operating at low SNRs the performance of receivers using this approximated approach may consequently move significantly away from the performance of the optimal receiver (3). Therefore, state-of-the-art receivers, which operate at very low SNR, more and more require methods to accurately solve the maximization problem (7). In the next section, we present an iterative synchronizer which enables to converge under mild conditions to the exact ML solution.

5. SYNCHRONIZATION BASED ON THE EM ALGORITHM

The expectation-maximization (EM) algorithm, first defined by Dempster, Laird and Rubin in [6], is a method which enables to iteratively solve ML estimation problems. In the particular case of the timing synchronization, it has been shown [7] that the sequence $\{\hat{\tau}^n\}_{n=0}^{\infty}$ defined as

$$\hat{\tau}^n = \arg\max_{\tilde{\tau}} \left|\sum_k \eta_k\, y(kT + \tilde{\tau})\right| \tag{9}$$

converges under fairly general conditions to the ML estimate (7). Notation η_k denotes the first order a posteriori average of symbol a_k given current estimate $\hat{\tau}^{n-1}$ i.e.

$$\eta_k \triangleq \sum_{a\in\Omega} a\; p(a_k = a|\mathbf{r}, \hat{\tau}^{n-1}) \tag{10}$$

Note that required a posteriori probabilities $p(a_k|\mathbf{r}, \hat{\tau}^{n-1})$ are not always available in the considered receivers. These probabilities have therefore to be approximated, leading to an non-exact implementation of the EM algorithm. However, although based on an approximation, this approach has already shown its efficiency in several papers (see [8] for example).

6. DESIGN OF AN ITERATIVE RECEIVER BASED ON THE SP ALGORITHM

The sum-product (SP) algorithm [2] is a message-passing algorithm which operates on factor graphs and enables to efficiently compute marginals of the function that the graph represents. In this section we show that the factor-graph framework is well-suited to the design of close-to-optimum iterative receivers in the presence of a timing uncertainty.

First, notice that probability $p(a_k|\mathbf{r})$ required to compute the symbol-wise optimal solution (3) may written as

$$\begin{aligned} p(a_k|\mathbf{r}) &= \sum_{\sim\{a_k\}} p(\mathbf{a}|\mathbf{r},\tau)\, p(\tau|\mathbf{r}) \\ &\sim \sum_{\sim\{a_k\}} p(\mathbf{r}|\mathbf{a},\tau)\, p(\mathbf{a})\, p(\tau) \end{aligned} \tag{11}$$

where the notation $\sim$ denotes equality up to a multiplicative normalization factor. Taking then into account that matched filter outputs $y_k \triangleq y(kT+\tau)$ are sufficient statistics of the received signal $r(t)$ and that the noise which affects matched filter outputs y_k is white, we respectively have

$$p(a_k|\mathbf{r}) \sim \sum_{\sim\{a_k\}} p(\mathbf{y}|\mathbf{a},\tau)\, p(\mathbf{a})\, p(\tau) \tag{12}$$

$$\sim \sum_{\sim\{a_k\}} \prod_k p(y_k|a_k,\tau)\, p(\mathbf{a})\, p(\tau), \tag{13}$$

where $\mathbf{y}$ denotes the vector which contains matched filter outputs y_k. Marginal probability $p(a_k|\mathbf{r})$ may therefore be computed by applying the SP algorithm to the factor graph relative to $\prod_k p(y_k|a_k,\tau)\, p(\mathbf{a})\, p(\tau)$ (see Fig. 1). The box referred to as *"Code and mapping factor graph"* accounts for the graph relative to the factorization of $p(\mathbf{a})$, which has not been represented here for the sake of conciseness. Note however that if we consider a coded transmission, the dependence between coded bits introduces cycles in the graph represented in Fig. 1. In this case, it can no longer be proved that the results delivered by the SP algorithm are exact. However, empirical results show that the SP algorithm yields very

[1] In the sequel we will refer to the device which computes $\hat{\tau}$ as a synchronizer.

