ALGORITHM FOR SIGNAL DECOMPOSITION BY USING THE S-METHOD

LJ. Stanković, T. Thayaparan**, M. Daković**

* Elektrotehnički fakultet, University of Montenegro, Podgorica, Serbia and Montenegro
** Radar Applications and Space Technology, Defence Research and Development, Ottawa, Ontario, Canada

ABSTRACT

Decomposition of multicomponent noisy signals is considered in the paper. A novel decomposition algorithm is presented and applied to the synthetic and real radar signals. The algorithm is based on time-frequency analysis and its eigenvalue decomposition. It has been statistically shown that the presented algorithm produces satisfactory results even in a very low signal to noise ratio. Obtained results are robust to the algorithm parameters.

1. INTRODUCTION

Eigenvalue decomposition of the inverse discrete rearranged Wigner distribution (WD) is used in signal synthesis [2]. If the considered two-dimensional function is a valid WD then this procedure produces exactly one eigenvalue different from zero, with eigenvector corresponding to the signal, up to a constant phase shift. Time-frequency representation referred to as the S-method has a property that, under certain assumptions, its value for multicomponent signals is equal (or very close) to the sum of the WDs of each component separately [5, 6]. In this paper we have proposed an algorithm that combines two previous facts. We introduced eigenvalue decomposition of the inverse discrete rearranged S-method. It can produce eigenvectors proportional to the signal components. In this way we achieved decomposition of a multicomponent signal.

After a review of the WD based synthesis in Section 2, Section 3 introduces and presents the decomposition algorithm. In Sections 4 and 5 numerical and statistical analysis of the algorithm on signals, including signals with a high amount of noise and real radar signals, are performed. An extended and detailed version of this algorithm is described in [7].

2. THEORY

A discrete form of the Wigner distribution is defined by [1, 3, 4, 9]

$$WD(n,k)=\sum_{m=-N/2}^{N/2} f(n+m)f^*(n-m)e^{-j\frac{2\pi}{N+1}m(2k)}, \quad (1)$$

where we assumed that the signal $f(n)$ is time limited within $|n|\leq N/2$ and omitted a constant multiplication factor of 2. Inversion relation for the Wigner distribution reads

$$f(n+m)f^*(n-m)=\tfrac{1}{N+1}\sum_{k=-N/2}^{N/2} WD(n,k)e^{j\frac{2\pi}{N+1}m(2k)}.$$

After substitutions $n_1=n+m$ and $n_2=n-m$ we get

$$f(n_1)f^*(n_2)=\tfrac{1}{N+1}\sum_{k=-N/2}^{N/2} WD(\frac{n_1+n_2}{2},k)e^{j\frac{2\pi}{N+1}k(n_1-n_2)}. \quad (2)$$

Here, we assumed that an appropriate interpolation is done, in order to calculate $WD((n_1+n_2)/2,k)$ for cases when $(n_1+n_2)/2$ is not an integer. Denoting by $R(n_1,n_2)$ the right-hand side of the previous equation, we get

$$R(n_1,n_2)=f(n_1)f^*(n_2). \quad (3)$$

Matrix form of (3) reads

$$\mathbf{R}=\mathbf{f}(n)\mathbf{f}^*(n), \quad (4)$$

where: $\mathbf{f}(n)$is a column vector whose elements are the signal values, $\mathbf{f}^*(n)$ is a row vector (Hermitian transpose of $\mathbf{f}(n)$) and $\mathbf{R}$ is the matrix whose elements are $R(n_1,n_2)$.

The eigenvalue decomposition of matrix $\mathbf{R}$ reads

$$\mathbf{R}=\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T=\sum_{i=1}^{N+1}\lambda_i\mathbf{u}_i(n)\mathbf{u}_i^*(n), \quad (5)$$

where λ_i are eigenvalues and $\mathbf{u}_i(n)$ are eigenvectors of the matrix $\mathbf{R}$. By comparing (4) and (5), it follows that the matrix $\mathbf{R}$ can be decomposed by using only one eigenvalue different from zero. Having in mind this fact we have $\mathbf{f}(n)\mathbf{f}^*(n)=\lambda_1\mathbf{u}_1(n)\mathbf{u}_1^*(n)$, resulting in $\lambda_1=E_f$, where E_f is energy of the signal $f(n)$.

Eigenvector $\mathbf{u}_1(n)$ is equal to the signal vector $\mathbf{f}(n)$ up to the constant amplitude and phase factor. It means that matrix $\mathbf{R}$, through its eigenvalue decomposition (5), can be used to check if an arbitrary 2D function $WD(n,k)$ is a valid Wigner distribution.

The same relations can be used in signal synthesis: starting from a given function $WD(n,k)$, calculating matrix $\mathbf{R}$, performing eigenvalue decomposition (5) and using the first (largest) eigenvalue and corresponding eigenvector we obtain signal such that its Wigner distribution is a valid Wigner distribution with minimal mean square error as compared to the given arbitrary function $WD(n,k)$, [2].

2.1 S-method

A definition of the STFT is

$$STFT(n,k)=\sum_{m=-N/2}^{N/2} f(n+m)w(m)e^{-j\frac{2\pi}{N+1}mk}. \quad (6)$$

where $w(n)$ is time window.

The S-method is defined as [5]

$$SM(n,k)=\tfrac{1}{N+1}\sum_{l=-L}^{L} STFT(n,k+l)STFT^*(n,k-l) \quad (7)$$

Basic property of the S-method is that it can produce time-frequency representation of a multicomponent signal equal to the sum of the WD of each component, avoiding cross-terms. Note that the spectrogram can be obtained for $L=0$ and the WD for $L=N/2$ [5].

Proposition 1: *Consider a multicomponent signal*

$$f(n) = \sum_{i=1}^{M} f_i(n),$$

where $f_i(n)$ are monocomponent signals. Assume that the STFT of each component lies inside the region $D_i(n,k)$, $i = 1, 2, ..., M$. Denote the length of $i-th$ region along k, for a given n, by $2B_i(n)$, and its central frequency by $k_{0i}(n)$. The S-method of $f(n)$ is equal to the sum of the Wigner distributions, $WD_i(n,k)$, $i = 1, 2, ..., M$, of each signal's component separately,

$$SM(n,k) = \sum_{i=1}^{M} WD_i(n,k), \tag{8}$$

if the regions $D_i(n,k)$, $i = 1, 2, ..., M$, do not overlap, $D_i(n,k) \cap D_j(n,k) = \emptyset$ for $i \neq j$, and the number of terms L in (7), for a point (n,k), is defined by:

$$L(n,k) = \begin{cases} B_i(n) - |k - k_{0i}(n)| & \text{for } (n,k) \in D_i(n,k) \\ 0 & \text{elsewhere .} \end{cases} \tag{9}$$

Proof is very similar to the one provided for the continuous S-method case. It can be found in [6].

Note: If we choose constant number of terms in (7) such that $L \geq \max_{n,k}\{L(n,k)\}$ we get $SM(n,k) = \sum_{i=1}^{M} WD_i(n,k)$, if the regions $D_i(n,k)$, $i = 1, 2, .., M$, are at least $2L$ apart along the frequency axis, i.e., $|k_{0i}(n) - k_{0j}(n)| > B_i(n) + B_j(n) + 2L$, for each i, j and n.

This is the **S-method with constant value of** L, as it was originally introduced in [5] and as it will be used in this paper. The signal dependent method (9) would be more accurate, but also more complex. Constant number of terms is used here in numerical implementation since it is much simpler for implementation, producing satisfactory and robust results.

3. DECOMPOSITION ALGORITHM

Let us consider multicomponent signal

$$f(n) = \sum_{i=1}^{M} f_i(n),$$

and assume that the signal components satisfy conditions mentioned in the Proposition 1. Then the S-method of the considered signal leads to

$$SM(n,k) = \sum_{i=1}^{M} WD_i(n,k). \tag{10}$$

Let us introduce the notation

$$R_{SM}(n_1,n_2) = \tfrac{1}{N+1} \sum_{k=-N/2}^{N/2} SM(\frac{n_1+n_2}{2}, k) e^{j\frac{2\pi}{N+1}k(n_1-n_2)}. \tag{11}$$

If we write inversion formula for each WD (2) in (10), after substitution in (11), we obtain

$$\mathbf{R}_{SM} = \sum_{i=1}^{M} \mathbf{f}_i(n)\mathbf{f}_i^*(n). \tag{12}$$

Using eigenvalue decomposition of the matrix $\mathbf{R}_{SM}$, whose elements are $R_{SM}(n_1,n_2)$, we get

$$\mathbf{R}_{SM} = \sum_{i=1}^{N+1} \lambda_i \mathbf{u}_i(n)\mathbf{u}_i^*(n). \tag{13}$$