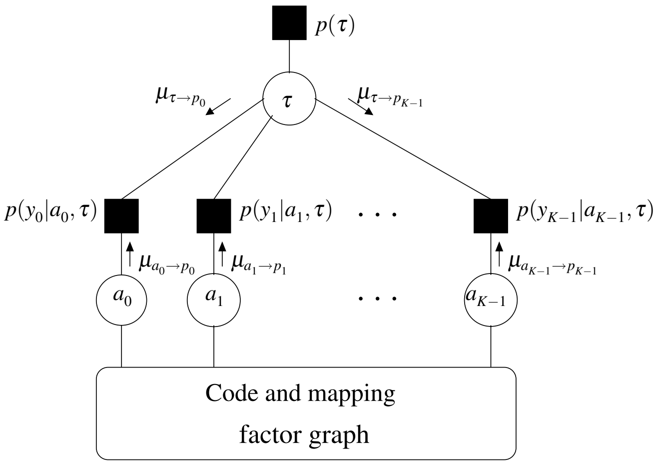

Figure 1: Factor graph representation of $\prod_k p(y_k|a_k,\tau)\, p(\mathbf{a})\, p(\tau)$. Factor nodes and variable nodes are respectively denoted by squares and circles.

good results even when the graph has cycles. Another consequence of the presence of cycles in the factor graph is that the application of the SP algorithm leads to an iterative algorithm with no natural termination. It is therefore required to define a message-passing schedule in order to specify the messages which are updated at each step of the algorithm. Denoting by $\mu^m_{a_k\to p_k}(a_k)$ (resp. $\mu^m_{\tau\to p_k}(\tau)$) the message passing from variable node a_k (resp. τ) to factor node $p(y_k|a_k,\tau)$ at iteration m, we define the following message passing schedule: at each iteration new messages $\mu^m_{\tau\to p_k}(\tau)$ are first computed by taking into account messages $\mu^{m-1}_{a_k\to p_k}(a_k)$; messages $\mu^m_{a_k\to p_k}(a_k)$ are then updated by applying the SP algorithm on the lower part of the factor graph in Fig. 1 i.e. by exploiting the code structure underlying transmitted symbols $\mathbf{a}$. It may be shown [2] that messages $\mu^m_{a_k\to p_k}(a_k)$ are actually equal to the so-called symbol extrinsic probabilities delivered by a turbo receiver. Considering then messages $\mu^m_{\tau\to p_k}(\tau)$, we have by applying SP algorithm update rules that

$$\mu^m_{\tau\to p_k}(\tau) \sim \sum_{\mathbf{a}} \prod_{l\neq k} p(y_l|a_l,\tau)\ \mu^{m-1}_{a_l\to p_l}(a_l)\ p(\tau). \qquad (14)$$

Note that since the number of messages arriving at variable node τ is large, messages $\mu^m_{\tau\to p_k}(\tau)$ may also be well-approximated by

$$\mu^m_{\tau\to p_k}(\tau) \sim \sum_{\mathbf{a}} \prod_{l} p(y_l|a_l,\tau)\ \mu^{m-1}_{a_l\to p_l}(a_l)\ p(\tau), \qquad (15)$$

i.e. by considering all the message arriving at node τ in the computation of message $\mu^m_{\tau\to p_k}(\tau)$. Doing this approximation, messages $\mu^m_{\tau\to p_k}(\tau)$ do no longer depend on index k and may simply be denoted by $\mu^m_{\tau\to p}(\tau)$. Messages $\mu_{p_k\to a_k}(a_k)$ transmitted to the code factor graph at each SP algorithm iteration may then be computed as

$$\mu^m_{p_k\to a_k}(a_k) \sim \int p(y_k|a_k,\tau)\ \mu^m_{\tau\to p}(\tau)\ p(\tau)\, d\tau. \qquad (16)$$

The integral in the right-hand side of (16) does unfortunately not have any simple analytical solution. In order to circumvent this problem , we consider in the sequel a modified version of the SP algorithm.