As in the case of Wigner distribution, we can conclude that $\lambda_i = E_{f_i}$, $i = 1, 2, ..., M$ and $\lambda_i = 0$ for $i = M+1, ..., N$, i.e.,

$$\lambda_i = \sum_{l=1}^{M} E_{f_l}\delta(i-l). \tag{14}$$

The eigenvectors $\mathbf{u}_i(n)$ will be equal to the signal components $\mathbf{f}_i(n)$, up to the phase and amplitude constants. Amplitude constants are again contained in the eigenvalues λ_i. Note that it is assumed that signal components do not overlap in the time-frequency plane what implies their orthogonality. Moreover if all nonzero eigenvalues are different from each other, then it can be easily proved that decomposition (13) is unique and that it can be related with (12) term by term. Thus, the reconstructed signal can be written as

$$f_{rec}(n) = \sum_{i=1}^{M} \sqrt{\lambda_i} u_i(n)$$

It is equal to the original signal, up to the phase constant in each component.

When there exists a very strong disturbing signal, like sea-clutter in HF radar signal, we can omit the first, strongest component, and define reconstructed signal as

$$f_{rec}(n) = \sum_{i=2}^{M_1} \sqrt{\lambda_i} u_i(n)$$

where M_1 is the expected number of components.

3.1 Calculation procedure

- Step 1. Choose appropriate time window $w(n)$.
- Step 2. Calculate the STFT of the zero-padded and oversampled signal by factor 2. Oversampling is neccesery in order to avoid noninteger indices in (11).
- Step 3. Choose value of L according to the conditions mentioned in Proposition 1.
- Step 4. Calculate the S-method of the signal according to (7) for a given L.
- Step 5. Calculate matrix $\mathbf{R}_{SM}$ according to (11).
- Step 6. Decompose $\mathbf{R}_{SM}$ into eigenvectors and eigenvalues.

First M eigenvectors, with corresponding eigenvalues are separated signal components. We can reconstruct whole signal by summing extracted components.

3.2 Examples

Let us consider four component signal, where each component is a Gaussian chirp, contaminated with a white Gaussian complex noise $\varepsilon(n)$

$$x(n) = \sum_{k=1}^{4} A_k e^{j\omega_k(n-d_k)+ja_k\frac{(n-d_k)^2}{2}} e^{-\frac{(n-d_k)^2}{256}} + \varepsilon(n)$$

where $d_1 = d_2 = -64$, $d_3 = 0$, $d_4 = 64$, , $\omega_1 = \omega_4 = 0$, $\omega_2 = -\frac{3\pi}{4}$, $\omega_3 = \frac{3\pi}{4}$, $a_1 = a_3 = \frac{1}{256\pi}$, $a_2 = a_4 = -\frac{1}{256\pi}$,and $-128 \leq n \leq 127$. Note that the signal components are separated in time-frequency plane.

In the noiseless case we will assume that $\varepsilon(n) = 0$ and $A_1 = 1.3$, $A_2 = 1.2$, $A_3 = 1.1$ and $A_4 = 1.0$. Rectangular window of 64 samples length is used and $L = 36$ is chosen in order to satisfy conditions form Proposition 1. The results are presented in Table 1 and Fig.1. Energy of each signal component and corresponding eigenvalue is presented in Table 1. Note that eigenvalues highly corresponds to the components energies. Fig.1 presents spectrogram of the original signal, TFRs of the first four eigenvectors and TFR of the reconstructed signal.

Component	1	2	3	4
Energy	67.8	57.7	48.5	40.1
Eigenvalue	67.5	57.5	48.3	39.9

Table 1: Component energies and corresponding eigenvalues

The decomposition algorithm is applied to the considered signal with $A_1 = A_2 = A_3 = A_4 = 1$ for signal to noise ratio 0dB (Fig.2) and -4dB (Fig.3). Hanning window, 128 samples length and $L = 16$ is used in booth cases. In booth cases all signal's components are separated, and reconstructed signal is obtained without noise in the parts of the time-frequency plane where there is no signal's components. In noisy cases equal components energies can be used because high noise introduce different eigenvalues in the decomposition process, avoiding possible ambiguity.

4. APPLICATION TO THE ANALYSIS OF HF RADAR SIGNALS IN STRONG CLUTTER

Proposed algorithm is also applied to the real radar signals. The signals considered here are experimental plane data, as used in [8]. The plane is a King-Air 200 performing maneuvers, tracked by a high frequency surface wave radar (HF-SWR), using a 10-element linear receiving antenna array. The radar carrier frequency is 5.672MHz and the pulse repetition frequency is 9.17762 Hz. Each trial corresponds to a block of 256 pulses. Therefore the CIT (coherent integration time) of each signal is 27.89 s. Here we deal with week target signal and very strong clutter (in the middle of the frequency axis). Since in this case signals components have long duration, rectangular window 256 samples length is applied, and $L = 32$ is chosen. Second eigenvector is used as reconstructed signal in all cases. In order to present booth signals in spectrogram (upper left graphics) high values of clutter component are cut off. Results are presented in Fig.4-6. As we can see in all cases target signal is successfully detected. More results and detailed analysis of radar signals can be found in [7, 8].

5. NOISE ANALYSIS

If the analyzed signal is corrupted with noise, then assumption that only M eigenvalues are different from zero is not valid. On the other hand noise components are distributed over all N eigenvectors.

Described decomposition algorithm is analyzed for various SNR and with different algorithm parameters (L in the S-method calculation and window length in the STFT calculation). The results are shown in tables 2 and 3. For each combination of SNR, L and window length h algorithm is repeated 100 times, giving total of 400 components for detection. The percentage of missed components is shown in tables and used as measure of algorithm robustness.

It is shown that decomposition algorithm is very robust to the parameters values. Heuristic analysis gives some estimations of parameter values. Namely if we want to obtain Wigner distribution than the parameter L should be equal to the half of the component frequency support (in discrete

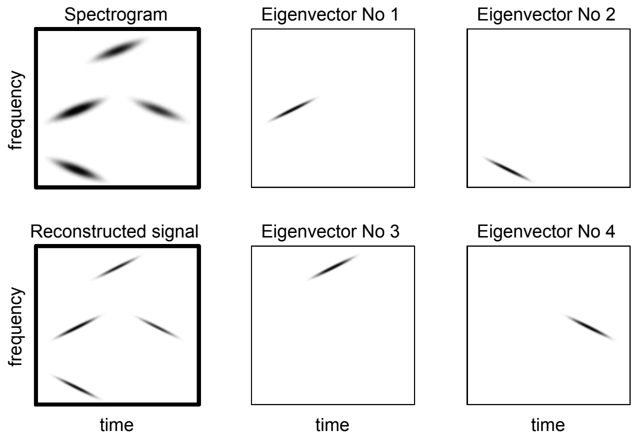

Figure 1: Spectrogram, reconstructed signal and TFRs of the first four eigenvectors - noiseless case

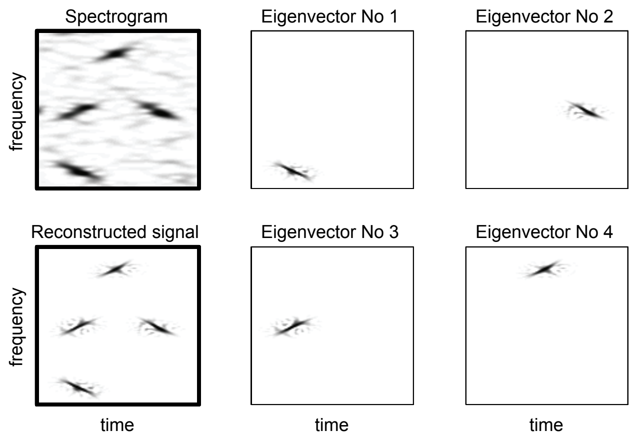

Figure 2: Spectrogram of noisy signal, eigenvectors TFRs and TFR of the reconstructed signal for 0dB SNR

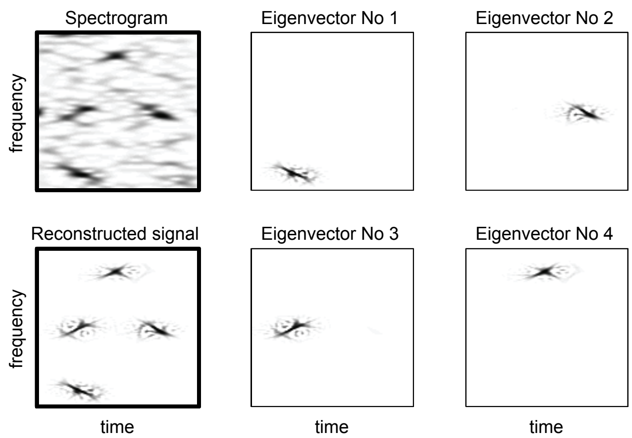

Figure 3: Spectrogram of noisy signal, eigenvectors TFRs and TFR of the reconstructed signal for -4dB SNR.