7. AN ITERATIVE RECEIVER BY COMBINING THE SP AND THE EM ALGORITHMS

As mentioned in section 6, the direct application of the SP algorithm to the factor graph represented in Fig. 1 is a computationally-complex task. In order to simplify the SP-algorithm implementation, we assume that message $\mu^m_{\tau\to p}(\tau)$ may be well-approximated by a delta function i.e. probability density $\mu^m_{\tau\to p}(\tau)$ will be assumed to be concentrated in a neighborhood of a point $\hat{\tau}^m$. Doing this approximation, messages $\mu^m_{p_k\to a_k}(a_k)$ may then be easily computed as follows

$$\mu^m_{p_k\to a_k}(a_k) \sim p(y_k|a_k,\hat{\tau}^m). \qquad (17)$$

We set $\hat{\tau}^m$ to the value which maximizes $\mu^m_{\tau\to p}(\tau)$ i.e.

$$\hat{\tau}^m = \arg\max_{\tilde{\tau}} \left\{ u^m_{\tau\to p}(\tilde{\tau}) \right\}. \qquad (18)$$

In [4] it is emphasized that message $u^m_{\tau\to p}(\tau)$ has exactly the same structure as likelihood function $p(\mathbf{r}|\tau)$. Equation (18) may therefore be regarded as a maximum-likelihood problem and the EM algorithm may be applied to solve it. Note that due to the structural similarity of likelihood functions $p(\mathbf{r}|\tau)$ and $u^m_{\tau\to p}(\tau)$, the application of the EM algorithm to the ML problem (17) leads to the same update equation that the one defined in (9). In this case however, exact symbol a posteriori probabilities required to implement the EM algorithm may be computed very easily as follows

$$p(a_k|\mathbf{y},\hat{\tau}^{m,n}) \sim p(y_k|a_k,\hat{\tau}^{m,n})\, \mu^{m-1}_{a_k\to p}(a_k), \qquad (19)$$

where $\hat{\tau}^{m,n}$ denotes n^{th} timing estimate generated by the EM algorithm at the m^{th} SP-algorithm iteration. Notice that both the timing update operation (9) and the evaluation of (19) are low-complexity operations. Therefore, the maximization of $u^m_{\tau\to p}(\tau)$ via the EM algorithm does not affect significantly the receiver complexity. The approach proposed in this section will be referred to as SP-EM in the sequel. In order to avoid confusion, the EM algorithm implemented in the SP-EM approach will be referred to as EM_{SP} in the sequel whereas the EM synchronizer presented in section 5 will be simply denoted EM.

8. SIMULATION RESULTS

In this section we compare the performance achieved by different kind of receivers in the presence of a timing uncertainty. We consider a rate-$1/3$ turbo coded transmission. The turbo encoder is made up with two rate-$1/2$ recursive systematic convolutional encoders with generator polynomials $(21,37)_8$, separated by an interleaver. The roll-of factor is set to $\alpha = 0.1$ and the timing offset to $\tau/T = 0.2$. The transmitted frames consist of 999 BPSK symbols.

Fig. 2 shows the estimation mean square error (MSE) and the bit-error rate (BER) achieved by different receivers versus the E_b/N_0-ratio. We consider the approaches mentioned above: the SP-EM approach, the EM-based synchronizer and the Oerder&Meyr (OM) synchronizer. Let us mention to avoid confusion that due to the particular message-passing schedule chosen in section 6, one SP-algorithm iteration corresponds to one turbo iteration. 15 EM_{SP} iterations are performed at each SP-algorithm iteration. The MSE's and

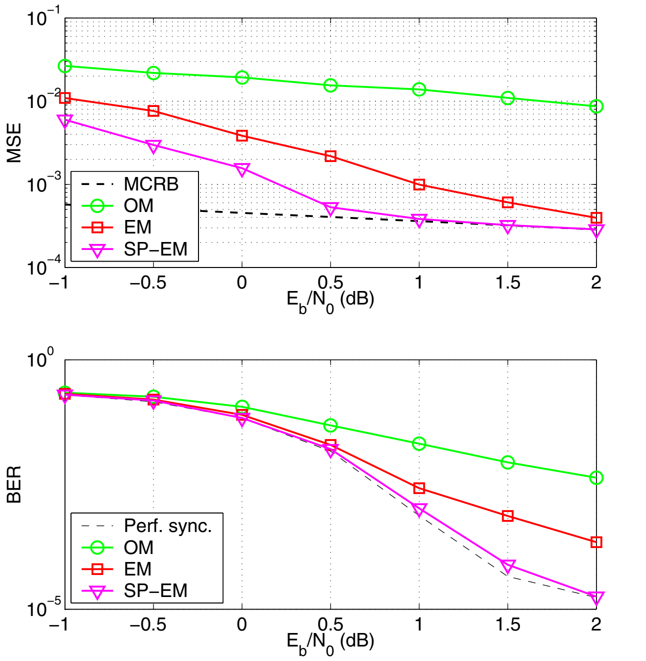

Figure 2: MSE and BER vs. the E_b/N_0-ratio at the 15th turbo iteration.

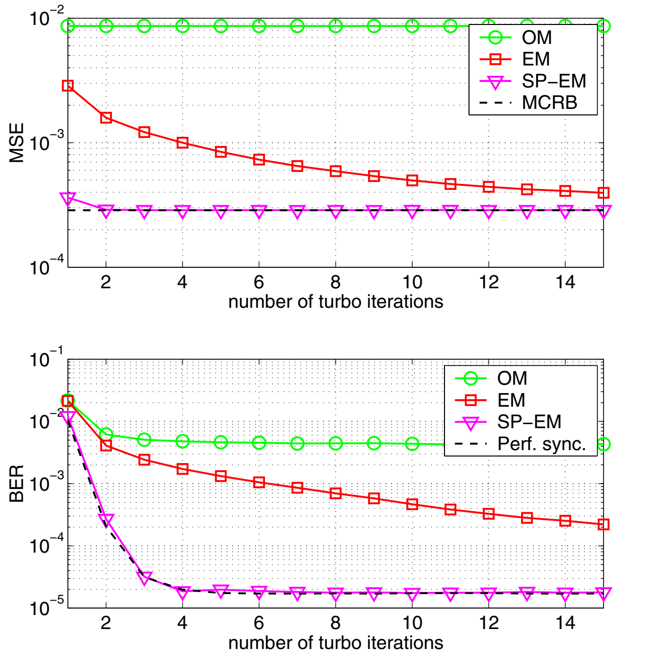

Figure 3: MSE and BER vs. the number of turbo iterations for $E_b/N_0 = 5$dB.

the BER's are respectively compared with the modified Cramer-Rao bound (MCRB) and the BER achieved by a perfectly synchronized system (Perf. sync.). We see from Fig. 2 the gain brought by the SP-EM approach. In particular, unlike the two other methods it enables to recover the BER of a perfectly synchronized system. Fig. 3 illustrates the system speed convergence in terms of MSE and BER for $E_b/N_0 = 5$dB. Let us insist on the fact that the EM_{SP} iterations performed at each turbo (or equivalently SP) iteration have a very low complexity with respect to the complexity of one turbo iteration. Comparing then the speed of convergence of the EM and the SP-EM approaches, we see that the SP-EM, which only requires a few turbo iterations to converge, leads to a receiver which has a much lower overall complexity than the one based on the EM synchronizer.

9. CONCLUSION

In this paper, we consider the design of iterative receivers in the presence of a timing uncertainty. The problem is placed into the SP algorithm and the EM algorithm framework. The proposed approach is compared with conventional methods such as methods based on Oerder&Meyr or EM-based synchronizers. Simulations results shows that the receiver design presented in this paper clearly outperforms classical approaches.

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