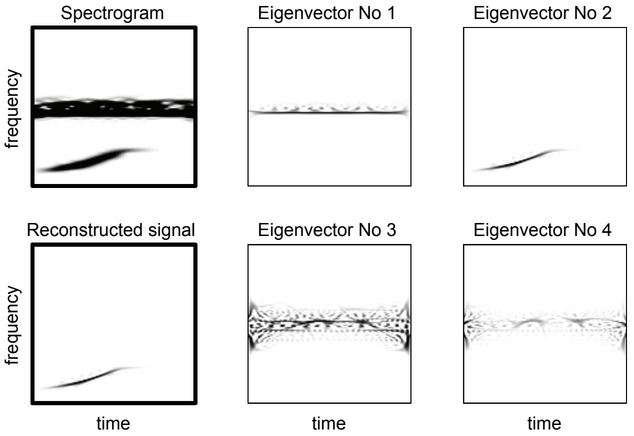

Figure 4: Decomposition of a real HF radar signal in a strong sea clutter (realization 1)

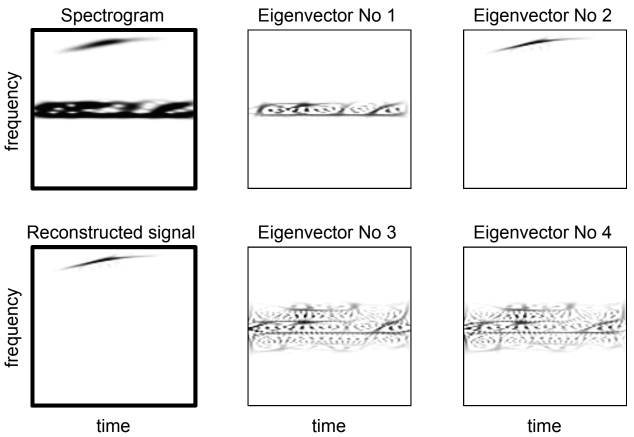

Figure 5: Decomposition of a real HF radar signal in a strong sea clutter (realization 2)

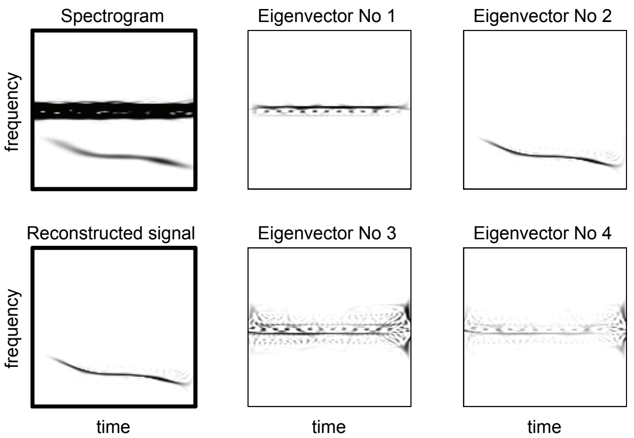

Figure 6: Decomposition of a real HF radar signal in a strong sea clutter (realization 3)

SNR	$L = 8$	$L = 16$	$L = 32$	$L = 64$
0dB	3.50%	0.00%	0.00%	0.00%
-2dB	6.50%	0.00%	0.50%	0.00%
-4dB	10.00%	3.25%	2.50%	3.50%
-6dB	20.25%	14.75%	11.50%	7.50%
-8dB	35.50%	29.50%	22.75%	21.00%

Table 2: Sensitivity of the proposed algorithm to the choice of L for various SNR. Window length is 128 samples.

SNR	$h = 160$	$h = 128$	$h = 64$
0dB	0.25%	0.00%	0.50%
-2dB	0.50%	0.50%	0.75%
-4dB	3.50%	2.50%	4.50%
-6dB	16.25%	11.50%	14.00%
-8dB	28.00%	22.75%	33.25%

Table 3: Sensitivity of the proposed algorithm to the choice of window length h for various SNR with $L = 32$

domain). Time window in STFT calculation should be long enough so the whole component is covered by window. On the other hand very large window can combine two components in one eigenvector.

6. CONCLUSION

Proposed decomposition algorithm is theoretically derived, applied to the synthetic signals with and without noise and to the real radar signals. In all considered cases decomposition is successfully done. It is shown that choice of algorithm parameters does not have high influence to the decomposition process. Results obtained in noiseless case numerically proves theoretically obtained conclusions.

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